PRIVATE
DEPOSITORY

SHL
WITHDRAWN

Digitized by the Internet Archive
in 2026

https://archive.org/details/london-mathematical-society-proceedings_1913_12

PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY

SECOND SERIES

VOLUME 12

LONDON:
PRINTED BY C. F. HODGSON & SON,
2 NEWTON STREET, KINGSWAY, W.C.;
AND PUBLISHED FOR THE SOCIETY BY
FRANCIS HODGSON, 89 FARRINGDON STREET, E.C.

1913

LONDON:
PRINTED BY C. F. HODGSON & SON,
2 NEWTON STREET, KINGSWAY, W.C.

BIBL.
LONDIN.
UNIV.

CONTENTS.

Records of Proceedings at Meetings, pp. iv–xlviii.

Library [Presents to Library (p. xlix), Exchanges (p. xlix)], pp. xlix–l.

Obituary Notices of Paul Gordan (p. li) and Sir G. H. Darwin (p. lv), pp. li–lix.

Papers published in the *Proceedings*, pp. 1–483.

Index of Proceedings, p. 484.

Index of Papers read and Communications made to the Society during the Session 1912–1913, p. 485.

Index of Papers published in this Volume arranged according to the names of Authors, pp. 486–487.

Index according to Subjects, p. 488.

RECORDS OF PROCEEDINGS AT MEETINGS

SESSION NOVEMBER, 1912–JUNE, 1913.

Thursday, November 14th, 1912.

ANNUAL GENERAL MEETING.

Dr. BAKER (President, 1910–11), and afterwards Prof. LOVE (newly elected President), in the Chair.

Present twenty-nine members and a visitor.

The minutes of the last meeting were read and confirmed.

Messrs. S. Lees, E. L. Watkin, P. L. Pethick, G. B. Jeffery, and Farid Boulad were nominated as members.

Messrs. J. R. Airey, F. E. Edwardes, A. B. Grieve, L. J. Mordell, and F. B. Pidduck were admitted into the Society.

The Treasurer, Sir Joseph Larmor, presented his Report. On the motion of Dr. Baker, seconded by Lt.-Col. Cunningham, the Report was received.

On the nomination of the President, Dr. J. G. Leathem was reappointed Auditor.

The Council and Officers for the ensuing Session were elected as follows:—President, Prof. A. E. H. Love; Vice-Presidents, Dr. H. F. Baker and Mr. J. E. Campbell; Treasurer, Sir Joseph Larmor; Secretaries, Mr. J. H. Grace and Dr. T. J. I'A. Bromwich; other members of the Council, Dr. W. Burnside, Mr. A. L. Dixon, Prof. L. N. G. Filon, Mr J. H. Jeans, Prof. E. W. Hobson, Mr. J. E. Littlewood, Prof. H. M. Macdonald, Major P. A. MacMahon, Mr. H. W. Richmond, Dr. A. E. Western.

Prof. Love (newly elected President) having taken the Chair, the following papers were communicated :—

*On Groups of Linear Substitutions of Finite Order which Possess Quadratic Invariants : Prof. W. Burnside.

*Some Points in the Geometry of Cubic Surfaces : Mr. A. B. Grieve.

*On the Representation of a Summable Function by Means of a Series of Finite Polynomials : Prof. E. W. Hobson.

*On the Irreducibility of Legendre's Polynomials : Mr. J. B. Holt.

*Theory of Functions of Real Vectors : Mr. E. Cunningham.

Some Solutions of Laplace's Equation : Mr. G. N. Watson.

*On the Determination of the Summability of a Function by means of its Fourier Constants : Prof. W. H. Young.

ABSTRACTS.

On Groups of Linear Substitutions of Finite Order which Possess Quadratic Invariants : Prof. W. Burnside.

A group of linear substitutions of finite order, the coefficients in which are all real, necessarily possesses a quadratic invariant. In this paper it is shewn that :—

A group of linear substitutions of finite order, which possesses a quadratic invariant of non-zero determinant, can always be transformed into a group with real coefficients.

This is proved by considering the simultaneous reduction of a definite positive Hermitian form and any quadratic form of non-zero determinant.

It is also shewn that an irreducible group of linear substitutions of odd degree, in which all the characteristics are real, necessarily possesses a quadratic invariant.

Some Points in the Geometry of Cubic Surfaces : Mr. A. B. Grieve.

The object of this paper is to indicate some results arising from the method of generation of a cubic surface by means of a network of cubic curves (Reye, *Math. Ann.*, Bd. 55).

(1) The first portion of the paper deals with the relation which Reye's "Conjugate Networks" bears to a particular double-six of lines—thus shewing that there are 36 pairs of such conjugate networks.

It proceeds to shew how the 27 lines arise as degenerate cases of the

* Printed in this volume.

curves of conjugate networks, and to use this result to find out how the various curves of the 72 networks intersect, and to deduce other results.

(2) In the second portion of the paper the equations of the 36 "Schür Quadrics" are found, though the form of some of the equations is not very satisfactory.

These equations are then used to deduce one or two interesting theorems regarding these "Schür Quadrics"; and, finally, two theorems regarding the straight lines are given.

On the Representation of a Summable Function by a Series of Finite Polynomials: Prof. E. W. Hobson.

If $f(x)$ denote any summable function, whether limited or not, defined in a finite interval of the real variable x, it is shown in the communication that it is possible to determine a series of which the terms are finite polynomials that has the following properties:—

(1) The series converges to the value $f(x)$ at every point of the interval with the possible exception of the points that belong to an exceptional set of which the measure is zero.

(2) For each point x, not belonging to the exceptional set, the terms of the series are all positive, from and after some fixed term, dependent in general on the value of x.

(3) It is possible to determine a set of points of which the measure is as near as we please to that of the whole interval such that the series converges uniformly in the set. This kind of convergence may be spoken of as quasi-uniform convergence of the series.

(4) The method of term-by-term integration is applicable to the series when the integration is through any measurable set whatever contained in the given interval. This also holds when the series and the function are first multiplied by any summable function $\phi(x)$, provided one at least of the two functions $f(x)$, $\phi(x)$ is limited in the interval of representation.

The above results hold for summable functions of any number of variables defined in a finite "cell."

The methods of this paper are applied to give a simple proof of a general convergence theorem which has been established by the author in an earlier paper and applied to questions of convergence of various well known types of series.

An application is also given to obtain certain very general results obtained by Haar and Mercer relating to the convergence of the Sturm-

Liouville series of normal functions corresponding to a given summable function.

On the Irreducibility of Legendre's Polynomials: Mr. J. B. Holt.

The present paper is a continuation of one read on May 5, in which it was proved that if p be an odd prime, and n have one of the forms

$$p+1, \quad p, \quad p-1, \quad p-2, \quad 2p-1, \quad 2p-2,$$

$P_n(x)$ is irreducible if we neglect the root $x=0$ for $n=$ odd. The same method shews that if n have one of the forms

$$p+3, \quad p+2, \quad p-3, \quad p-4, \quad 2p-1, \quad 2p-4,$$

then $P_n(x)$, if reducible, has a rational factor ax^2+b. It is here shewn that such a factor must be $3x^2-1$; so that, unless $P_n(x)=0$ when $3x^2=1$, P_n is irreducible when n is one of the above forms.

It is usually possible to shew that $P_n(x)$ and $3x^2-1$ cannot vanish together. This is done by means of the congruences

$$P_{p+r} \equiv x^p P_r \pmod{p},$$

$$P_{p-r} \equiv P_{r-1} \pmod{p},$$

r being $< p$. In this way it appears that $n=p+3$, $p-4$, $2p-4$ are irreducible cases, and also that if p, q be consecutive primes whose difference is less than ten, each of the cases $p, p+1, \ldots, q$ is irreducible. This proves that for $n < 114$ P_n is irreducible, and further examination shews that for $n < 200$ there are only three exceptional cases, namely $n=122, 185, 186$.

Another proof of the fundamental lemma is given. It appears that if the first r coefficients of a polynomial be divisible by p, whilst the first is not divisible by p^{s+1}, there is an irreducible factor of degree r/s at least, and that, if every coefficient but one be divisible by p, every irreducible factor has the same property.

Theory of Functions of Real Vectors: Mr. E. Cunningham.

The paper deals with the functional relation between two vector quantities. A vector quantity having a definite value at each point of space is called a vector function of the vector which determines the position of a point relative to a fixed point. The theory of functions of a single variable consists mainly in classifying functions under more or less comprehensive headings — *e.g.*, continuity, differentiability, integrability.

These classifications are extended to the more general case of functions of a vector variable. The analogue of differentiability is found in the existence of the *divergence* and *curl* of a vector, defined respectively as $\underset{V\to 0}{\mathrm{Lt}} \frac{1}{V}\int_S (\mathbf{fdS})$ and $\underset{\delta S\to 0}{\mathrm{Lt}} \frac{1}{S}\int (\mathbf{fds})$, where in the first S is any closed surface including a volume V, and the limit is assumed to be a definite limit which is independent of the way in which V shrinks to zero, and in the second s is a closed curve including an area S, and the limit is assumed to be a definite limit which is independent of the way in which S shrinks to zero.

The main purpose of the paper is to introduce certain mean-value theorems which are generalizations of the well known mean value theorems

$$f(x+h) = f(x)+hf'(x+\theta h),$$

$$\int_a^b f(x)\,dx = (b-a)\,f\{a+\theta(b-a)\},$$

$$\int_a^b f(x)\,\phi(x)\,dx = f\{a+\theta_1(b-a)\}\int_a^b \phi(x)\,dx,$$

$$\int_a^b f(x)\,\phi(x)\,dx = f(a)\int_a^\xi \phi(x)\,dx+f(b)\int_\xi^b \phi(x)\,dx,$$

the conditions under which these theorems are true having complete analogues in each case.

Some Solutions of Laplace's Equation: Mr. G. N. Watson.

The result which Dr. Bromwich recently communicated* to the Society that

$$F_n(r, z) = r^n\{\log(r+z)\,P_n - 2\phi_n(\mu)\},$$

where $\phi_n(\mu) = \frac{2n-1}{1.2n}P_{n-1} - \frac{2n-3}{2.2n-1}P_{n-2} + \frac{2n-5}{3.2n-2}P_{n-3} - \ldots,$

the arguments of the Legendre functions being

$$\mu = \cos\theta = z/r,$$

is a solution of Laplace's equation, suggested to me the obvious fact that

$$\frac{\partial}{\partial n}(r^n P_n), \quad \frac{\partial^2}{\partial n^2}(r^n P_n), \quad \ldots,$$

* "Records of Proceedings at Meetings," June, 1912.

are solutions of Laplace's equation and remain solutions when n is made an integer after differentiation.

The expression $\frac{\partial^m}{\partial n^m}(r^n P_n)$ suggests a trial solution of the form

$$V_{m,n} = \sum_{p=0}^{m} r^n (\log r)^{m-p} m! \, \Theta_{p,n} \div (m-p)!,$$

where Θ_p is a function of μ and not of r; it is found, by substitution in Laplace's equation, that Θ_p satisfies the equation of mixed differences

$$\nabla_n \Theta_{p,n} + (2n+1)\,\Theta_{p-1,n} + \Theta_{p-2,n} = 0 \quad (\Theta_{-2,n} = \Theta_{-1,n} = 0),$$

where ∇^n is Legendre's operator $\frac{d}{d\mu}\left\{(1-\mu^2)\frac{d}{d\mu}\right\} + n(n+1)$.

If we suppose that $\Theta_{p,n}$ reduces to T_p when we put $n = 0$, we get

$$\nabla_0 T_p + T_{p-1} + T_{p-2} = 0.$$

Solutions of this mixed difference equation are given by

$$T_0 = 1, \qquad T_1 = \log(1+\tfrac{1}{2}t), \qquad T_2 = \sum_{s=1}^{\infty} \frac{(-)^{s-1} t^s}{2^s . s^2},$$

$$T_{2p+1} = \sum_{s=p+1}^{\infty} \frac{(-)^{s+p-1}}{2^s . s} \sigma_{s-1,p}\, t^s, \qquad T_{2p+2} = \sum_{s=p+1}^{\infty} \frac{(-)^{s+p-1}}{2^s . s^2} \sigma_{s-1,p}\, t^s,$$

where $$t = \mu - 1 = -2\sin^2 \tfrac{1}{2}\theta,$$

while $\sigma_{s,p}$ is the sum of the products p at a time of $1^{-2}, 2^{-2}, \ldots, s^{-2}$; the functions T_p have no singularities in real space except on the negative part of the axis of harmonics; also*

$$tT'_{2p+2} = T_{2p+1}, \qquad (t+2)\,T'_{2p+1} = T_{2p}.$$

The expressions for $V_{m,0}$ give the following solutions of Laplace's equation

(i) 1, (ii) $\log r + \log(\cos^2 \tfrac{1}{2}\theta)$,

(iii) $(\log r)^2 + 2\log r \log\cos^2(\tfrac{1}{2}\theta) - 2\sum_{s=1}^{\infty} s^{-2} \sin^{2s}(\tfrac{1}{2}\theta)$,

(iv) $(\log r)^3 + 3(\log r)^2 \log\cos^2(\tfrac{1}{2}\theta) - 6\log r \sum_{s=1}^{\infty} s^{-2}\sin^{2s}(\tfrac{1}{2}\theta)$
$+ 6\sum_{s=2}^{\infty} s^{-1}\{1^{-2} + 2^{-2} + \ldots + (s-1)^{-2}\}\sin^{2s}(\tfrac{1}{2}\theta)$.

... ...

* Dashes denote differentiations with regard to t or μ.

Further, it can be verified that solutions of the equation

$$\nabla_n \Theta_{p,n} + (2n+1)\,\Theta_{p-1,n} + \Theta_{p-2,n} = 0,$$

are given by the formulæ

$$\Theta_{2p,n} = \sum_{k=0}^{2p} v_k T_{2p-k}, \qquad \Theta_{2p+1,n} = \sum_{k=0}^{2p+1} u_k T_{2p+1-k},$$

where u_k, v_k are functions of μ chosen to satisfy the equations

$$\nabla_n u_{k+1} - u_k - 2\mu u_k' + (-)^k u_k' + (2n+1)\,v_k = 0,$$

$$\nabla_n v_{k+1} - v_k - 2\mu v_k' - (-)^k v_k' + (2n+1)\,v_k = 0.$$

The solution of Bromwich suggests that we should try to find solutions of these equations in the form of series of Legendre functions; and we may verify that for $p = 1, 2, 3, \ldots$, we may write

$$u_p = \sum_{q=1}^{n} \delta_{p,q} (-)^{(p-1)q} P_{n-q}(\mu), \qquad v_p = \sum_{q=1}^{n} \delta_{p,q} (-)^{pq} P_{n-q}(\mu),$$

where $\delta_{p,q}$ depends on n, p and q only (not on μ), provided that $\delta_{p,q}$ satisfies the relations

$$\delta_{1,q} = 2(-)^q \frac{2n-2q+1}{q\,(2n-q+1)},$$

$$(-)^q q\,(2n-q+1)\,\delta_{p+1,q} + \{(-)^q (2n+1) - (2n+1-2q)\}\,\delta_{p,q} = 2(2n+1-2q)\{\delta_{p,q-1} + \delta_{p,q-2} + \ldots + \delta_{p,1}\}.$$

And it may be verified, by some tedious algebra that a solution of this partial difference equation is given by

$$\delta_{p,q} = \sum_{s=1}^{q} \epsilon_{q,s} \lambda_s^{p-1},$$

where $$\lambda_q = \frac{(-)^q (2n+1-2q) - 2n - 1}{q\,(2n-q+1)},$$

and, if $\alpha(\alpha+1)\ldots(\alpha+s-1) = \alpha^{(s)}$, $\quad \alpha(\alpha-1)\ldots(\alpha-s+1) = \alpha^{(-s)}$,

then

$$\epsilon_{2q,2s} = -\frac{(\frac{1}{2})^{(q)} (\frac{1}{2})^{(q-s)} (\frac{1}{2})^{(s)}}{1^{(q)} 1^{(q-s)} 1^{(s-1)}} \frac{(n-\frac{1}{2})^{(-q+1)} (n-s-\frac{1}{2})^{(-q+1)}}{n^{(-q)} (n-s)^{(-q)}} \frac{n-2q+\frac{1}{2}}{n-s+\frac{1}{2}},$$

$$\epsilon_{2q-1,2s} = \frac{(\frac{1}{2})^{(q-1)} (\frac{1}{2})^{(q-s-1)} (\frac{1}{2})^{(s)}}{1^{(q-1)} 1^{(q-s-1)} 1^{(s-1)}} \frac{(n-\frac{1}{2})^{(-q+1)} (n-s-\frac{1}{2})^{(-q+1)}}{n^{(-q)} (n-s)^{(-q)}} \frac{n-2q+\frac{1}{2}}{n-s+\frac{1}{2}},$$

$$\epsilon_{2q,\,2s+1} = \frac{(\frac{1}{2})^{(q)}\,(\frac{1}{2})^{(s-1)}\,(\frac{1}{2})^{(q-s)}}{1^{(q)}\,1^{(s-1)}\,1^{(q-s)}}\;\frac{(n-\frac{1}{2})^{(-q+1)}\,(n-s+\frac{1}{2})^{(-q)}}{n^{(-q)}\,(n-s)^{(-q)}}\;\frac{n-2q+\frac{1}{2}}{s-\frac{1}{2}},$$

$$\epsilon_{2q-1,\,2s-1} = -\frac{(\frac{1}{2})^{(q-1)}\,(\frac{1}{2})^{(s-1)}\,(\frac{1}{2})^{q-s}}{1^{(q-1)}\,1^{(s-1)}\,1^{(q-s)}}\;\frac{(n-\frac{1}{2})^{(-q+1)}\,(n-s+\frac{1}{2})^{(-q+1)}}{n^{(-q)}\,(n-s)^{(-q+1)}}\;\frac{n-2q+\frac{3}{2}}{s-\frac{1}{2}}.$$

In particular, harmonics with $n = 2$ are given by

(i) $r^2 P_2$,

(ii) $r^2\left[P_2 \log r + P_2 \log \tfrac{1}{2}(1+\cos\theta) - 2.\tfrac{3}{4}P_1 + 2\,\frac{1}{2.3}\,P_0\right]$,

(iii) $r^2\Big[P_2(\log r)^2 + 2\log r\,\Big\{P_2 \log \tfrac{1}{2}(1+\cos\theta) - 2.\tfrac{3}{4}P_1 + 2\,\frac{1}{2.3}\,P_0\Big\}$

$$+2\Big\{-P_2 \sum_{s=1}^{\infty} s^{-2}\sin^{2s}(\tfrac{1}{2}\theta) + \log\Big(2.\tfrac{3}{4}P_1 + 2\,\frac{1}{2.3}\,P_0\Big)\log\tfrac{1}{2}(1+\cos\theta)$$

$$+\Big(3P_1 - \frac{26}{36}P_0\Big)\Big].$$

The solution $V_{n,\,m}$ which has now been constructed completely is the most general symmetrical solution of Laplace's equation which reduces to $z^n(\log z)^m$ on the positive part of the axis of harmonics.

These potential functions occur in connection with circular discs of variable surface density. If the disc be of radius a and lie in the plane xOy, the centre of the disc being at O, and the surface density be

$$\Big(\frac{r}{a}\Big)^k \Big(\log\frac{r}{a}\Big)^{m-1},$$

its potential at $(O,\ O,\ z)$ is

$$2\pi a \sum_{M=0}^{\infty} \frac{\Gamma(m)}{(k+2+2M)^m}(-)^M \frac{\Gamma(M+\frac{1}{2})}{\Gamma(\frac{1}{2})\,\Gamma(M+1)}\Big(\frac{a}{z}\Big)^{2M+1},$$

when $z > a$. The analytic continuation of this function when $0 < z < a$ is found by the Barnes-Mellin methods of contour integration to be

$$\frac{2\pi a}{\Gamma(\frac{1}{2})}\Gamma(m)\Big[\sum_{N=0}^{\infty}{}' (-)^N \frac{\Gamma(\frac{1}{2}+N)}{\Gamma(1+N)}\,\frac{1}{(k+1-2N)^m}\Big(\frac{z}{a}\Big)^{2N}$$

$$+\Big(\frac{z}{a}\Big)^{k+1}\sum_{M=0} y_M(\log z)^M\Big].$$

The Σ' means that the term for which $N = \frac{1}{2}(k+1)$ has to be omitted

from the summation; and $\left(\frac{z}{a}\right)^{k+1} \sum_{M=0} y_M (\log z)^M$ is the residue of

$$\frac{\Gamma(s+\frac{1}{2})\,\Gamma(-s)}{(k+2+2s)^m} \left(\frac{a}{z}\right)^{2s+1}$$

at the point $s = -\frac{1}{2}(k+2)$, the quantities y_M being independent of z.

A Note on the Determination of the Summability of a Function by Means of its Fourier Constants: Prof. W. H. Young.

The present paper is one of several dealing with the possibility of generalising Parseval's Theorem and its converse. Parseval's Theorem enunciates a property of the coefficients of a Fourier series of a function of a certain type, and the theorems obtained in previous papers by the author fall into the same category. The converse of Parseval's Theorem, due to F. Riesz and Fischer, on the other hand, may be regarded as a contribution to the important and difficult problem. *Given the Fourier constants of a function, what can we say with regard to the function itself?* In particular, the converse in question tells us when the function has its square summable. The present paper gives us a sufficient condition in order that the function may have any even power summable. The theorem obtained is as follows:—

If the Fourier constants of a function be such that the sum of their $(1+1/p)$*-th powers, where* P *is an odd positive integer, converges, the function necessarily has its* $(1+p)$*-th power summable.*

In a concluding article a necessary and sufficient condition is given in order that a Fourier series should correspond to a function whose $(1+p)$-th power is summable, p being any positive quantity.

Thursday, December 12th, 1912.

Prof. A. E. H. LOVE, President, in the Chair.

Present twenty-four members and two visitors.

The minutes of the last meeting were read and confirmed.

Mr. E. H. Neville was nominated as a member.

Messrs. S. Lees, E. L. Watkin, P. L. Pethick, G. B. Jeffery, and Farid Boulad were elected members.

The President presented the Auditor's Report. On the motion of Mr. R. J. Dallas, seconded by Mr. F. M. Saxelby, the Treasurer's Report was adopted and the thanks of the Society were voted to the Treasurer and to the Auditor.

The President referred to the death of Mr. C. J. T. Sewell, elected a member on the 11th January, 1912.

The President spoke on Sir George Darwin's connexion with the Society, and on his scientific work, and moved:—

> That this meeting of the London Mathematical Society desires to express its sense of the great loss sustained by the scientific world through the death of Sir George Howard Darwin, who was a member of the Society for forty-four years, and who, by his continued activity in the application of mathematics to the fundamental problems of astronomy and cosmogony, achieved results which take rank among the chief scientific advances of the time. That the Society offers to Lady Darwin, and the family, an expression of the respectful and sympathetic condolence of the members.

This resolution was seconded by Sir Joseph Larmor, who also spoke on Sir George Darwin's scientific services, and was carried unanimously.

The President spoke on Prof. Henri Poincaré, Honorary Member of the Society, and on his mathematical achievements, and moved:—

> That an address be presented to the Académie des Sciences in Paris, expressing on behalf of the London Mathematical Society their sense of the loss to mathematical science occasioned by the death of Jules Henri Poincaré and their admiration of his researches.

This resolution was seconded by Dr. H. F. Baker and carried unanimously.

Dr. H. F. Baker, retiring President, gave his Presidential Address, entitled, "On Recent Advances in the Theory of Surfaces." On the motion of the President it was agreed that the address should be printed in the *Proceedings* of the Society.

The following papers were communicated:—

*On a Connexion between the Functions of Hermite and those of Legendre: Mr. H. E. J. Curzon.

*An Extension of a Theorem on Oscillating Series: Mr. G. H. Hardy.

*The Equations of the Theory of Electrons transformed relative to a System in Accelerated Motion: Mr. H. R. Hassé.

*On the Convergence of Series of Orthogonal Functions: Prof. E. W. Hobson.

---

* Printed in this volume.

On Mersenne's Primes: Mr. J. McDonnell.

The Diophantine Equation $y^2 = x^3 + k$: Mr. L. J. Mordell.

*(1) Derivates and their Primitive Functions, *(2) On Functions and their Associated Sets of Points: Prof. W. H. Young.

ABSTRACTS.

On a Connexion between the Functions of Hermite and those of Legendre: Mr. H. E. J. Curzon.

The differential equation

$$\frac{d^2u}{dx^2} - 2x\frac{du}{dx} + 2nu = 0$$

arises in connexion with the harmonic functions associated with the parabolic cylinder. See Whittaker, *Proc. London Math. Soc.*, 1903, pp. 420, &c. One solution of this equation is

$$U_n(x) = \frac{\Gamma(n+1)}{2\pi i}\int_{\Gamma_n} e^{-2ux-u^2}u^{-n-1}\,du,$$

where Γ_n is a contour beginning at positive infinity on the real axis, circulating positively round the origin, and returning again to its starting point.

A second independent solution is

$$V_n(x) = e^{x^2}U_{-n-1}(-ix).$$

The object of this paper is to establish the following relations between these Hermite functions and the Legendre functions:

$$P_n(x) = \frac{1}{\pi^{\frac{1}{2}}i\sin n\pi\,\Gamma(n+1)}\int_{\Gamma_t} e^{-t^2}t^n U_n(xt)\,dt,$$

if $|x| < 1$,

$$Q_n(x) = \frac{2^n i e^{\mp\frac{1}{2}(n+1)\pi i}}{e^{2n\pi i}\sin n\pi}\int_{\Gamma_t} e^{-t^2}t^n V_n(\pm xt)\,dt,$$

if the real part of x is numerically less than $\frac{1}{\sqrt{2}}$, where the top or bottom

---

* Printed in this volume.

signs are to be taken according as x is above or below the real axis

$$U_n(x) = -\frac{\Gamma(n+1)}{e^{n\pi i}\pi^{\frac{1}{2}}}\int_{+\infty}^{-\infty} e^{-t^2}t^{-n}P_n\left(\frac{ix}{t}\right)dt,$$

where the origin is avoided by an anti-clockwise semicircle,

$$V_n(x) = \frac{e^{\frac{3}{2}n\pi i}}{2^n\pi}e^{\pm\frac{1}{2}(n+1)\pi i}\int_L e^{-t^2}t^{-n}Q_n\left(\frac{ix}{t}\right),$$

if the argument of x lies between $\pm\frac{\pi}{4}$ and $\pm\frac{3\pi}{4}$ (the plus sign being retained throughout the ambiguities or else the minus sign), where L is an infinite straight line parallel to the straight line joining O to ix, such that the argument of $\frac{ix}{t}$ always lies between 0 and π as t travels along L, and where t traverses L, so that its angular motion about the origin is counter-clockwise.

An Extension of a Theorem on Oscillating Series: Mr. G. H. Hardy.

This paper contains a direct and general proof that a series Σa_n, for which

$$a_n = O\left(\frac{\lambda_n-\lambda_{n-1}}{\lambda_n}\right)$$

cannot be summable by Riesz's means of type λ unless it is convergent. This was proved in a former paper (*Proc. London Math. Soc.*, Ser. 2, Vol. 8, p. 314), when the means are of order 1 and the λ-sequence is subject to the restriction $\lambda_{n+1}/\lambda_n \to 1$; and (subject always to this last restriction) the extension to means of arbitrary order is a corollary of (1) a generalisation by Riesz of a theorem of Hölder, and (2) Littlewood's generalisation of Tauber's theorem (*Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 438). This method of procedure, however, besides introducing an unnecessary restriction, makes the theorem depend on another theorem of much greater intrinsic difficulty, and is therefore logically objectionable.

The Equations of the Theory of Electrons transformed relative to a System in Accelerated Motion: Mr. H. R. Hassé.

The paper consists essentially in the application of the general solution of the transformation

$$dx_1^2+dy_1^2+dz_1^2-dt_1^2 = \lambda^2(dx^2+dy^2+dz^2-dt^2),$$

to the motion of an electron according to the theory of relativity.

The type of motion is one in which the centre of the electron describes a hyperbola, the shape of the electron depending on its motion in accordance with the transformation.

The electromagnetic forces acting on the electron due to the reaction of the æther are calculated, and are expressible in terms of the velocity and acceleration of the centre of the electron at the instant, a further constant (δ), giving the motion relative to the centre, being also necessary. When the accelerations are small, the values of the electromagnetic masses are obtained, the results depending on the values given to the constant δ. Other general results for the accelerated motion are also found and compared with those obtained by Lorentz, Abraham, and others, by means of the theory of quasi-stationary motion.

On the Convergence of Series of Orthogonal Functions: Prof. E. W. Hobson.

The functions $\phi_1(x)$, $\phi_2(x)$, ..., $\phi_n(x)$, ... denoting a set of normal orthogonal functions, for the interval (0, 1) of x, and thus satisfying the conditions

$$\int_0^1 \{\phi_n(x)\}^2 dx = 1, \quad \int_0^1 \phi_n(x)\,\phi_m(x) = 0,$$

for $m \neq n$, it is proved that the series

$$c_1\phi_1(x) + c_2\phi_2(x) + \ldots + c_n\phi_n(x) + \ldots = \ldots$$

is convergent at every point of (0, 1), with the exception at most of the points of a set of measure zero, if a number k (>0) exists, such that the series

$$1^k c_1^2 + 2^k c_2^2 + \ldots + n^k c_n^2 + \ldots$$

is convergent.

For the case $k = \frac{1}{2}$, the theorem has been proved by Weyl, who also proved it for the case $k = \frac{1}{3}$ when the restriction is made that $|\phi_n(x)|$ is less than some fixed number for all the values of n and x.

The condition of the theorem is satisfied if a number $\lambda\,(>0)$ exists such that

$$\lim_{n=\infty} n^{\frac{1}{2}+\lambda} c_n = 0.$$

In the case of the ordinary trigonometrical series

$$\tfrac{1}{2}a_0 + \sum_{n=1} (a_n \cos nx + b_n \sin nx),$$

the conditions $\lim_{n=\infty} n^{\frac{1}{2}+\lambda} a_n = 0$, $\lim_{n=\infty} n^{\frac{1}{2}+\lambda} b_n = 0$,

are sufficient to ensure the convergence of the series at all points of the interval $(-\pi, \pi)$ with the possible exception of those belonging to a set of measure zero. This condition is wider than the conditions

$$\lim_{n=\infty} na_n = 0, \quad \lim_{n=\infty} nb_n = 0,$$

which were found by Fatou to be sufficient. The convergence, when the condition of the general theorem is satisfied, is found to be of the kind which may be described as quasi-uniform convergence, in the sense that, if ϵ be an arbitrarily prescribed positive number, a set of points of measure differing from that of the whole interval by less than ϵ can be so determined that the series converges uniformly in that set.

On Mersenne's Primes: Mr. J. McDonnell.

This paper is concerned with a method published by M. E. Lucas in 1878 for testing whether the number $2^n-1 = N$ is prime or not. Lucas shewed that if P^2-1 is a quadratic residue of $-N$ and we form the series of numbers $r_0, r_1, r_2, \ldots, r_m$, where

$$r_0 = 2p, \quad r_1 = r_0^2-2, \quad \ldots, \quad r_m = r_{m-1}^2-2,$$

then if N be prime some one of these numbers will be divisible by N for $m \not> n-2$. This is a necessary and sufficient condition, but leaves m rather indefinite. It is here shewn that if $p+1$ is apparently a non-residue of N and $p-1$ a residue, then the necessary and sufficient condition that N should be prime is $r_{n-2} \equiv 0 \pmod{N}$. Numbers of the form $A+B(p^2-1)^{\frac{1}{2}}$ are used in the process.

The Diophantine Equation $y^2 = x^3+k$: Mr. L. J. Mordell.

The paper may be divided into three sections. In the first section we find very general formulæ for k of the form $a^3+\lambda b^2$, for which the equation has no integral solutions.

In the second section we apply ideal numbers, and, in many cases, we can ascertain directly whether or not a finite number of integral solutions exist.

In the third section we apply the theory of the binary cubic, and shew, under certain conditions, the solution of the above equation is identical with the representation of numbers by binary cubics of determinant $4k$.

Derivates and their Primitive Functions: Prof. W. H. Young.

In this paper it is shown that if the lower derivate of a function, known to be upper semi-continuous on the right and lower semi-continuous on the left, assumes the value $+\infty$ at a countable set of points only, and if it is summable over the set of points at which it is positive, then the function is a lower semi-integral, whose positive variation is the integral of any one of its derivates over the set of points at which the lower derivate in question is positive.

This result is a generalisation of Lebesgue's well known theorem that, if a derivate of a continuous function is finite and summable, the function is the integral of its derivates and has as total variation the integral of the absolute value of any one of its derivates.

The author first proves a lemma which is an extension of that given by himself and his wife in a paper on "The Existence of a Differential Coefficient," and then applies the extended lemma to show that, if a function is upper semi-continuous on the right and lower semi-continuous on the left, and has one of its lower derivates greater than zero at a set of zero content only, then this derivate assumes the value $+\infty$ at a more than countable set of points.

Both this theorem and the lemma are proved without the use of Cantor's numbers.

From the theorem the main result of the paper above stated then follows by means of the known theorem that the derivates of an integral agree with the integrand, except at a set of content zero, a theorem of Lebesgue's shown on a previous occasion by the author to be obtainable without the use of Cantor's numbers.

In particular, it appears that Lebesgue's result is capable of an important generalisation involved in allowing the derivate there considered to assume the value infinity with determinate sign at any countable, not necessarily reducible, set of points.

With the object of showing how powerfully Cantor's numbers may be used as a method of research, alternative proofs by means of Lebesgue's chain are given, both of the lemma and the theorem based on it.

On Functions and their Associated Sets of Points: Prof. W. H. Young.

The paper begins by employing the method of monotone sequences to obtain a theory of sets of points precisely analogous to that given previously by the author for classes of functions. Wide generalisations are

thus obtained of results already proved by the author for ordinary inner and outer limiting sets.

These sets of points are then shown to present themselves naturally in the Theory of Functions of one or more Real Variables.

The problem which the author proposed to himself is, in fact, to determine the nature of the set of points at which a function of given type assumes a particular value or a particular range of values. Almost the only results known hitherto are those which state that the points at which an upper semi-continuous functionis $\geqslant k$, and the points at which a lower semi-continuous function is $\leqslant k$ form a closed set, while the points at which a continuous function is $= k$ also form a closed set. It is shown in the paper that the points at which any semi-continuous function assumes the value k form an ordinary inner limiting set, and the points at which a *lu*-function $> k$ and an *ul*-function $< k$ form an ordinary outer limiting set, and these results are generalised for the various functions of the types characterised by alternate *l*'s and *u*'s. Thus, in particular, if we denote an outer limiting set of ordinary inner limiting sets by the symbol *oi*-set, and an inner limiting set of ordinary outer limiting sets by the symbol an *io*-set, we have the result that the points at which a *lu*- or an *ul*-function assumes the value k is an *io*-set, and the points at which a *lul*-function is $> k$, or an *ulu*-function $< k$, form an *oi*-set.

The functions considered are shown to possess other properties which constitute direct generalisations of known properties of semi-continuous functions. Thus it is shown that the converse of certain of the theorems referred to above also hold. For example, it is a necessary and sufficient condition for a function to be a *lu*-function that the points at which it is $> k$ form an ordinary outer limiting set. Hence, or otherwise, it follows that, if at the points at which such a function is $\leqslant k$ any lower value be assigned to the function, its type is unaltered. The same is true for every function whose denomination, in the author's notation, begins with an *l*; the corresponding property holds for all the functions beginning with a *u*.

Thursday, January 9th, 1913.

Prof. A. E. H. LOVE, President, and temporarily Sir JOSEPH LARMOR, Treasurer, in the Chair.

Present also thirteen members.

The minutes of the last meeting were read and confirmed.

Derivates and their Primitive Functions: Prof. W. H. Young.

In this paper it is shown that if the lower derivate of a function, known to be upper semi-continuous on the right and lower semi-continuous on the left, assumes the value $+\infty$ at a countable set of points only, and if it is summable over the set of points at which it is positive, then the function is a lower semi-integral, whose positive variation is the integral of any one of its derivates over the set of points at which the lower derivate in question is positive.

This result is a generalisation of Lebesgue's well known theorem that, if a derivate of a continuous function is finite and summable, the function is the integral of its derivates and has as total variation the integral of the absolute value of any one of its derivates.

The author first proves a lemma which is an extension of that given by himself and his wife in a paper on "The Existence of a Differential Coefficient," and then applies the extended lemma to show that, if a function is upper semi-continuous on the right and lower semi-continuous on the left, and has one of its lower derivates greater than zero at a set of zero content only, then this derivate assumes the value $+\infty$ at a more than countable set of points.

Both this theorem and the lemma are proved without the use of Cantor's numbers.

From the theorem the main result of the paper above stated then follows by means of the known theorem that the derivates of an integral agree with the integrand, except at a set of content zero, a theorem of Lebesgue's shown on a previous occasion by the author to be obtainable without the use of Cantor's numbers.

In particular, it appears that Lebesgue's result is capable of an important generalisation involved in allowing the derivate there considered to assume the value infinity with determinate sign at any countable, not necessarily reducible, set of points.

With the object of showing how powerfully Cantor's numbers may be used as a method of research, alternative proofs by means of Lebesgue's chain are given, both of the lemma and the theorem based on it.

On Functions and their Associated Sets of Points: Prof. W. H. Young.

The paper begins by employing the method of monotone sequences to obtain a theory of sets of points precisely analogous to that given previously by the author for classes of functions. Wide generalisations are

thus obtained of results already proved by the author for ordinary inner and outer limiting sets.

These sets of points are then shown to present themselves naturally in the Theory of Functions of one or more Real Variables.

The problem which the author proposed to himself is, in fact, to determine the nature of the set of points at which a function of given type assumes a particular value or a particular range of values. Almost the only results known hitherto are those which state that the points at which an upper semi-continuous functionis $\geqslant k$, and the points at which a lower semi-continuous function is $\leqslant k$ form a closed set, while the points at which a continuous function is $= k$ also form a closed set. It is shown in the paper that the points at which any semi-continuous function assumes the value k form an ordinary inner limiting set, and the points at which a *lu*-function $> k$ and an *ul*-function $< k$ form an ordinary outer limiting set, and these results are generalised for the various functions of the types characterised by alternate *l*'s and *u*'s. Thus, in particular, if we denote an outer limiting set of ordinary inner limiting sets by the symbol *oi*-set, and an inner limiting set of ordinary outer limiting sets by the symbol an *io*-set, we have the result that the points at which a *lu*- or an *ul*-function assumes the value k is an *io*-set, and the points at which a *lul*-function is $> k$, or an *ulu*-function $< k$, form an *oi*-set.

The functions considered are shown to possess other properties which constitute direct generalisations of known properties of semi-continuous functions. Thus it is shown that the converse of certain of the theorems referred to above also hold. For example, it is a necessary and sufficient condition for a function to be a *lu*-function that the points at which it is $> k$ form an ordinary outer limiting set. Hence, or otherwise, it follows that, if at the points at which such a function is $\leqslant k$ any lower value be assigned to the function, its type is unaltered. The same is true for every function whose denomination, in the author's notation, begins with an *l*; the corresponding property holds for all the functions beginning with a *u*.

Thursday, January 9th, 1913.

Prof. A. E. H. LOVE, President, and temporarily Sir JOSEPH LARMOR, Treasurer, in the Chair.

Present also thirteen members.

The minutes of the last meeting were read and confirmed.

Mr. E. H. Neville was elected a member.

Professors P. J. Heawood, J. W. Nicholson, and Mr. A. M. Grundy were admitted into the Society.

Dr. Bromwich, as Secretary, reported that during the Session 1911–12 three members had died, five had resigned, the name of one had been removed, and 22 new members had been elected. The number of members at the end of the Session was 305 as against 292 at the beginning.

The President read a letter of thanks from Lady Darwin. He mentioned the death of Professor Gordan, an Honorary Member of the Society, and spoke on his scientific work. On the motion of Sir Joseph Larmor, seconded by Professor Hobson, it was agreed that a letter of condolence should be sent to the family of Professor Gordan.

The following papers were communicated :—

*Proofs of certain General Theorems relating to Orders of Coincidence : Prof. J. C. Fields.

*The Reduction of Ideal Numbers : Mr. W. E. H. Berwick.

*Notes on the Dynamical Theory of the Tides : Prof. A. E. H. Love.

*On Uniform Oscillation of the First and Second Kind : Prof. W. H. Young.

Some Definite Integrals occurring in the Harmonic Analysis connected with a Circular Disc : Mr. H. Bateman.

Prof. Hobson spoke on "The Convergence of Series of Orthogonal Functions."

ABSTRACTS.

Proofs of certain General Theorems relating to Orders of Coincidence : Prof. J. C. Fields.

Consider any algebraic equation

$$f(z, u) = u^n + f_{n-1}u^{n-1} + \ldots + f_0 = 0.$$

We suppose the n branches corresponding to a value $z = a_\kappa$ to group themselves into r_κ cycles of orders $\nu_1^{(\kappa)}, \ldots, \nu_{r_\kappa}^{(\kappa)}$ respectively. The orders of coincidence of the branches of these cycles with the function $f'_u(z, u)$ we indicate by $\mu_1^{(\kappa)}, \ldots, \mu_{r_\kappa}^{(\kappa)}$. A set of orders of coincidence $\tau_1^{(\kappa)}, \ldots, \tau_{r_\kappa}^{(\kappa)}$ imposes certain conditions on the general rational function which can be represented in the form $(z-a_\kappa)^{-i_\kappa}((z-a_\kappa, u))$. The number of these condi-

* Printed in this volume.

tions is shewn by a simple method to be given by the expression

$$ni_\kappa + \sum_{s=1}^{r_\kappa} \tau_s^{(\kappa)} \nu_s^{(\kappa)} - \tfrac{1}{2} \sum_{s=1}^{r_\kappa} \left(\mu_s^{(\kappa)} - 1 + \frac{1}{\nu_s^{(\kappa)}}\right) \nu_s^{(\kappa)},$$

when the integer i_κ is chosen sufficiently large. In particular the number of the conditions imposed on the function by adjointness relative to the value $z = a_\kappa$ is

$$ni_\kappa + \tfrac{1}{2} \sum_{s=1}^{r_\kappa} \left(\mu_s^{(\kappa)} - 1 + \frac{1}{\nu_s^{(\kappa)}}\right) \nu_s^{(\kappa)}.$$

Construct the general rational function of (z, u) whose representations in the neighbourhoods of the individual finite values $z = a_\kappa$ have the forms $(z - a_\kappa)^{-i_\kappa}((z - a_\kappa, u))$, while it is represented by the form $z^{i_\infty}((1/z, u))$ in the neighbourhood of the value $z = \infty$. Any individual integer i_κ is here subject to the one condition that it must be at least as large as would be compatible with the function being adjoint for the corresponding value of the variable z. This condition only requires that a finite number of these integers should have values other than 0. The number of arbitrary constants involved in the general function in question is

$$n + \sum_\kappa ni_\kappa.$$

If the number of the conditions imposed on these arbitrary constants by adjointness relative to all values of the variable z is given by the sum

$$\sum_\kappa ni_\kappa + \tfrac{1}{2} \sum_\kappa \sum_{s=1}^{r_\kappa} \left(\mu_s^{(\kappa)} - 1 + \frac{1}{\nu_s^{(\kappa)}}\right) \nu_s^{(\kappa)},$$

the conditions of adjointness for the different values of z are independent of one another. This will be the case if the number of the arbitrary constants involved in the general adjoint function is

$$n - \tfrac{1}{2} \sum_\kappa \sum_{s=1}^{r_\kappa} \left(\mu_s^{(\kappa)} - 1 + \frac{1}{\nu_s^{(\kappa)}}\right) \nu_s^{(\kappa)},$$

obtained on subtracting the last expression from the expression preceding. That this, however, is actually the number of the arbitrary constants involved in the general adjoint function is an immediate consequence of the complementary theorem.

The Reduction of Ideal Numbers: Mr. W. E. H. Berwick.

The reduction of an indefinite binary quadratic form, or of a real quadratic ideal, is known to depend on the expansion of a quadratic surd as a periodic continued fraction. In this memoir an extension of continued fractions is given by which rational approximations to the mutual

ratios of three quantities are determined. A previous method of obtaining such approximations is due to Jacobi (*Crelle*, 68), who showed that a certain periodicity may, and in some numerical cases does, exist when the three quantities are taken to be integers in a cubic corpus of negative discriminant. By forming the approximations according to the methods developed in the present paper, it is shown that periodicity must ensue in every case. The process of forming the approximations gives rise to a method of linking up an ideal of the corpus with an equivalent ideal. When the expansion is continued a finitely periodic closed chain of equivalent ideals is discovered, two equivalent ideals always giving rise to the same chain. The unique primitive unit of the corpus can be derived from the periodic part of the chain. The ideals which belong to the closed chain are defined to be reduced and a necessary and sufficient condition for an ideal to be reduced is given. Examples are appended showing the application of the method to numerical cases. The paper is developed on strictly arithmetical considerations throughout.

Notes on the Dynamical Theory of the Tides: Prof. A. E. H. Love.

The paper contains a simple method of obtaining by a direct process the sequence equations, found by Hough, which connect the coefficients in the spherical harmonic series for the elevation. The method rests on the expression of the horizontal velocity as compounded of two constituents: one derived from a potential, the other from a stream-function. The paper also contains a discussion of the relations that connect the phases of the elevation at different places, and the exhibition of these relations by tracing loci of high water and low water.

On Uniform Oscillation of the First and Second Kind: Prof. W. H. Young.

In previous papers the author has developed the theory of uniform oscillation for oscillating successions of continuous functions by means of what he has called the peak and chasm functions. The theory so developed is, as he has pointed out, applicable in general only for continuous functions. On the other hand, another mode of treatment which suggested itself was remarked to have the serious disadvantage that, in the case of continuous functions, uniform oscillation, so defined, would not as a rule be present anywhere, whereas in the treatment by peak and chasm functions, the facts observed with regard to the presence of points of uniform convergence in any converging series of continuous functions, found their natural generalisation.

In the present communication it is shown how, by the use of monotone

successions, the definition by means of the peak and chasm functions can be transformed, so as to form a definition of uniform oscillation above and below, on the left and on the right available for successions of any kind of functions. The concept so introduced, the author denotes by the term uniform oscillation of the first kind. The second mode of generalisation above alluded to as available as it stands for any kind of functions, is then called uniform oscillation of the second kind.

By means of the simultaneous use of these two kinds of uniform oscillation, the subject is found to form an organic whole. Coupled with the theorem, of which a proof is here given, that a uniform convergent succession of functions of any monotone class has a function of the same class as limiting function, and the known properties of monotone sequences, these definitions supply us with theoretically very convenient tests that the upper and lower functions of a succession should have each of the various forms which are permissible.

The importance of the consideration of uniform oscillation appears also from the theorem proved in the paper that a set of content as near as we please to that of the continuum can always be found with respect to which the succession oscillates uniformly above (below) whenever the upper (lower) function assumes the value $+\infty$ $(-\infty)$ at most at the points of a set of zero content; here the oscillation considered is equally of the first or of the second kind, or both.

Some Definite Integrals occurring in the Harmonic Analysis connected with the Circular Disc: Mr. H. Bateman.

The values of some definite integrals used by Prof. H. M. Macdonald (*Proceedings*, Ser. 1, Vol. xxvi, 1895, p. 257) can be found by using Neumann's peri-polar coordinates. If

$$\rho = \frac{a \sinh \sigma}{\cosh \sigma - \cos \psi}, \qquad z = \frac{a \sin \psi}{\cosh \sigma - \cos \psi},$$

it can be shown that

$$\cos \frac{\psi}{2} \sqrt{(\cosh \sigma - \cos \psi)} \tanh^m \frac{\sigma}{2}$$
$$= \sqrt{\pi}\, a^{\frac{1}{2}} \int_0^\infty e^{-\kappa z} J_m(\kappa \rho) J_{m-\frac{1}{2}}(\kappa a)\, \kappa^{\frac{1}{2}}\, d\kappa \qquad (m > -\tfrac{1}{2}),$$

$$\sin \frac{\psi}{2} \sqrt{(\cosh \sigma - \cos \psi)} \tanh^m \frac{\sigma}{2}$$
$$= \sqrt{\pi}\, a^{\frac{1}{2}} \int_0^\infty e^{-\kappa z} J_m(\kappa \rho) J_{m+\frac{1}{2}}(\kappa a)\, \kappa^{\frac{1}{2}}\, d\kappa \qquad (m > -1),$$

where a and ρ are real and $z \geqslant 0$.

The following integrals are related to the preceding ones

$$\int_0^\pi \frac{\sin^{2m-1}\theta\, d\theta}{[\rho^2+(z-ia\cos\theta)^2]^{m+\frac{1}{2}}}$$
$$= \tfrac{1}{2}\sqrt{\pi}\,\frac{\Gamma(m)}{\Gamma(m+\frac{1}{2})}\left[\{\rho^2+(z+ia)^2\}^{-\frac{1}{2}}+\{\rho^2+(z-ia)^{-\frac{1}{2}}\right]$$
$$\times\left[\frac{2}{a^2+z^2+\rho^2+\sqrt{\{(a^2+z^2+\rho^2)^2-4a^2\rho^2\}}}\right]^m \quad (m>-\tfrac{1}{2}),$$

$$\int_0^\pi \frac{(z+i\rho\cos\theta)\sin^{2m}\theta\, d\theta}{[a^2+(z+i\rho\cos\theta)^2]^{m+1}}$$
$$= \tfrac{1}{2}\sqrt{\pi}\,\frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+1)}\left[\frac{2}{a^2+z^2+\rho^2+\sqrt{\{(a^2+z^2+\rho^2)^2-4a^2\rho^2\}}}\right]^m$$
$$\times\left[\{\rho^2+(z+ia)^2\}^{-\frac{1}{2}}+\{\rho^2+(z-ia)^2\}^{-\frac{1}{2}}\right] \qquad (m>-1).$$

When ρ and z are real, the first of these equations holds for all complex values of a, provided cuts are made from the points $\rho+iz$, $\rho-iz$, $-\rho+iz$, $-\rho-iz$ respectively, to infinity along lines through the origin, and these cuts are excluded from the domain of the complex variable a. The discussion of the second equation proceeds on similar lines.

Thursday, February 13th, 1913.

Prof. A. E. H. LOVE, President, in the Chair.

Present also twenty-two members.

The minutes of the last meeting were read and confirmed.

The following papers were communicated :—

*Figures in n-Dimensional Space analogous to Orthocentric Tetrahedra : Mr. T. C. Lewis.

Two Properties of the Riemann ξ-Function : Mr. J. E. Littlewood.

*On the Summability of a Fourier's Series : Mr. G. H. Hardy.

Trigonometrical Series which Converge Nowhere or almost Nowhere : Messrs. G. H. Hardy and J. E. Littlewood.

A Theorem concerning Power Series : Dr. H. Bohr.

---

* Printed in this volume.

*On a Graphical Demonstration of the Fundamental Properties of Quadratic Residues: Prof. P. J. Heawood.

The Irreducibility of Legendre's Polynomials (Third Paper): Mr. J. B. Holt.

*On the Mode of Oscillation of a Fourier Series and its Allied Series: Prof. W. H. Young.

*Some Non-primary Perpetuant Syzygies of the Second Kind: Mr. H. T. H. Piaggio.

ABSTRACTS.

Figures in n-Dimensional Space analogous to Orthocentric Tetradedra: Mr. T. C. Lewis.

Consider a figure formed by lines joining $n+1$ vertices $A, B, \ldots, N$, such that the perpendiculars from the vertices upon the opposite $n-1$ dimensional faces meet in a point P, viz., the orthocentre. Let R be the circumradius, O the circumcentre, G the centre of gravity, and let $OP = l$.

Any $r+1$ of the vertices determine a face of r dimensions, which is an orthocentric figure. Let P_r, G_r be its orthocentre and centre of gravity.

A hypersphere of n dimensions passes through the points P_r, G_r of every r-face and intersects such face in an r-hypersphere on P_r, G_r as diameter.

There is a series of $n+1$ such n-hyperspheres corresponding to values 0 to n of r. Each of these $n+1$ hyperspheres has its centre on PO at a distance $PO_r = PO/r+1$, and its radius ρ_r is given by

$$4(r+1)^2\rho_r^2 = \Sigma a + (n-2r-1)^2\varpi, \qquad (1)$$

where $\alpha, \beta, \ldots, \nu, \varpi$ are the squares of the radii of hyperspheres having their centres at $A, B, \ldots, N, P$ and forming an orthotomic system. The adoption of these constants instead of the lengths of the edges of the figure very greatly simplifies the investigation of expressions for lengths, areas, volumes, &c., related to the orthocentric figure, making some of them easy which before seemed well-nigh impossible, and leading to the discovery of new properties.

As a particular case of (i),

$$4R^2 = \Sigma a + (n-1)^2\varpi,$$

---

* Printed in this volume.

and it may also be shewn that

$$4l^2 = \Sigma a + (n+1)^2 \varpi.$$

The $n+1$ hyperspheres determined by (i) belong to a family generated by the continuous variation of r, any two members of which intersect in a common hyperspherical section of $n-1$ dimensions whose centre is T on OP produced, where $PT = \dfrac{-(n+1)\varpi}{2l}$; and radius y, where $4l^2y^2 = \varpi \Sigma a$.

Further, the circle in the plane GPP_r which passes through the three points P_r, G, P_{n-r-1}, of which P_r, P_{n-r-1} belong to complementary figures, passes through T. Thus 2^n-1 circles are determined having a common chord GT. The cases of two and three dimensions serve as examples.

The following results may also be noted.

The sum of squares of edges of any polygon, or set of polygons, with one and only one angular point at each vertex is constant. The same is true if the orthocentre be treated as an additional vertex. Therefore if n is odd the sum of the squares of edges no two of which intersect is constant. If n is even the sum of the squares of $n/2$ non-intersecting edges none of which passes through A, together with the square on PA, is constant. These are exemplified in figures of two or three dimensions by known properties.

$$\Sigma PA^2 = 4R^2 + (n-3)(R^2 - OP^2),$$

$$(n+2)R^2 - (n-2)OP^2 = \Sigma a + \varpi,$$

is constant for all the $n+2$ figures having as orthcentre any one of the $n+2$ points $A, \ldots, N, P$, and the remaining points as vertices, *e.g.*, when $n = 2$ the value of R is the same for the four triangles concerned.

If r be here taken to be the radius of the inscribed hypersphere,

$$\frac{1}{r} = \Sigma \sqrt{\frac{\varpi}{a\varpi_a}},$$

where ϖ_a is the orthocentric constant associated with the face of $n-1$ dimensions which is opposite to the vertex A.

Let Δ_n be the n-dimensional content of the figure, and Δ_r the r-dimensional content of a face of r-dimensions. Then

$$(n!)^2 \Delta_n^2 + \frac{a\beta\gamma \ldots \nu}{\varpi} = 0,$$

whence

$$n!\, \Delta_n = \frac{\rho_a \rho_b \rho_c \ldots \rho_n}{\rho_p},$$

where ρ_a is the radius of the fundamental hypersphere whose centre is A, and so for the other vertices and P, made rational and positive in any case in which it is irrational.

$$(r!)^2 \Sigma \Delta_r^2 = (n-r+1)\Sigma\alpha\beta \ldots \text{(to } r \text{ factors)}.$$

Also any point O_r is the centroid of masses proportional to

$$\Sigma \frac{1}{\alpha} - \frac{n-2r-1}{\alpha}, \ldots,$$

at A and the other vertices, where r may have any value, whether integral or not, provided O_r is on PO, and $PO_r = PO/r+1$, as above.

Two Properties of the Riemann ζ-Function : J. E. Littlewood.

The first of my results is concerned with the order of the function $1/\zeta(s)$ on the line $\Re s = 1$. It has long been known that

$$\zeta(1+ti) = O(\log t).$$

The best known upper limit for $1/\zeta(1+ti)$ was found comparatively recently by Prof. Landau and is given by

$$1/\zeta(1+ti) = O(\log t \log\log t) \qquad (1)$$

It is further known that, on the well known hypothesis of Riemann concerning the zeros of $\zeta(s)$, both $\zeta(s)$ and $1/\zeta(s)$ are of the form $O(\log\log t \log\log\log t)$ on $\Re s = 1$; but it appears unlikely that limits of this degree of precision could be obtained independently except by means of new theorems of very great depth. Various considerations suggest, however, that it should be possible to replace the right-hand side of (1) by $O(\log t)$. This I have succeeded in doing by adopting a line of argument rather different from Prof. Landau's.

My other result is concerned with the distribution of the imaginary zeros of $\zeta(s)$. It is known that they lie inside the strip $0 < \Re s < 1$, and that the number of them which have their imaginary parts between 0 and T is given by the equation

$$N(T) = \frac{1}{2\pi} T \log T + \frac{1+\log(2\pi)}{2\pi} T + O(\log T).$$

It is a trivial deduction from this equation that fewer than $K \log T$ of

the zeros have their imaginary parts between T and $T+a$, where a is any constant, and also that for sufficiently large values of a at least one zero corresponds to the interval T to $T+a$ for every value of T. I have proved that, for sufficiently large values of T, this is the case whatever be the value of the constant a, and, indeed, even if a is a certain function of T which tends to zero with $1/T$.

We may consider, instead of the zeros of $\zeta(s)$, the points at which it takes some constant value other than O. It has been shown by Dr. Bohr that an infinity of these points lie to the right of $\Re s = 1$; in fact, that an infinity of them lie in the strip $1 < \Re s < 1+\delta$. (Dr. Bohr informs me that he and Prof. Landau have proved that, on the Riemann hypothesis, the same thing is true of the strip $\frac{1}{2}-\delta < \Re s < \frac{1}{2}+\delta$.) Just as in the case of the zeros of $\zeta(s)$ it is possible to show that, for all sufficiently large values of T, at least one point at which $\zeta(s)$ takes an assigned value has its ordinate between T and $T+\delta$.

On the Summability of Fourier's Series: Mr. G. H. Hardy.

It was proved by Fejér that the Fourier's series of any summable function $f(x)$ is summable $(C1)$ for any value of x for which the limits $f(x-0)$ and $f(x+0)$ exist. This result was afterwards generalised independently by M. Riesz and Chapman, who proved that, for such values of x, the series is summable $(C\delta)$ for any positive value of δ.

It has also been shown by Lebesgue that the series is summable $(C1)$ *almost everywhere*; that is to say, for all values of x except, perhaps, for a set of values of measure zero. These various results at once suggest the truth of the following theorem :—

1. *The Fourier's series of any summable function is summable* $(C\delta)$ *almost everywhere for any positive value of* δ.

This theorem having been established, it follows from a theorem of M. Riesz that, *if* a_n *and* b_n *are the Fourier constants of* $f(x)$, *and*

$$A_n = a_n \cos nx + b_n \sin nx,$$

then the series $\Sigma n^{-\delta} A_n$ *converges almost everywhere.* This result is due to Prof. W. H. Young (*Comptes Rendus*, December 23rd, 1912). It was the desire to connect Prof. Young's result with the theorems referred to above which suggested this investigation.

Theorem 1 cannot be deduced from Prof. Young's. On the other hand, Prof. Young has gone further and shown that

$$\Sigma(\log n)^{-1-\delta} A_n, \quad \Sigma(\log n)^{-1}(\log\log n)^{-1-\delta} A_n, \quad \ldots,$$

are also convergent almost everywhere. These results cannot be deduced from Theorem 1. It is, however, possible to prove another theorem of a different character which enables us to go still further.

2. *If s_n is the sum of the first n terms of the series, then $s_n = o(\log n)$ almost everywhere.*

From this we can deduce Prof. Young's further results, and a still more delicate result, the probability of which had already suggested itself to Prof. Young, viz.,

3. The series $\Sigma(\log n)^{-1}A_n$ *is convergent almost everywhere.*

These various theorems suggest the attempt to apply to Fourier's series methods of summation of "infinitely small" order, such as have been considered by Hardy and Chapman (*Quarterly Journal*, Vol. 42, p. 181). But the results of this investigation are not yet complete.

Trigonometrical Series which Converge Nowhere or almost Nowhere: Messrs. G. H. Hardy and J. E. Littlewood.

The problem of finding a trigonometrical series whose coefficients tend to zero, and which converges, if ever, only for a set of values of the argument whose measure is zero, was suggested by Fatou (*Acta Mathematica*, Vol. 30) and first solved by Lusin (*Rendiconti del Circolo Matematico di Palermo*, Vol. 32). In the course of our investigations concerning certain problems of "Diophantine Approximation" we have been led to consider certain trigonometrical series derived from the elliptic ϑ-functions, and have found solutions of Fatou's problem which seem to us to have considerable advantages over Lusin's.

We have, in fact, by a suitable adaptation of the formulæ for the linear transformation of the ϑ-functions, proved the following theorem.

The series $\quad \Sigma n^{-\alpha}\cos 2n^2\theta\pi, \quad \Sigma n^{-\alpha}\sin 2n^2\theta\pi,$

where $0 < \alpha \leqslant \frac{1}{2}$, *are convergent if, and only if,* θ *is a rational number of the form* $(2p+1)/(4q+2)$. *Further, when the series are not convergent, they are not summable by any of Cesàro's means.*

Considered simply as solutions of Fatou's problems, these series have, as against Lusin's, two advantages. In the first place, they are series of a simple and natural analytical form. In the second place, the problem of their convergence is solved *completely;* there is no exceptional set of values of θ concerning which doubt remains.

A more important point is this: as the series are in general not summable they cannot, by a well known theorem of Lebesgue, be Fourier's series. If, indeed, we use a theorem of Prof. Young, to which we shall refer again, this might be deduced from their non-convergence. Thus, they show that trigonometrical series $\Sigma(a_n \cos nx + b_n \sin nx)$ exist for which

$$\Sigma(|a_n|^{2+\delta} + |b_n|^{2+\delta}),$$

where δ is an arbitrarily small positive number, is convergent, or even for which

$$\Sigma \frac{a_n^2 + b_n^2}{(\log n)^{1+\delta}}$$

is convergent, and which are not Fourier's series. The well known theorem of Riesz-Fischer asserts that a series for which $\Sigma(a_n^2 + b_n^2)$ is convergent is necessarily the Fourier series of a function whose square is summable. Our examples show the impossibility of certain extensions of this theorem which might otherwise naturally suggest themselves.

If $\alpha > \frac{1}{2}$ it follows, from the Riesz-Fischer theorem and from a theorem of Prof. Young (*Comptes Rendus*, December 23, 1912), that the series converge for almost all values of θ. The methods which we have used in our work on Diophantine Approximation show that the point of convergence include all irrationals of certain general types; for example, all irrationals

$$\theta = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots}}},$$

for which $$a_n = O(n^\kappa).$$

A Theorem concerning Power Series: Dr. H. Bohr.

Proof of the following theorem: *If* $f(x) = \Sigma a_n x^n$ *is regular, and* $|f(x)| < 1$, *for* $|x| \leqslant 1$, *then*

$$\Sigma |a_n x^n| < 1$$

for $|x| \leqslant \frac{1}{8}$.

On a Graphical Demonstration of the Fundamental Properties of Quadratic Residues: Prof. P. J. Heawood.

In this paper use is made of a circle of circumference p, with arcs each

$= q$ measured continually round it, where p, q are integers, the positions of the points of division exhibiting the values of the least residues of q, $2q$, $3q$, ... This fact is used, *in connection with the principle of symmetry*, to prove properties of quadratic residues.

It is shown (supposing p, q prime and $p > q$) that when $\frac{p-1}{2}$ points have been inserted, $(q/p) = (-1)^r$ and $(p/q) = (-1)^{r'}$, where r, r' are the numbers of points inserted on two adjacent arcs YX, XZ, of which the former is a semicircle and the latter $= \frac{q}{2}$.

The diameter bisecting YZ is seen to be an axis of symmetry, so that $r+r'$ is necessarily an even number, *unless one of the* $r+r'$ *points lies on the axis*. The conditions for this being that $\frac{p-1}{2}$ is odd and $\frac{q-1}{2}$ is odd, the reciprocal property of quadratic residues follows.

A like method leads at once to the result $(2/p) = (-1)^{\frac{1}{8}p^2-1}$.

The Irreducibility of Legendre's Polynomials: Mr. J. B. Holt.

The first part of the paper contains the solution of the problem to find all values of n for which the coefficients of $P_n(x)$ satisfy the conditions

(1) All but one are divisible by p;

(2) The first and last are not divisible by p^2,

p being a given odd prime.

It is first shewn that if n be expressed in the scale of p, so that

$$n = a + bp + cp^2 + \ldots,$$

then
$$P_n(x) \equiv P_a(x)\, P_b(x^p)\, P_c(x^{p^2}) \ldots \pmod{p}.$$

From this it follows that if the first condition hold, each of a, b, ..., must have one of the values 0, 1, $p-2$, $p-1$. This is necessary and sufficient. When $p = 3$ it is always satisfied, so that, when $P_n(x)$ is integralised by multiplication with the proper power of 2, every coefficient but one is divisible by 3.

The first coefficient of $P_n(x)$ is prime to p, when each of a, b, ..., is $< p/2$; when one only is $> p/2$, and the next is not $\frac{1}{2}(p-1)$, the coefficient is divisible by p, but not by p^2. Again, when each of the numbers

$$n-a,\ b,\ c,\ \ldots,$$

is even, the last coefficient of $P_n(x)$ is prime to p; when two consecutive ones are odd, and the rest even, the coefficient is divisible by p, but not by p^2. These conditions are all necessary and sufficient, and by means of them the solutions of the problem may be written down.

If n have one of these values, and either the first or the last coefficient of $P_n(x)$ be prime to p, Eisenstein's conditions are satisfied, and $P_n(x)$, or $P_n(x)/x$ when n is odd, is irreducible. There are eight solutions of this kind, four of which are new, namely,

$$n = p^r(p \pm 1), \quad n = p^r(p \pm 1)+1,$$

where r is any positive integer.

When the coefficient prime to p is that of x^m, where $n > m > 1$, there are sixteen solutions, all but two being new. For any one of these, if λ be that one of n, $n-m$, which is not the greater, $P_n(x)$ can have no factor of degree less than $\lambda-1$; on the other hand, if $P_n(x)$ be reducible, such a factor must exist if there be a prime between the limits $n \pm \lambda$ when n is even, or $n+1 \pm \lambda$ when n is odd. When such a prime exists $P_n(x)$ is irreducible. In certain cases this condition is always fulfilled, as, for instance, when $n = 2p^r(p-1)$, $p > 5$, or 16.5^r, r being any positive integer.

Besides these there is a class of cases in which the leading coefficient of $P_n(x)$ is prime to p, the other coefficients being divisible by p, the last m divisible by p^2, but the last not divisible by p^3. In such a case $P_n(x)$ can have no factor of degree between 1 and m. This result still holds if the words "first" and "last" be interchanged in the above conditions. There are twelve cases of this kind, each giving rise to a theorem of which the following is typical: If $n = p^2+1$, and there be a prime between the limits $n \pm (p+1)$, $P_n(x)$ is irreducible.

It appears that, for $n \leqslant 1000$, $P_n(x)$ or $P_n(x)/x$ is certainly irreducible in 952 cases, half of which are of even degree. The two lowest values of n for which irreducibility is not proved are 186 and 206.

On the Mode of Oscillation of a Fourier Series and of its Allied Series: Prof. W. H. Young.

The object of the present communication is to give indications of a method by means of which the order of infinity everywhere of the partial summations of the Fourier series and allied series of a function $f(x)$ may be determined. Unlike what might be inferred from results hitherto

obtained with respect to the order of infinity *almost everywhere*, all depends in the present instance on the class of summable functions to which $f(x)$ belongs. The method is illustrated by reference to particular classes of functions. Thus, when $|f(x)|^{1+p}$ is summable, the order is infinitely lower than n^{1+p}. If $e^{|f(x)|}$, or $E_1(|f(x)|)$, is summable it is of order not higher than $(\log n)^2$ or $l_1(n)^2$. More generally, if $E_p(|f(x)|)$, that is, $e^{E_{p-1}(|f(x)|)}$, is summable, the order is not higher than that of $l_1(n)l_p(n)$, where $l_p(n) = \log l_{p-1}(n)$.

These results may be compared with the known fact that, if $f(x)$ is bounded, the order is not higher than $\log n$, and with the further result, here pointed out, that when the function has only simple discontinuities it is of an infinitely lower order. It is then explained how these results may be employed to obtain information as to the convergence factors which will transform a Fourier series and allied series into series which converge everywhere, when $f(x)$ belongs to a particular class of summable functions. In this way results previously given by the author are confirmed and shown to be part of a general theory.

Some Non-primary Perpetuant Syzygies of the Second Kind: Mr. H. T. H. Piaggio.

In a previous paper syzygies of the n-th "kind" were shewn to exist among perpetuant "types" for all degrees greater than $n+1$. It was also shewn that for degrees less than $2n+2$ these syzygies were all formed by one simple method, namely, symbolical multiplication of the single syzygy of the $(n+2)$-th degree by perpetuant products.

It was proved that *non-primary* syzygies (*i.e.* those not formed by this method) certainly do arise of degree $2n+2$. For the first kind ($n = 1$) non-primary syzygies thus arise of degree 4. These are the well known Stroh syzygies. For the second kind ($n = 2$) non-primary syzygies arise of degree 6. The present paper deals with these. It is found that three distinct species of such syzygies exist. In terms of these (and of certain primary ones) all other syzygies of the second kind of degree 6 can be expressed linearly.

Thursday, March 13th, 1913.

Prof. A. E. H. LOVE, President, in the Chair.

Present also sixteen members.
The minutes of the last meeting were read and confirmed.
Mr. S. Lees was admitted a member.

The following papers were communicated:—

*The Diffraction of Light by an Opaque Prism: Prof. H. M. Macdonald.

The Natural Radiation from Transparent Media: Mr. S. Lees.

Indeterminate Equations of the Third and Fourth Degrees: Mr. L. J. Mordell.

On Mersenne's Numbers: Lt.-Col. A. Cunningham.

(1) On a Two-Dimensional Potential Problem with Applications to Hydrodynamics and Elasticity: Mr. J. Proudman.

*(2) On Tidal Motion in Rotating Sheets of Water: Mr. J. Proudman.

ABSTRACTS.

The Diffraction of Light by an Opaque Prism: Prof. H. M. Macdonald.

This subject has been discussed by Jackson,† who compares a perfectly conducting prism with a perfectly absorbing prism. The solution for the perfectly absorbing prism is the usual one which is obtained on the assumption that the electric and magnetic forces vanish over the part of the surface of the absorbing body which is in the shadow, and that the electric and magnetic forces have the same values just outside the part of the surface not in the shadow as in the incident waves. This assumption neglects the creeping effect at the edge of the shadow. The object of the following note is to obtain the exact solution for a perfectly absorbing prism.

The Natural Radiation from Transparent Media: Mr. S. Lees.

In this paper, the author attempts to determine the distribution of

* Printed in this volume.

† *Proc. London Math. Soc.*, Ser. 2, Vol. 1, p. 393, 1904.

energy in the spectrum for a black body, by considering the natural radiation emitted at constant temperature by a transparent medium. A mass of transparent substance is supposed placed in an enclosure kept at constant temperature, and the quality of the radiation is investigated when statistical equilibrium has taken place. The investigation involves finding (i) the energy emitted per second per unit volume between wave-lengths λ and $\lambda+d\lambda$, (ii) the corresponding energy absorbed. These are equated for equilibrium.

To determine (i), it is assumed that the medium consists of molecules constantly impinging on one another. The molecules contain vibrating electrons, and the impacts of the molecules cause impulsive velocities to be communicated to the electrons. The motion of the electrons under the given conditions is first investigated, and the energy emitted can be then calculated. The emitted energy has to be analysed, and this is done by first analysing (say) the $\dot{\xi}$ component of the velocity of an electron. Assuming that the energy emitted by an electron is of the form

$$2mk\int_{-\infty}^{\infty}(\dot{\xi}^2+\dot{\eta}^2+\dot{\zeta}^2)\,dt,$$

we can use a theorem of Lord Rayleigh's, viz., that if

$$\dot{\xi}(t)=\frac{1}{\pi}\int_0^{\infty}\{\psi_1(p)\sin pt+\psi_2(p)\cos pt\}\,dp,$$

then

$$\int_{-\infty}^{\infty}\dot{\xi}^2dt=\frac{1}{\pi}\int_0^{\infty}\{[\psi_1(p)]^2+[\psi_2(p)]^2\}\,dp.$$

The results of the investigation are of two forms, depending upon whether kT is large or small compared with unity, T being an average time between two successive impacts of a molecule. The most interesting result is that if kT is small, the Lorentz formula for the energy between wave-lengths λ and $\lambda+d\lambda$, viz.,

$$E_\lambda d\lambda=D\frac{d\lambda}{\lambda^4}$$

(which holds for large wave-lengths), is here found to hold for small wave-lengths, but as we make the electrical conductivity of the medium larger and larger, this formula can be extended to larger and larger wave-lengths. To this extent, the result would appear to be a generalisation of Lorentz's result, but it must be pointed out that the dependence of D on the temperature is not discussed in the paper.

Indeterminate Equations of the Third and Fourth Degrees: Mr. L. J. Mordell.

It is shewn that the complete integral solution of

$$z^2 = (A, B, C, D)(x, y)^3,$$

where x is prime to y, is given by a finite number of expressions of the form $x = f_1(p, q)$, $y = f_2(p, q)$, where f_1 and f_2 are binary quartics in p and q. In particular when $y = 1$, the solution depends upon the representation of unity by the finite number of non-equivalent quartics with the same invariants.

Applying these results to the equation $z^2 = f(x, y)$, where $f(x, y)$ is a binary quartic, it follows that being given a solution of this equation, and also one when $f(x, y)$ is replaced by another quartic $f_1(x, y)$ with the same invariants, we can in general find two more solutions for the original equation; and hence if the original equation has only a finite number of solutions, but at least one, *e.g.*, $f(x, y) = x^4 + y^4$, then the second equation cannot have more than a finite number of solutions, and if the first equation has an infinite number of solutions, so will the second equation if it has any.

It is then shewn that corresponding results exist for the solution of the general ternary cubic equation $f(x, y, z) = 0$.

Finally, applications of the above results are given.

On Mersenne's Numbers: Lt.-Col. A. Cunningham.

The author announced the recent determination by Mr. R. E. Powers, of Denver, Colorado, that $2^{257} - 1$ has no divisor less than 10,017,000.

On a Two-Dimensional Potential Problem, with Applications to Hydrodynamics and Elasticity: Mr. J. Proudman.

The solution of a number of problems in applied mathematics can be reduced to finding a function χ of x and y, which satisfies the equation

$$\frac{\partial^2 \chi}{\partial x^2} + \frac{\partial^2 \chi}{\partial y^2} = -2, \tag{1}$$

at all points inside a given boundary, and which is zero at all points on

the boundary. These problems are :—

(i) *The steady flow of viscous liquid in a pipe of uniform section;* then the velocity w is connected with χ by the relation*

$$w = P\chi/2\mu,$$

where P is the pressure-gradient along the pipe and μ is the coefficient of viscosity. Thus the total efflux is $PF/2\mu$, where

$$F = \iint \chi \, dx \, dy \tag{2}$$

integrated over the section.

(ii) *The torsion of an isotropic elastic cylinder :*† here if τ is the rate of twist along the cylinder, the couple to be applied is found to be

$$G = 2\eta\tau F,$$

where η is the rigidity and F is given by (2).

(iii) *The irrotational motion of a perfect incompressible fluid in a rotating cylinder.*‡

The kinetic energy and angular momentum per unit length of the fluid are then

$$\tfrac{1}{2}\rho\omega^2(I-2F) \quad \text{and} \quad \rho\omega(I-2F),$$

where ρ is the density, ω the angular velocity, and I is the moment of inertia of the section about the axis of rotation.

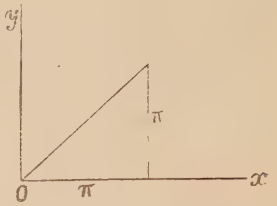
(iv) *The circulation of a perfect incompressible fluid with constant vorticity ζ inside a fixed cylinder.*§

The kinetic energy and angular momentum per unit length of the fluid are then expressible as

$$\rho\zeta^2 F \quad \text{and} \quad 2\rho\zeta F.$$

* Lamb, *Hydrodynamics*, Art. 320, p. 545, 3rd edition.
† Love, *Theory of Elasticity*, Vol. I, p. 157.
‡ Lamb, *Hydrodynamics*, Art. 72, p. 81.
§ *Ibid.*, Art. 154, p. 211.

We proceed to find the function χ in the case of an *isosceles right-angled triangle;* and suppose first that the equal sides of the triangle are of length π, so that in the given figure the equations of the sides are



$$y = 0, \quad x = \pi, \quad y = x.$$

Then the series

$$\chi = xy - y^2 + \sum_1^\infty A_n (\sinh nx \sin ny - \sin nx \sinh ny), \tag{3}$$

satisfies (1) and is zero on $y = 0$ and $y = x$; and on the side $x = \pi$, the series (3) reduces to

$$\pi y - y^2 + \sum_1^\infty (A_n \sinh n\pi) \sin ny,$$

which must also vanish between $y = 0$ and π.

Now, by Fourier's theorem, or by direct summation, it is easy to see that, if $0 \leqslant y \leqslant \pi$,

$$\pi y - y^2 = \frac{8}{\pi} \sum_0^\infty \frac{\sin (2s+1)y}{(2s+1)^3},$$

and so, in series (3), n must be of the form $(2s+1)$ and *the final solution is*

$$\chi = xy - y^2 - \frac{8}{\pi} \sum_0^\infty \frac{\sinh (2s+1)x \sin (2s+1)y - \sin (2s+1)x \sinh (2s+1)y}{(2s+1)^3 \sinh (2s+1)\pi}. \tag{4}$$

On integrating χ over the section it is easy to verify that

$$F = \iint \chi \, dx \, dy = \frac{\pi^4}{24} - \frac{8}{\pi} \sum_0^\infty \frac{1}{(2s+1)^5} \coth \{\tfrac{1}{2}(2s+1)\pi\}.$$

In the more general case when the equal sides of the triangle are of length c, F is multiplied by $(c/\pi)^4$; thus in general

$$\left.\begin{aligned} \frac{F}{c^4} &= \frac{1}{24} - \frac{8}{\pi^5} \sum_0^\infty \frac{1}{(2s+1)^5} \coth \{\tfrac{1}{2}(2s+1)\pi\} \\ \text{or} \qquad F/c^4 &= \cdot 01305 \end{aligned}\right\}. \tag{5}$$

The numerical value of the series can be found to five places of decimals (as given) from the first three terms: an application of Euler's method of

approximate summation* shews that the sum of the remainder is less than 3×10^{-6}.

In the torsional problem (ii), Saint-Venant pointed out that for the majority of simple sections the couple could be written in the form†

$$\kappa\eta\tau\,(\text{area})^4/I,$$

where κ is a numerical coefficient varying between the limits ·0228 and ·026. In the present example, we have

$$\text{area} = \tfrac{1}{2}c^2, \quad I = \tfrac{1}{18}c^4,$$

and so
$$2F = \kappa\,(\tfrac{1}{2}c^2)^4/(\tfrac{1}{18}c^4).$$

Thus here
$$\kappa = \tfrac{16}{9}(F/c^4) = \cdot0232,$$

which falls between Saint-Venant's limits, but is less than the value ·02533 ($= 1/4\pi^2$), which is found for an elliptic or circular section.

On some Cases of Tidal Motion of Rotating Sheets of Water: Mr. J. Proudman.

The results contained in this paper refer to both free and forced tidal motion, and also to the limiting forms of forced tides as the period of the disturbing force tends to become infinite. The depths of the sheets of water are taken to be uniform throughout. Those results which are not limiting forms are either approximations, or else the sheets to which they refer are not bounded in all directions. Those forms of the disturbing force are taken which appear to be of most interest in connection with terrestrial tides.

The limiting forms are found to exist uniquely for flat closed sheets of all shapes, and to be always different from the "equilibrium forms," when the sheets are rotating.

All the problems considered refer to flat sheets except one, which concerns the free symmetrical oscillations of a small circular sea on a sphere. For this the periods and the corresponding forms of the disturbance are obtained as the next approximation after that which regards the sea as flat.

The other special results are as follows.

---

* Bromwich, *Infinite Series*, p. 238.

† Saint-Venant, *Comptes Rendus*, t. 88, 1879, pp. 142–147; Love, *l.c.*, p. 171.

For the free and forced oscillations of a nearly circular sheet, in the modes which are nearly symmetrical, the forms of the disturbance are determined, and, to the approximation considered, the free periods are found to be the same as those for a circular sheet of the same area. The limiting form is obtained in two ways.

A complete expression is found for the limiting form in the case of a rectangular sheet.

Indications are given for the determination of the limiting form for a sheet of the shape of a circular sector.

Some possible solutions of the equations for free motion are obtained for a sea covering a semi-infinite plane, and for an infinite channel.

Thursday, April 24th, 1913.

Prof. A. E. H. LOVE, President, in the Chair.

Present also twelve members and a visitor.

The minutes of the last meeting were read and confirmed.

Messrs. H. E. J. Curzon and J. Rose-Innes were elected members.

The President stated that the Society had received an announcement of a prize offered for a work on elliptic functions, treating the subject from the critical and historical points of view from 1700 to the present day. Essays are to be sent to the Instituto di Bologna before the end of next year, and the amount of the prize is 500 lire. Further particulars may be had from Prof. Pincherle of Bologna.

The following papers were communicated:—

The Projective Geometry and Covariant Theory of a Ternary Quadratic Form, modulo 2: Prof. L. E. Dickson.

The Application of the Calculus of Finite Differences to certain Trigonometric Series: Messrs. A. E. Jolliffe and S. T. Shovelton.

On an Assumption contained in Euclid's Proofs of certain Propositions in his Twelfth Book: Mr. J. Rose-Innes.

Dr. H. F. Baker made an informal communication on "Potential Theory and the Theory of Functions of several Complex Variables."

ABSTRACTS.

The Application of the Calculus of Finite Differences to certain Trigonometrical Series: A. E. Jolliffe and S. T. Shovelton.

The similarity of the expansions of $\cos m\theta$, $\frac{\sin m\theta}{\cos\theta}$, $\sin m\theta$ and $\frac{\cos m\theta}{\cos\theta}$ in ascending powers of $\sin\theta$ to certain central-difference formulæ suggested the possibility of obtaining the former by means of central-differences. Two central-difference formulæ are used and the four expansions are obtained from them. The remainders are investigated in each case, and by considering the remainders in special cases proofs of the expansions of $\sin x$ and $\cos x$ in factors are obtained. The methods employed for the series are all elementary and involve little more than the elements of finite differences and the meaning of a definite integral. The results are shown to be true for all values of m, real or complex.

On Potential Theory and the Theory of Functions of several Complex Variables: Dr. H. F. Baker.

It is known that the equation of the zero-construct of an integral function of the complex variables $T_1 = t_1+it_2, \ldots, T_p = t_{n-1}+it_n$, where $n = 2p$ may be calculated from the potential of the construct. I have recently had occasion to work out the simple case where the construct is linear, $T_p = mT_1+kT_2+\ldots+u+iv$, where $m, k, \ldots, u, v$ are real constants. Putting

$$P(x, t) = -\frac{1}{n-2}[(x_1-t_1)^2+\ldots+(x_n-t_n)^2]^{-\frac{1}{2}n+1},$$

or

$$\tfrac{1}{2}\log[(x_1-t_1)^2+(x_2-t_2)^2],$$

and considering the integral

$$\int dx_1 \ldots dx_{n-2}\,[P(x, t)-P(x, t_0)],$$

wherein $x_{n-1} = mx_1+kx_3+\ldots+u, \quad x_n = mx_2+kx_4+\ldots+v,$

we are reduced to the consideration of an integral

$$\int dx_1 \ldots dx_m\,(\phi^{-\frac{1}{2}m}-\phi_0^{-\frac{1}{2}m}),$$

wherein ϕ is a definite positive quadratic form in $x_1, x_2, \ldots, x_m, 1$, and

ϕ_0 a similar form, wherein the terms of order 2 are the same as in ϕ. This can be shewn to be a multiple of

$$\frac{1}{\sqrt{C}} \log \frac{\Delta}{\Delta_0},$$

where C is the discriminant of the terms of order 2 in ϕ and ϕ_0, Δ is the discriminant of ϕ and Δ_0 of ϕ_0, when ϕ is derived as above, the discriminant Δ_1 save for a factor also entering in Δ_0, is the sum of the two squares

$$(-t_{n-1}+mt_1+kt_3+\ldots+u)^2+(-t_n+mt_2+kt_4+\ldots+v)^2.$$

This furnishes an exact verification of a general theory. See, for example, Baker, *Trans. Camb. Phil. Soc.*, Vol. XVIII (1900), p. 426.

Thursday, May 8th, 1913.

Prof. A. E. H. LOVE, President, in the Chair.

Present also ten members.
The minutes of the last meeting were read and confirmed.
Mr. H. E. J. Curzon was admitted into the Society.

The following papers were communicated :—

On some Properties of Groups whose Orders are Powers of Primes (Second Paper): Prof. W. Burnside.

The Green's Function for the Equation $\nabla^2 u+k^2 u = 0$: Prof. H. S. Carslaw.

The Usual Convergence of a Class of Trigonometrical Series: Prof. W. H. Young.

(i) Factorial Moments in Terms of Sums or Differences, (ii) Fitting of Polynomials by the Method of Least Squares: Mr. W. F. Sheppard.

The Effect of Internal Friction on Stress-Strain Relationships for Elastic Solids: Mr. S. Lees.

ABSTRACTS.

On some Properties of Groups whose Orders are Powers of Primes (Second Paper): Prof. W. Burnside.

In this paper a determination is made of all possible sets of generating operations of any given group of prime power order. Several properties are proved of the series of derived groups. In particular it is shewn that if the n-th derived group is not identity, the order of the group must be greater than $p^{\frac{1}{2}(n+1)(n+2)}$, except when n is 1 or 2. Some further properties of the maximum Abelian self-conjugate sub-groups are given.

The Green's Function for the Equation $\nabla^2 u + k^2 u = 0$: Prof. H. S. Carslaw.

This paper forms the continuation of a preliminary note given in the Report of Proceedings at Meetings (April 11th, 1912).

The Usual Convergence of a Class of Trigonometrical Series: Prof. W. H. Young.

Lebesgue's theorem that every Fourier series converges almost everywhere to the function with which it is associated when summed in the Cesàro manner is one of the most useful theorems in the modern theory of trigonometrical series. Since the publication of Lebesgue's papers the concept of fractional Cesàro summation has been introduced, and Mr. G. H. Hardy has recently shown that Lebesgue's theorem is still true when fractional Cesàro summation is employed. In the present paper it is shown that both Lebesgue's important theorem, and Mr. G. H. Hardy's interesting extension are true for the derived series of the Fourier series of a function of bounded variation, the convergence being to a function which except at a set of content zero is equal to the differential coefficient of the function of bounded variation. The convergence almost everywhere when summed in the ordinary manner of a Fourier series has been considered by various writers, and it has been shown that, by the use of certain convergence factors, every Fourier series and allied series can be transformed into a Fourier series of this type. In the present paper it is shown that all the results hitherto obtained of this nature remain true when the derived series of the Fourier series of a function of bounded variation, and of its allied series, take the place of the general Fourier series.

Factorial Moments in Terms of Sums or Differences: Mr. W. F. Sheppard.

A factorial moment is, *e.g.*, of the form $\Sigma(\theta+q, g)\, u_q$, where

$$(\theta+q, g) \equiv (\theta+q)!/\{g!\,(\theta+q-g)!\}\ ;$$

the data being $u_0, u_1, u_2, \ldots, u_m$, and the summation being for

$$q = 0, 1, 2, \ldots, m.$$

Two sets of formulæ are considered. (i) Starting with $u_0, u_1, u_2, \ldots, u_m$, successive summations made in a particular way give sums $\Sigma'' u_0, \Sigma''^2 u_0, \ldots$, such that

$$\Sigma''^{g+1} u_0 = \sum_{q=0}^{q=m} (q+g, g)\, u_q,$$

$$\Sigma''^{g+1} u_0 = (m+g, g) \sum_{s=0}^{s=m} \frac{m+g+1}{s+g+1} (m, s)\, \Delta^s u_0.$$

Hence, if any expression of the form $\alpha_0 M_0+\alpha_1 M_1+\ldots+\alpha_j M_j$, where

$$M_g \equiv \sum_{q=0}^{q=m} q^g u_q,$$

is expressed in terms of $u_0, \Delta u_0, \ldots, \Delta^m u_0$, the coefficient of $\Delta^s u_0$ will be of the form

$$(\beta_0+\beta_1 s+\ldots+\beta_j s^j)(m, s)/\{(s+1)(s+2)\ldots(s+j+1)\}\ ;$$

and conversely, if

$$\Phi \equiv \sum_{s=0}^{s=m} \alpha_s \Delta^s u_0,$$

and α_s is of this latter form, with known β's, we can express Φ in terms of $\Sigma'' u_0, \Sigma''^2 u_0, \ldots, \Sigma''^{j+1} u_0$. There are similar properties for central sums. (ii) Where the $m+1$ u's form part of a larger sequence, the above partial sums can be expressed in terms of the successive sums of the sequence as a whole by means of the formula

$$\sum_{q=p}^{q=P} (\theta-q, g)\, u_q = L_{P+1}-L_p,$$

where L_x denotes the ordinary interpolation-formula for $\Sigma^{g+1} u_{\theta+1}$ in terms of $\Sigma^{g+1} u_x, \Sigma^{g+1} u_{x+1}, \ldots, \Sigma^{g+1} u_{x+g}$; Σ here being the inverse of Δ.

Fitting of Polynomials by Method of Least Squares: Mr. W. F. Sheppard.

The problem is to fit a polynomial

$$v_q \equiv v_0+(q, 1)\,\Delta v_0+(q, 2)\,\Delta^2 v_0+\ldots+(q, j)\,\Delta^j v_0$$

to data $u_0, u_1, u_2, \ldots, u_m$ so that $\Sigma(v_q - u_q)^2$ shall be a minimum. (i) The general solution in terms of $u_0, \Delta u_0, \ldots, \Delta^m u_0$ is obtained; the corresponding solution in terms of central differences had already been given elsewhere. (ii) The solution is also given, both for advancing differences and for central differences, in terms of the successive sums of a larger sequence of which $u_0, u_1, u_2, \ldots, u_m$ may form a part. This gives a more practical method for reducing the errors of terms of a long sequence than is given by the formula in terms of central differences.

The Effects of Internal Friction on Stress-Strain Relationship for Elastic Solids: Mr. S. Lees.

In this paper the view is taken that an elastic viscous solid consists (i) of a perfectly elastic framework obeying the generalised Hooke's law, (ii) of some kind of viscous fluid. The views of Maxwell are taken for viscous fluids, but the treatment (as highly viscous fluids must be taken into account) is somewhat different from that given in Butcher's paper (*Proc. London Math. Soc.*, Vol. VIII, p. 108, 1876). The type of stress-strain equations arrived at is

$$X_x(1+aD) = (B+bD)\Delta + 2(C+cD)e_{xx},$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$Z_y(1+aD) = (C+cD)e_{zy},$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

where a, B, b, C, c are constants and D denotes d/dt.

Applications to some simple cases are given.

Thursday, June 12th, 1913.

Prof. A. E. H. LOVE, President, in the Chair.

Present also sixteen members and a visitor.

The minutes of the last meeting were read and confirmed.

The following papers were communicated:—

The Electromagnetic Force on a Moving Charge in Relation to the Energy of the Field: Sir J. Larmor.

Einige Ungleichungen für zweimal differentiierbare Funktionen: Prof. E. Landau.

(i) On the Fractional Part of $n^k\theta$, (ii) The Trigonometrical Series Associated with the Elliptic θ-Functions: Messrs. G. H. Hardy and J. E. Littlewood.

On Foucault's Pendulum: Dr. T. J. I'A. Bromwich.

On a certain Definite Integral: Mr. J. Hammond.

ABSTRACTS.

The Electromagnetic Force in a Moving Charge in Relation to the Energy of the Field: Sir J. Larmor.

In purely formal treatment of the electric field, the circuital relations and the forces on a moving charge are usually taken as two independent foundations. It is known that they are connected dynamically, through the expression for the energy. But the interaction of a moving charge with an unlimited magnetic field involves infinite terms; and therefore the deduction of the electromagnetic force has to be made through the Principle of Least Action, as part of the general dynamical analysis involving radiation, &c., in a form which is rather intricate and not very amenable to direct intuition.

The infinite terms can be evaded by consideration of a pair of conjugate charges, positive and negative. An ideally simple system for the purpose in view is that of a charged condenser, with uniform flat dielectric of any form, convected through a uniform magnetic field. The motion of the charges superposes another uniform magnetic field between the plates, but none outside them. Thus the magnetic energy of the field is altered by the motion in a very simple manner: and from its variation the forcive sustained by the condenser, or even by one plate of it separately, owing to the magnetic field in which it moves, can readily be deduced. The result is the same as follows from considering each element of charge of the condenser to be acted on by its own electrodynamic force. Thus the two methods are in necessary relation, and the law of electrodynamic force on an element of charge, if not strictly deduced, is at any rate closely linked up with the relations of the field, by elementary considerations.

On Foucault's Pendulum: Dr. T. J. I'A. Bromwich.

Let axes of x, y be drawn in a horizontal plane, so that the sense of rotation from x to y is that of $NWSE$; then the approximate equations

of motion of the bob of Foucault's pendulum can be written

$$\frac{d^2x}{dt^2} - 2\omega \frac{dy}{dt} + p^2x = 0,$$

$$\frac{d^2y}{dt^2} + 2\omega \frac{dx}{dt} + p^2y = 0,$$

where ω is the component of the earth's angular velocity about the vertical and $2\pi/p$ is the period of the pendulum, when ω is neglected. To solve these write $z = x + \iota y$ and then the two equations can be combined in one

$$\frac{d^2z}{dt^2} + 2\iota\omega \frac{dz}{dt} + p^2z = 0,$$

which gives $$z = e^{-\iota\omega t}(Ae^{\iota qt} + Be^{-\iota qt}),$$

where $q^2 = p^2 + \omega^2$, or practically $q = p$.

In Foucault's actual experiment, the bob is let go from relative rest, so that $dz/dt = 0$ at $t = 0$; and suppose that then $z = a$, so that the axis of x passes through the initial starting point. Then we find that

$$z = \frac{a}{2q}\left\{(q+\omega)\, e^{\iota(q-\omega)t} + (q-\omega)\, e^{-\iota(q+\omega)t}\right\}.$$

This equation represents a hypocycloid, with cusps on the circle $|z| = a$, and vertices on the circle $|z| = a\omega/q$. At a cusp, the bob is instantaneously at rest, so that $dz/dt = 0$; and then $t = n\pi/q$, where n is an integer, so that the cusps are given by

$$z = ae^{n\pi\iota(1-\omega/q)}.$$

Hence the distance between adjacent cusps is $2\pi a\omega/q$: and so the number of cusps in a complete circuit of the figure is q/ω, and is very large.

In Foucault's own experiments at the Panthéon, q/ω was about 7200, and a about 3 metres; thus the inner circle had a diameter of about $\frac{4}{5}$ mm., and the advance of the cusps along the outer circle was about $2\frac{1}{2}$ mm.

Another case is often described in which the bob is projected so as to pass through the origin (the point vertically below the point of suspension). It does not appear that this could be obtained practically; but its mathematical theory is easy. We have initially

$$z = 0, \quad \frac{dz}{dt} = \frac{dx}{dt} = n, \text{ say},$$

and then we find the solution

$$z = \frac{n}{q} \sin(qt)\, e^{-i\omega t}.$$

Or, using polar coordinates in the horizontal plane, we can write

$$r = (n/q) \sin(qt), \quad \theta = -\omega t,$$

or

$$r = -a \sin(q\theta/\omega), \quad \text{if} \quad a = n/q.$$

The path is then a kind of rosette, and the distance between adjacent vertices is $2\pi a\omega/q$.

It is necessary to push the approximations further than is here indicated, in order to take account of terms corresponding to those which produce the known progression of the apses in the ordinary conical pendulum: but the corrections so obtained do not lead to any material alteration of the foregoing results.

LIBRARY

Presents.

Between December, 1912, and August, 1913, the following presents were made to the Library, from their respective Authors or Publishers :—

Bachelier, L.—"Calcul des Probabilités, Les Probabilités semi-uniformes," 1913.
———— "Les Probabilités Cinématiques et Dynamiques," 1913.
Collins, C. E.—"Matrices and Determinoids," Vol. 1, 1913.
Mallik, Prof. D. N.—"Fermat's Law," 1913.
Marsh, H. W.—"Industrial Mathematics," 1913.
Martin, Artemas.—"Some Properties of Rational Plane Triangles whose Sides are Whole Numbers," 1912.
———— "On Powers of Numbers whose Sum is the same Power of some Number," 1912.
———— "On Rational Right-Angled Triangles," 1912.
Riabouchinsky, D.—"La Fonction $|X|$," essai d'un calcul des valeurs absolues, 1913.

Calcutta : Indian Association for the Cultivation of Science, Bulletin, no. 8, 1913.
Calcutta : Indian Engineering, vol. 52, nos. 15–26, 1912 ; vol. 53 ; vol. 54, nos. 1–8, 1913.
Cape Town : South African Journal of Science, vol. 9, nos. 4–13, 1912–13.
Coimbra : Accademia Polyt. de Porto, Ann. Scientificos, vol. 7, nos. 3, 4, 1912 ; vol. 8, nos. 1, 2, 1913.
Hamburg : Math. Gesellschaft, Mittheilungen, bd. 5, heft 2, 1913.
Liége : Société Royale des Sciences, Memoirs, ser. 3, tom. 9, 1912.
London : Educational Times, vol. 65, no. 620, 1912 ; vol. 66, nos. 621–628, 1913.
London : Educational Times, Math. Questions and Solutions, new series, vol. 22, 1912.
London : Mathematical Gazette, vol. 6, no. 101, 1912 ; vol. 7, nos. 102–104, 1913.
London : Nautical Almanac for 1915.
Madras : Indian Mathematical Society, Journal, vol. 4, no. 6, 1912 ; vol. 5, nos. 1, 2, 1913.
Paris : L'Enseignement Math., ann. 14, no. 6, 1912 ; ann. 15, nos. 1, 2, 1913.
Prag : Časopis Mathematiky a Fysiky, ročnik 39–42, 1909–13.
Sendai : Tôhoku Mathematical Journal, vol. 2, nos. 3, 4, 1912 ; vol. 3, no. 1, 1913.
Tokyo : Sūgaku-Butsurigaku, vol. 6, nos. 20–22, 1912 ; vol. 7, nos. 1–5, 1913.
Warsaw : Wiadomości Matem., tom. 16, zeszyt 5, 6, 1912 ; tom. 17, zeszyt 1–4, 1913.
Washington : Journal of the Academy of Sciences, vol. 3, no. 1, 1913.

Exchanges.

Between December, 1912, and August, 1913, the following exchanges were received :—

American Journal of Mathematics, vol. 34, nos. 2–4, 1912 ; vol. 35, nos. 1, 2, 1913.
American Mathematical Society, Transactions, vol. 13, nos. 3, 4, 1912 ; vol. 14, nos. 1, 2, 1913.
American Mathematical Society, Bulletin, vol. 19, nos. 3–9, 1912–13.

American Philosophical Society, Proceedings, vol. 51, nos. 206, 207, 1912.
Amsterdam: Nieuw Archiev, deel 10, stuk 2, 1912.
Amsterdam: Revue Semestrielle, tome 21, pt. 1, 1913.
Belgique: Académie Royale des Sciences, Bulletin, 1912, nos. 8–12; 1913, nos. 1–3.
Berlin: Jahrbuch über die Fortschritte, bd. 41, 1912–13.
Berlin: Journal für die Mathematik, bd. 142, 1913.
Berlin: Sitzungsberichte der K. Preuss. Akademie, 1912, nos. 39–53; 1913, nos. 1–22.
Boston: American Academy of Arts and Sciences, vol. 47, no. 22; vol. 48, nos. 4, 5, 7–17.
Cambridge Philosophical Society, Proceedings, vol. 13, pt. 6, 1906; vol. 16, pts. 3–8, 1911–12; vol. 17, pts. 1, 2, 1913.
Cambridge, Mass.: Annals of Mathematics, vol. 14, nos. 2–4, 1912.
Canadian Institute, Transactions, no. 22, 1912.
Catania: Accademia Gioenia, Atti, ser. 5, vol. 5, 1912.
Catania: Accademia Gioenia, Bollettino, ser. 2, fasc. 24, 25, 1912.
Erlangen: Physik.-medicin. Societät, Sitzungsberichte, bd. 44, 1912.
France: Société Mathématique, Bulletin, tome 41, fasc. 1, 1913.
France: Société Mathématique, Comptes Rendus des Séances, 1912.
Göttingen: Königl. Gesell. der Wissenschaften, Nachrichten, Math. Klasse, 1912; heft 7, and beiheft.
Holland: Archives Néerlandaises, ser. A, tome 2, 1912.
København: Kgl. Danske Vidensk. Selsk., Forhandlinger, Oversigt, 1912, nos. 3–6; 1913, nos. 1, 2.
Leipzig: Beiblätter zu den Annalen der Physik, bd. 36, hefte 23, 24, 1912; bd. 37, hefte 1–11, 1913.
Leipzig: K. Sächsische Gesell., Math. Klasse, Abhandlungen, bd. 32, no. 6, 1912.
Leipzig: K. Sächsische Gesell., Math. Klasse, Berichte, 1912, nos. 3, 4.
Livorno: Periodico di Matematica, anno 28, fasc. 2–5, 1912–13.
Livorno: Periodico di Matematica, Supplemento, anno 16, fasc. 1–7, 1912–13.
London: Nature, vol. 90, nos. 2247–2261, 1912–13; vol. 91, 1913.
London: Physical Society, Proceedings, vol. 25, pts. 1–3, 1913.
London: Royal Society, Proceedings, Series A, nos. 547, 574, 598–609, 1912–13.
London: Royal Society, Transactions, Series A, vol. 212, 1913.
Marseille: Annales de la Faculté des Sciences, tome 21, 1913.
Modena: Regia Accademia, Memorie, vol. 10, no. 1, 1912.
München: K. Bayer Akad. d. Wissen., Math.-Phys. Kl., Abhandlungen, bd. 26, abh. 1, 1913.
München: K. Bayer Akad. d. Wissen. Math.-Phys. Kl., Sitzungsberichte, 1912, abh. 3.
Napoli: Accademia delle Scienze, Rendiconti, vol. 18, fasc. 10–12, 1912.
Palermo: Rendiconti del Circolo Matematico, tome 34, fasc. 3, 1912; tome 35, 1913.
Paris: Bulletin des Sciences Mathématiques, Oct., 1912–Sept., 1913.
Roma: Reale Accademia dei Lincei, Rendiconti, vol. 22, sem. 1, 1913.
Stockholm: Acta Mathematica, bd. 36, pts. 3, 4, 1912.
Torino: R. Accademia delle Scienze, Atti, vol. 48, disp. 4–10, 1913.
Toulouse: Faculté des Sciences, Annales, ser. 3, tome 3, 1911.
Upsala: Reg. Soc. Scientiarum, Nova Acta, ser. 4, vol. 3, fasc. 1, 1913.
Warsaw: Prace Matematyczno-Fizyczne, tome 23, 1912; tome 24, 1913.
Wien: Monatshefte für Mathematik, jahr. 24, viertel. 1, 2, 1913.
Zurich: Vierteljahrsschrift, hefte 3, 4, 1912.

OBITUARY NOTICES

PAUL GORDAN.

The death of Paul Gordan on December 21st, 1912, in the seventy-sixth year of his age, removed one who for over thirty-four years had been an honorary member of the London Mathematical Society. He was born at Breslau on April 27th, 1837 ; he studied at the University of Giessen, and there took the degree of Doctor of Philosophy in the year 1862. The following year saw the publication of the first of his writings, which were to raise him very rapidly to a position of eminence. From that time there flowed from his pen a steady stream of work, which ceased only three years before his death. Altogether we owe over eighty publications to him—including two books and the famous *Programm.*

The coming of Clebsch to the University of Giessen in the year 1862 was an event of the greatest importance in determining the direction in which Gordan's genius should be applied. His interest at that time centred round the theory of Abelian functions ; and he introduced Clebsch to Riemann's work in this subject. The two entered into a fruitful partnership, producing among other things their treatise *Abel'sche Functionen.* At a time when Riemann's ideas were only imperfectly understood, and the accuracy of his reasoning open to question, this little volume, based on familiar geometrical reasoning, but informed with a rare insight into the principles of the subject, was able to exert a great influence. What are the precise contributions of Gordan and Clebsch perhaps the authors themselves would not say. But that between them, in addition to Clebsch's undoubted geometric genius, they were in touch with the work of Weierstrass and Kronecker, as well as intent on bringing Riemann's results into common use, no reader can fail to see. The book is of permanent value.

Clebsch on his side made known to Gordan the new theory of modern algebra which he was presently to make his own in so remarkable a manner ; and to this theory Gordan made his first contributions in the year 1867.

It was natural that these should have in view applications to his old study of Abelian functions, in the first place. But very soon Gordan

applied himself to the elucidation of the theory of forms for its own sake. Two years later (1869) he published* the first proof of his great theorem of the "Finiteness":—*i.e.*, "Every invariant or covariant of a binary quantic can be expressed rationally and integrally in terms of a finite number of them." A theorem which at once placed the theory of modern algebra on an entirely fresh basis. The proof was built up by an elaborate and somewhat difficult system of inductions; it possessed the great merit that it gave a practical method of constructing the "complete system" in any given case. The original proof was accompanied by the construction of the complete system for the cases of the quintic and the sextic, by way of illustration—this had not been accomplished before, in fact it was generally accepted at the time that no *finite* complete system existed in either case. A variety of other proofs of this theorem have since been put forward by various writers; and while these bring into prominence important and interesting facts, there still remains no other proof which actually provides the means of constructing the finite system.

Gordan next devoted his energies to the simplification of his method of proof; and to the extension of his theorem to other cases. Thus he shewed that it was true for any finite system of binary quantics as well as for a single quantic; he proved it for combinants, also for "types," and finally for ternary forms of the lowest orders. The most notable advance in this direction was the publication of his *Programm* at Erlangen in 1875, entitled "Ueber das Formensystem Binaerer Formen," which contained his original proof of the finiteness, modified and extended, and further developments in the theory of binary forms.

The proof of the finiteness produced no small sensation amongst the other mathematicians who were engaged in building up the theory of modern algebra. In England, Cayley, to whom the foundations of the subject were so largely due, had stated his conclusion in the second Memoir on Quantics (1855) that the number of covariants of a binary quantic is infinite first in the case of the quintic; and that the number of invariants is first infinite in the case of the septic. This statement coming from so great an authority, given with apparently adequate reasons, was accepted as correct, until Gordan's paper was published. In 1870, Cayley published his tenth Memoir on Quantics as the result of Gordan's theorem; in this he shewed the flaw in his own reasoning by which he arrived at the above false conclusion, and gave a *résumé* of Gordan's proof of his theorem, "the importance of which," he went on to state, "in reference to the whole theory of forms it is impossible to estimate too highly."

---

* *Crelle*, Vol. LXIX, pp. 323–354.

A friendly rivalry ensued between the English mathematicians and Gordan and his followers. To one party was due the honour of laying the foundations of the subject, to Gordan the honour of proving the fundamental theorem. The methods of attack were different; in fact, the one was the complement of the other. For while Gordan and his school obtained a system in terms of which all other invariants and covariants could be rationally and integrally expressed; there remained some uncertainty as to whether some members of the system might not be reducible. They, in fact, obtained an upper limit to the irreducible system, while the methods of the English school gave a lower limit to this system. Thus when the two came to an agreement an absolute certainty was established that the system in question was complete and irreducible. Sylvester, in particular, applied himself to attacking the same systems by the method of generating functions which Gordan had discussed by his symbolical methods: and in several instances succeeded in reducing covariants which had been included in the systems. He speaks of the matter thus in discussing the system for two biquadratics:* "J'ajouterai seulement que cette preuve éclatante de l'insuffisance de la méthode de M. Gordan et de son école, pour séparer les formes véritablement élémentaires des formes superflues qui s'y rattachent (insuffisance reconnue par M. Gordan lui-même de la manière la plus loyale dans son discours inaugural prononcé à Erlangen), n'ôte rien à la valeur immense du service qu'il a rendu à l'Algèbre, en ayant le premier démontré l'existence d'une limite au nombre de ces formes."

The untimely death of Clebsch, in the year 1872, brought to an end a partnership which had been most fruitful. The last of the memoirs† published under the joint names of Clebsch and Gordan did not appear until the following year.

In 1873, Gordan was placed on the staff of the *Mathematische Annalen*, which he had already enriched with a good many contributions; and which continued to be the principal means by which he made the results of his labours known to the world.

The year 1875 saw Gordan appointed to a professorship of mathematics at Erlangen; and at Erlangen he remained for the rest of his life. His arrival there was marked by one of his most memorable publications, the *Programm*, "Ueber des Formensystem Binaerer Formen," which has already been mentioned.

In 1885–1887 there appeared his book on *Invarianten-Theorie*, in two

* *Comptes Rendus*, LXXXIV, 1877; *Collected Papers*, Vol. III, p. 66.

† "Ueber cubische ternäre Formen," *Math. Ann.*, Vol. VI.

volumes; in writing which he had the assistance of Dr. G. Kerchensteiner. The second volume contained an exposition of his theorem of the finiteness and of his own work in that region; an exposition of the greatest value to those who wished to learn the subject. A third volume was announced and eagerly anticipated by many, but it has never appeared.

It is not possible in a short obituary notice to give a full account of all the writings of Prof. Gordan. Naturally most of his researches followed on the foundation of the "finiteness." Many of his papers dealt with resultants and discriminants, and also with the solution of quintic and sextic equations. He made important contributions to the theory of ternary forms, particularly to the cases of the ternary cubic and biquadratic. He also wrote valuable expositions of already known theorems, notably that on the transcendency of e and π.*

A. Y.

The above notes are written by one who had not the privilege of knowing the man, but only his writings. Prof. M. Noether has courteously informed the writer that an account of the life and work of Prof. Gordan is about to appear in the Mathematische Annalen.

* *Math. Ann.*, Vol. XLIII.

GEORGE HOWARD DARWIN.

George Howard Darwin, the second son of Charles Darwin, was born at Down, Kent, on July 9, 1845. He received his school education at the hands of the Rev. Charles Pritchard, afterwards Savilian Professor of Astronomy at Oxford, entered Trinity College, Cambridge, as a Scholar, in 1864, graduated as Second Wrangler in 1868, and was elected a Fellow of Trinity in the same year. In 1874 he was called to the Bar, but soon afterwards returned to Cambridge and began the long course of investigations in mathematics and astronomy which made him famous. In 1883 he was elected to the Plumian Professorship of Astronomy and Experimental Philosophy, in succession to James Challis, and continued to hold the Chair till the time of his death, which took place on December 7, 1912. He was re-elected Fellow of Trinity, as professor, in 1884, and was married in the same year. He joined the London Mathematical Society in 1868, and some of his earliest papers are published in the *Proceedings*.

The thread which runs through Darwin's researches and renders them a coherent whole is the design of applying mathematical analysis, supported by arithmetical computation, to develop a reasoned cosmogony—a physically intelligible history of the earth, the earth-moon system, and the solar system. The dynamical causes and effects of geological change, the modifications of rotational speeds, mean motions, eccentricities, inclinations, and distances in the earth-moon system and the solar system that can have been brought about by the agency of tidal friction, the conditions for the breaking up of a gravitating fluid mass in rotation as throwing light on the nebular hypothesis and alternative hypotheses, the theory of a swarm of meteorites simulating a gas by collisions, the possible condensation of meteorites to increase two pre-existing nuclei and the determination of localities in which they could escape capture and go to form other nuclei, as offering a possible explanation of the distances of the planets from the sun: these are the main subjects of his work. Naturally there were by-paths. Geological dynamics requires some hypothesis as to the internal constitution of the earth, and Darwin's calculation of the actual fortnightly tide contributed in no small measure to establish the doctrine of the rigidity of the earth. To the same domain belong his design of a bifilar suspension to measure the lunar deflexion of gravity, his reports on observed earth movements, and his papers on "The Stresses produced in the Interior of the Earth by

the Weight of Continents and Mountains," and on "The Dynamical Theory of the Tides of Long Period." The theory of tidal friction required an analysis of the tide-generating potential into constituent terms, each proportional to a spherical surface harmonic and to a simple harmonic function of the time, and a provisional development of such an analysis, sufficient for the purpose in hand, was given in the paper on "The Precession of a Viscous Spheroid and the Remote History of the Earth" (1878). When, under the influence of Lord Kelvin, the task of developing a complete analysis and utilizing it for tidal prediction was undertaken by a Committee of the British Association, the work fell to Darwin. He did it thoroughly. His analysis, printed in *Brit. Assoc. Rep.* 1883, is a masterpiece of lucidity, and his final results were put in such a form that comparatively unskilled persons can use them to predict the tides of a port in their most intricate details. Incidentally, as it were, he thus became the greatest authority on the tides, contributing comprehensive articles upon them to the *Admiralty Scientific Manual* and the *Encyclopædia Britannica*, delivering a course of lectures concerning them at Boston, U.S.A., afterwards published as a popular treatise, now in its third edition, and writing the greater part of the article devoted to them in the *Encyclopädie der Mathematischen Wissenschaften.*

The theory of tidal friction was developed in a series of memoirs published in the *Philosophical Transactions of the Royal Society* between the years 1878 and 1881. In these the earth is regarded as a nearly spherical body composed of a material which reacts both by elasticity and by viscosity, the latter property involving the requisite degradation of energy, so that the tidal friction considered is internal friction within the elastico-viscous material distorted by tide-generating forces. In accordance with this supposition, the history of the earth-moon system is traced backwards in time until a critical stage is reached in which the period of rotation of the earth was about five and a half of our present hours, the moon was distant but a few thousand miles from the surface of the earth, and the earth always turned the same face to the moon. Darwin did not hesitate to draw the inference that the moon was once part of the earth and became detached from it, not in the form of a ring as in Laplace's nebular hypothesis, but as a globe. He pointed out that his theory would account effectively for the present relative lengths of the day and month, the inclination and eccentricity of the lunar orbit, the obliquity of the ecliptic, and the fact that the moon now always presents the same face to the earth. He further investigated the extent to which tidal friction can have been a cause of change in other members of the solar system, and found that, except in the unique system of earth and moon, a planet attended by

a single satellite of relatively large mass, the effect would be insignificant. These memoirs on tidal friction constitute the work by which Darwin is best known. He himself has declared that it was inspired by a conversation with Lord Kelvin on the subject of the early paper on "The Influence of Geological Changes on the Earth's Axis of Rotation" (1877), for which Lord Kelvin had acted as referee.

The inspiration of the most important work on figures of equilibrium of rotating fluid came from the famous memoir of Poincaré, "Sur l'équilibre d'une masse fluide animée d'un mouvement de rotation" (1885). A mass of homogeneous gravitating liquid, rotating slowly about an axis passing through its centre of gravity, takes the form of an oblate spheroid. If the angular velocity increases gradually, the figure becomes more and more oblate until a critical stage is reached beyond which it can no longer be an ellipsoid of revolution, but becomes an ellipsoid with three unequal axes. If the angular velocity is increased sufficiently, the ellipsoid tends to pass into a surface which has been described as "pear-shaped." The supposition that further change might result in the rupture of the pear by shedding the stalk end at the waist is inevitable, and suggests a previously unsuspected possibility in regard to the origin of planetary systems and double stars. Darwin determined to probe this suggestion. Realizing that he must needs use ellipsoidal harmonic analysis, and finding that the current formulæ relating to ellipsoidal harmonics were not well suited to arithmetical computation, he proceeded to develop the theory of these functions in a fashion appropriate to his purpose, and to apply them to the problem of determining the conditions of stability of the pear-shaped figure. These researches were published in a series of memoirs printed in the *Philosophical Transactions* between the years 1901 and 1903. His conclusion that the pear-shaped figure is stable has been disputed by Liapounoff, and the matter has not yet been settled satisfactorily. A subsequent paper on "The Figure and Stability of a Liquid Satellite" (1906) was devoted to the extension and verification of the work of E. Roche, according to which a planet and satellite can be too close together for the figure of either of them to be an ellipsoid. The determination of the figures of both when within this limit offers a problem of great interest in its bearing on questions of cosmogony, and, if Darwin did not solve it, he has gone further than any one else towards obtaining a solution.

But the perennially fascinating problem which seems to have dominated his attention in the intervals allowed by other occupations was the problem of three bodies. In the simplest form of the problem known as "the restricted problem of three bodies" two gravitating

particles, say a sun and a planet, revolve in circles around their common centre of gravity, and a third particle, too small in mass to affect this regular revolution, moves in the plane of motion of the first two. It may ultimately collide with one of them, or it may revolve in an orbit as an inferior planet, or as a superior planet, or as a satellite, or its motion may change from that answering to one of these descriptions to that answering to another. The integration of the equations of motion, even for the restricted problem, has hitherto defied mathematical analysis. But G. W. Hill showed in 1878 how to construct a particular solution in which the third particle describes a periodic orbit as a satellite. This result was the starting point of Darwin's work. Observing that what was required was essentially the determination of a trajectory in which the curvature was known as a function of the position and the direction of motion, he set about a systematic search for periodic orbits by a step-by-step method of approximate integration. The stronghold, impregnable to analysis, was to be besieged by arithmetic. Some years later Poincaré published his remarkable researches on the problem of three bodies, emphasizing *inter alia* the importance of periodic orbits, but these researches did not inspire the work of Darwin who had for years been occupied with the problem. His results were ready for publication in 1897, when they appeared in the *Acta Mathematica*, t. 21, in a famous memoir on "Periodic Orbits." Others promptly took up the problem, among them notably S. S. Hough, and his discussion led Darwin to further investigations which were published some years later. The courage and perseverance required for this work are perhaps not less conspicuous than the interest of the results obtained.

The distinguishing quality that manifests itself in all Darwin's writings is an immense patience. He did not seek after short cuts or elegant methods, and would sometimes reach general conclusions by examining the numerical details of multitudes of special cases, expending upon his calculations a labour and care that would appal most mathematicians. His papers have been republished by the Cambridge University Press in four massive volumes. Yet, in spite of weak health and continual occupation with research, he found time to fill with distinction several official and quasi-official positions. A few examples of this kind of activity must suffice. Soon after his return to Cambridge he was nominated by the Royal Society a member of the Meteorological Council, and at a later date, when the Council was re-arranged as a Committee under the Treasury, he became one of the two representatives nominated by the Royal Society to serve on that body. He succeeded Sir G. G. Stokes as adviser of the Indian Government and referee in regard to the

Indian survey, and when the International Geodetic Association was established he was appointed by the Foreign Office as the representative of Great Britain. When the triennial meeting of the Association was held in London in 1909, he acted as Vice-President, and had a large share in organizing the gathering. He was President of Section A of the British Association in 1886, when he took as the subject of his address "Geological Time," and was elected President of the Association in 1905, the year in which he received the title of K.C.B. He devoted his address on this occasion to the theme of evolution in the physical universe. When it was decided to hold the fifth International Congress of Mathematicians in Cambridge in 1912, he became chairman of the organizing committee, and was subsequently elected president of the Congress. The success of the meeting was due largely to his initiative and to the tact with which he discharged the duties of the chair. Although such offices as this are sometimes supposed to be of the nature of compliments, they involve the expenditure of time and trouble in mastering the details of business and in preparation for the inevitable speech-making in which it is most essential not to say the wrong thing. Many honours of a less responsible kind were conferred upon Sir George Darwin. He was the recipient of honorary degrees from many Universities in Europe and America, and became a corresponding member of many of the most notable foreign academies and learned societies, but it is said that there was no distinction which he appreciated more highly than the award to him in 1911 of the Copley Medal by the Royal Society. It was well earned. Few men have attempted so much. Rare indeed are those endowed with the faith, the courage, the insight, the industry, and the sobriety of judgment that issue in such solid achievement.

A. E. H. L.

ERRATUM.

In Mr. Hardy's paper, p. 368,

$$\Sigma \frac{A_n}{(\log n)^{2+\delta}}$$

should be

$$\Sigma \frac{A_n}{(\log n)^{1+\delta}}.$$

PAPERS

PUBLISHED IN THE

PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY

ON SOME RECENT ADVANCES IN THE THEORY OF ALGEBRAIC SURFACES

(*Presidential Address.*)

By H. F. Baker.

[Read December 12th, 1912.]

Mr. President, Ladies, and Gentlemen,—

In accordance with a custom which I have no right to seek to evade, I have to-night to make some remarks to you on the occasion of my giving up the Presidential chair, which I have had the honour to occupy for two years. If you will believe me, it would in many ways be most agreeable to me merely to acknowledge the kindness I have received from the Council, and to emphasize the fact, which is obvious enough, that the success of a society such as this depends for the most part on the Secretaries and Treasurer ; and, having said this, to sit down. Tradition, however, requires that something more than this should be attempted. And I wish to recommend to your notice a department of mathematics which has already reached a stage in which the results, as it seems to me, are extremely remarkable, though our English mathematical literature bears as yet very little sign of its influence. I wish to speak to you of some modern developments of the theory of algebraical surfaces. As so often happens in the progress of science, these have come about by the union of two subjects, which, though their origin was largely identical, had drifted somewhat apart—namely, the general algebraical theory of surfaces, as we know it in books on Solid Geometry, and the theory of Abelian functions.

But before I come to my theme, which is, I fear, somewhat technical, I wish to make some remarks in which I may hope to have the sympathy

of all. I come before you to-night not in the character of discoverer, but in the character of dealer or importer. What I wish to speak about has been made for the most part outside these islands. And I want to ask the question: Why is it so often the case that the early history in England of a department of Pure Mathematics is a history of importation? We have teachers of Pure Mathematics in various places; our students are as quick and as ready to learn the subject as foreign students are; and it is not due to a reasoned feeling that Pure Mathematics is unworthy of study, for when a theory has acquired a certain age we require our students to learn at least the elements of it. Is the explanation to be found in our examination system, which rests like a network of red tape upon all our teaching, or is it to be found in a backward state of our civilisation? Pure Mathematics has the importance belonging to a subject which has interested the human mind for thousands of years. As a science it is both the basis and the model for all exact knowledge, and there can be no safer guide to how much of it is necessary for this purpose than the curiosity of those who know its relations. It is, however, as an art that the subject must appeal to those who practise it. And what an art it is, how vast, how delicate! With problems of every degree of difficulty, its humblest devotee may make something beautiful with his own hand; with conceptions of every degree of abstractness, it can allay the most restless imagination. If the student years are to leave an inspiriting background for the daily task, if creative effort be any solace when life is beset with meanness, Pure Mathematics surely has peculiar claims in any scheme of intellectual education. Surely, in an advanced community there would be, in every University of the land, a school of students working at mathematics for mathematics' sake, in addition to working at something else, perhaps, for livelihood's sake. We, as a mathematical society, ought not to rest content with things as they are. We ought constantly to urge that our Science is of the same sort of importance for the intellectual health of the community as are, for instance, the arts of literature, and painting, and music for the emotional health of the community.

Further, I ought, perhaps, to make an apology to those members of the Society who have devoted themselves far more continuously than I have to the study of geometrical questions, for venturing to speak on a geometrical theme. It will indeed appear quite sufficiently that the matter is a branch of the theory of Abelian functions, with which the few publications for which I have been responsible have been mostly concerned. But perhaps I can make my apology best, at the risk of being personal, by explaining how I came to study the matter. The narrative, as it deals

with a familiar subject, may serve the purpose of eliciting your interest, in the same way as mine was secured. I was concerned, now five years ago, with the proof of the theorem, that any function of several variables, of certain characteristics, which is simultaneously periodic in regard to the variables, can be expressed in terms of proper combinations of theta functions. For this I was led to consider a corpus of algebraical functions of a certain number of independent variables—a corpus here meaning a set defined to include all functions obtainable from any functions of the set by one of the four fundamental algebraical processes, addition, multiplication, and so on. The problem was to prove the existence of certain elements in terms of which all functions of the corpus could be rationally expressed. In the case of a corpus of rational functions of one variable, it is known that all the functions are rationally expressible in terms of one function of the corpus. In the case of a corpus of rational functions of two independent variables, it is also true that two functions of the corpus can be chosen in terms of which all other functions of the corpus can be rationally expressed. A particular application of this is a theorem in regard to surfaces in three dimensions. Suppose that the equation to a surface can be reduced to an identity by substituting for each of the three coordinates a proper rational function of two independent variables or parameters. It may then happen that these parameters are incapable of being each expressed as rational functions of the coordinates. But when this is so, it is still possible to choose two other parameters, instead of these, for which this unique reversion, and rational expression, shall be possible. This seemed an obvious enough conclusion, and extremely probable. But I could not find a proof of it, and the enquiries I made were fruitless. Finally, after much searching, I came, by a sort of accident, across a memoir written in Italy, in which a proof was contained, written only in 1893. On examining this, I found that many of the ideas with which I had struggled had presented themselves already from a geometrical point of view. And I was thus led to study the matter. You may perhaps think that the particular instance to which I have referred is mainly of formal interest, and that the result might well have been assumed without proof. Indeed this remark was made to me. On the same general ground one would perhaps regard it as obvious that if a rational relation connecting four variables can be satisfied identically by replacing the variables by rational functions of three parameters, then these parameters can be taken so as to be rational functions of the four variables. This, however, appears not to be necessarily the case, though, in spite of various attempts, no proof of this statement was given until this year, 1912. Even so, the manner of the proof is instructive as to the difficulty of such

questions; it depends upon an elaborate and detailed study of the cubic complex of lines.

Passing from this particular problem, I proceed now to outline the general idea of what I am dealing with.

If we try to compare the theory of conic sections, as it would present itself to the mind of a Greek, and as it presents itself to us, we shall find, if my impression is just, that to us the theory is simpler because of the introduction of general principles, which enable us to collect under a few heads what were to him a vast number of disconnected propositions. And these general principles are, speaking roughly, consequences of only one—namely, of the principle of correspondence, whereby we consider two figures as identical when to any point of either corresponds a single point of the other. Geometrically, that is, we consider the older metrical geometry as a particular case of projective geometry. Analytically, this is one of the consequences of the enormously important step made by Descartes in the introduction of coordinates, and arises by the free use of linear transformation of coordinates. In brief, Projective Geometry is the study of the invariants of the group of linear transformations.

After the conic sections, and with the introduction of coordinates, other curves naturally come into consideration, and it is possible to fill many pages with the metrical and projective properties of special curves. But this alone would be of small interest to us to-day. The subject of all possible plane curves is so vast that we need a wider principle of identification, of correspondence, than is furnished by linear transformation. Any two conics, leaving aside straight lines, can be changed into one another by linear transformation; this is not the case for any two plane curves. But two curves which cannot be linearly transformed into one another, as, for instance, two curves of different orders, may nevertheless be capable of being transformed into one another by relations which ensure an unambiguous correspondence of point to point, the coordinates of the point of either curve being rational functions of the coordinates of the corresponding point of the other curve. And this is the leading idea of any general theory of plane curves—that two curves are to be regarded as identical when they are capable of such birational transformation into one another. And the properties of a curve that are considered are, thus, the invariants of birational transformation.

And I want particularly to emphasize some historical facts which ensured that this point of view, which has become the dominant one, should receive due recognition. The equation of a plane curve is merely an algebraical relation connecting two variables; it is therefore fundamental in a wide field of considerations. Euler and Legendre, in attempt-

ing to extend the possibilities of the integral calculus, were led to consider certain plane curves, and Cramer's *Introduction to the Analysis of Algebraic Curves* is as early as 1750. But it is to Abel's and Riemann's investigations in regard to Abelian integrals that we owe the due recognition of the modern point of view. Abel considered sums of algebraical integrals, and shewed that these are expressible by sums of a certain number P, of integrals, just as a sum of elliptic integrals is expressible in terms of only *one* integral, beside algebraic and logarithmic functions. This number P may be said, with certain conventions, to be what is now called the genus or deficiency of a curve, and a large part of Abel's great paper is concerned with shewing how this number P can be found from the equation of the curve. Riemann, both explicitly in words and implicitly by the number of results of his theory of Abelian integrals which can only be expressed in terms of the genus—as, for instance, by his remarkable result that the number of irreducible constants for a curve of genus p is $3p-3$, when $p > 1$—further emphasized the same idea, and proposed to say that all curves capable of being birationally transformed into one another belong to the same *class*. From the point of view adopted by these writers, it appears at once that the genus of a curve is an invariant of birational transformation. It is, without doubt, the most important of such invariants.

Riemann's discoveries were made largely by help of a theorem of existence based on the theory of double integrals, and in connexion with his conception of a Riemann surface. But the geometrical results so obtained were expounded from a purely algebraical point of view later on, by Clebsch and Brill and Noether in Germany, and afterwards, with added refinement and generalisation, by a numerous school of Italian geometers. To-day none of us can think of a theory of plane curves without the apparatus of point sets, which gives to the theory a unity and simplicity which is very remarkable.

One of the obvious suggestions of such a theory of plane curves is that it ought to be capable of being extended to surfaces. This is a particularly welcome suggestion to anyone who knows by trial the difficulties of formulating a theory of functions of several variables in extension of the ordinary theory of functions of one variable. The idea presented itself to Clebsch. As the genus of a plane curve can be defined by the number of everywhere finite algebraical integrals attached thereto, so, if we consider everywhere finite integrals, double integrals however, attached to a surface, we obtain a number which is unaltered by birational transformation. Like many important steps in discovery, this idea was published in a brief note, at the end of 1868.

The suggestion involved in this remark found two different lines of development. On the one side, by Noether, in Germany, and afterwards by a powerful school in Italy, having its roots far down in the investigations which had been undertaken there for plane curves, a theory of the invariants of the birational transformation of surfaces was elaborated, working with an apparatus of sets of curves lying on surfaces, analogous to, but far more intricate than, the preceding theory of sets of points of curves, and the elaboration is as yet far from complete. No Englishman who attempts to acquaint himself with the exuberance, the power and the beauty of these enquiries, can fail to be struck with admiration, even with envy.

But this is not all. In the theory of plane curves it had happened, historically, that the theory of the rational functions was subsequent to a transcendental theory of the detailed properties of algebraic integrals. Possibly there might be an analogous circumstance in the case of surfaces. At any rate Picard, in France, set himself to imitate, on a surface, the theory of the various kinds of integrals known to exist for a plane curve, namely, those which are everywhere finite, those which possess algebraical infinities, and those which possess logarithmic infinities. The integrals he investigated in this regard which up to the present have led to the most definite results, were not the double integrals previously considered by Clebsch, though these also enter in important ways, but single integrals of complete differentials. To many, Picard's problem must have seemed an artificial and profitless one, a mere slavish imitation of Riemann's work in a field where analogous success was not to be looked for. It was, moreover, a problem of enormous labour, extending over many years, and sustained more by faith than by any striking definite result, as at least it would appear to some.

To-day, however, the point of view is different. For it now appears that the results obtained by the Italian school, and the results obtained by Picard, are complementary to one another in a most wonderful way. By the union of these two diverse theories, of common ancestry, has arisen a remarkable series of consequences which really seem to bring the problem, of a classification of all possible surfaces, into manageable shape. To-day, as it appears to me, we are far advanced in such a general outlook as will make possible a general theory of surfaces, and of algebraic constructs of any number of dimensions.

This is the central proposition to which I wish to invite your consideration to-day.

So much for general exposition. I propose now to attempt to describe in more detail, but still roughly, some of the results which seem to me the

more important or interesting of those obtained. I must naturally give more attention to introductory notions than their final importance warrants, in order to try to be clear. You will understand also that what I say is necessarily incomplete, and in several important respects deliberately so. My object is not to make a scientific report, but to suggest the interest of the matter.

In the theory of a plane curve $f(y, x) = 0$, the genus may be defined as the number of integrals of the form

$$\int \frac{\phi(y, x)}{\partial f/\partial y}\, dx,$$

which are everywhere finite, $\phi(y, x)$ being an integral polynomial. If $f(y, x)$ is of order n, the polynomial ϕ must be of order $n-3$ at most, and must satisfy certain conditions at the multiple points of f in order to compensate the zeros of $\partial f/\partial y$, and certain conditions at infinity. These conditions we summarise by saying that ϕ must be an adjoint polynomial. It can be shewn that any everywhere finite integral must necessarily be of the form written down. It is an obvious fact that any everywhere finite integral is changed into such an integral by birational transformation. Thus the genus, the number of homogeneous coefficients left arbitrary in the polynomial ϕ, is an invariant of birational transformation.

Clebsch's proposal for a surface $f(z, x, y) = 0$ was similarly to consider everywhere finite integrals of the form

$$\iint R(z, x, y)\, dx\, dy,$$

in which R is rational in z, x, y. The number of such integrals is evidently unaltered by birational transformation. It can be shewn that such an integral must have the form

$$\iint \frac{\phi(z, x, y)}{\partial f/\partial z}\, dx\, dy,$$

in which ϕ is a polynomial, of order $N-4$, at most, if N be the order of f, subject to certain conditions at the multiple points and multiple curves of f, and at infinity, which we summarise by saying that ϕ is to be an adjoint polynomial; and conversely that, subject to these conditions, such an integral is everywhere finite. The number of such integrals is a first invariant for the birational transformation of surfaces. It is equal to the number of adjoint surfaces ϕ, of order $N-4$. It is interesting to-day, in the light of subsequent qualifications, to see the easy confidence which

was at first reposed in the finality of the step forward undoubtedly made by this remark.

The conditions of adjointness being stated, it becomes possible to calculate a formula for the number of surfaces ϕ. This formula applied to a rational surface is found to give zero, as it evidently should; for it is at once clear that if $F(\xi, \eta)$ be a rational function of two independent variables ξ, η, there is no integral of the form $\iint F(\xi, \eta)\, d\xi d\eta$ which is everywhere finite. But this same formula applied to a ruled surface gives a negative number; indeed, the negative of the genus of the plane section of the surface. This fact was remarked by Cayley in 1871. The formula is thus at fault in some cases, since there cannot be a negative number of everywhere finite integrals. We must hence conclude that for a surface of so low a comparative order as $N-4$, the conditions of adjointness that have been assumed in calculating it are not independent; and the circumstance is a warning always to revert to the geometrical conditions and not trust the general formula. We might be further tempted to conclude that, in the cases where the formula gives a negative number, there are no everywhere finite double integrals, and also that the general formula is of no use. Both these conclusions would be wrong. As regards the first, though it is true that for ruled surfaces there exist no adjoint polynomials of order $N-4$, yet this is not always so for the cases in which the general formula gives a negative number. For instance, the surface of which the coordinates are quadruply periodic functions of two variables, the so-called hyperelliptic surface, possesses one everywhere finite double integral, though the general formula leads to the value -1. As regards the second conclusion, it was shewn by Zeuthen that, even though the general formula for the number of adjoint surfaces of order $N-4$ may have a negative value, the expression is, nevertheless, one which is unaltered by birational transformation of the fundamental surface, the proof being very much on the lines of the analytical proof that the expression for the genus of a plane curve is unaltered by birational transformation of the curve. In the case of a plane curve this expression always gives the number of linearly independent adjoint curves of order $n-3$, if n be the order of the fundamental curve. In the case of a surface, as we have been explaining, the facts shew that we have two invariants of birational transformation, which need not be equal. The first, the Clebsch invariant, now called the geometrical genus, and denoted by p_g, is the actual number of adjoint surfaces of order $N-4$. The second, the Zeuthen invariant, now called the arithmetical genus, and denoted by p_a (or p_n), is the number given by the general expression for the number of adjoint surfaces of

order m, when in this expression m is replaced by $N-4$. In all cases $p_g \geqslant p_a$.

It is extremely fortunate that neither of these genera was discarded in the earlier investigations; for subsequent work has shewn that it is their difference $p_g - p_a$ which is the most important invariant of the surface. In very many respects it is this difference which is the analogue of the genus of a plane curve, as we shall illustrate immediately. This difference is called the irregularity of the surface, and a surface for which it is zero is called regular. Familiar examples of regular surfaces are (1) rational surfaces, (2) surfaces the most general of their order, (3) surfaces whose only multiple line is a double curve which is a complete intersection of two surfaces. On the other hand, every ruled surface is irregular, the irregularity being the genus of the plane section of the surface; so also a surface representing the aggregate of the pairs of points of any two curves (not both rational) is irregular. I believe that, save for surfaces reducible to ruled surfaces, no surface of order less than 8 has yet been found which is irregular. Among the earliest English publications in this subject are Mr. Berry's investigations on elementary lines of the cases of the quartic and quintic surfaces (1899, *Camb. Phil. Trans.*, XVIII, XIX, XX).

Other invariants of birational transformation, after the genera, may now be mentioned. The notion of a linear system of curves on a plane is familiar. On any surface a linear system of curves is given by the intersection of the surface with a family of surfaces of the form

$$\lambda\phi + \lambda_1\phi_1 + \ldots + \lambda_r\phi_r = 0,$$

in which $\lambda, \lambda_1, \ldots, \lambda_r$ are arbitrary parameters. The surfaces $\phi = 0$, $\phi_1 = 0, \ldots, \phi_r = 0$, which are supposed of the same order, may all pass through a common curve lying on the fundamental surface; this, which then belongs to every curve of the system independently of the values of the parameters $\lambda, \lambda_1, \ldots, \lambda_r$, is not in general to be reckoned as part of the linear system considered. It may happen that the curves considered belong to a more general system,

$$\lambda\phi + \ldots + \lambda_r\phi_r + \lambda_{r+1}\phi_{r+1} + \ldots + \lambda_s\phi_s = 0,$$

of curves of the same order, depending, however, on $r+s$, instead of r, parameters. When this is the case the original system is called incomplete. In what we are to say we speak only of complete systems unless the contrary is asserted. The distinction is of the most fundamental importance. An analogous distinction arises for linear series of point sets upon a curve. A linear series of sets of points is then called complete when it is not contained in a series of sets, each of the same

number of points as the original sets, given by a variable curve containing linearly a larger number of parameters than in the original series. Consider, for instance, a rational quartic curve in space, defined by the partial intersection of a cubic and a quadric surface whose remaining intersection consists of two skew lines. We have, then, upon the quartic curve a linear series of sets of four points, obtained by the intersections of the curve with variable planes. This, however, is not complete, being contained in the series obtained by variable quadric surfaces passing through four fixed points of the curve. The former series depends on only three parameters, the latter on four.

It is clear that a linear system of curves on a surface changes, by birational transformation, into a linear system of curves on the transformed surface, depending on the same number of linear parameters. If this number, entering homogeneously, be $r+1$, the system is said to be of *freedom*, or multiplicity, or dimension, r; and a curve of the system is identified by assigning r arbitrary points of it. Further, the number of intersections of any two curves of the system, called the *grade* of the system, is also evidently unaltered by the transformation. Further, as two curves which are in birational relation with one another are of the same genus, this genus is a third invariant of the birational transformation.

In making these remarks about linear systems we have avoided reference to the possibility that all the curves of the system may have certain fixed common points, which may be multiple points of the same multiplicity for all the curves of the system. It is evident that a system of which all the curves have certain base points prescribed can be regarded as included in a system of greater freedom, having no prescribed base points. Whether a system is complete or not depends, therefore, on what base points are supposed to be prescribed. The definition of the grade and genus of the system is also in general modified. Further, we have taken no account of the fact, which is a capital source of difficulty in the theory of the birational transformation of surfaces, that a curve may transform to a point and a point transform to a curve. It appears undesirable in a brief survey to attempt to deal with this matter.

Beside the numbers associated with a linear system which are invariant in a birational transformation, there are certain linear series of sets of points upon any curve of the system which are also invariant. Chief among these is the so-called canonical series existing upon any curve, having, when the curve is of genus π, a freedom $\pi-1$ and, in each set, $2\pi-2$ points, these being the zeros of the differential of an everywhere finite integral associated with the curve; that is, of an adjoint polynomial when the curve is plane. Another series, in the case of a curve belonging

to a linear system of freedom r, is the series of freedom $r-1$ cut upon that curve by all the other curves of the same system, of which each set contains a number, n, of points equal to the grade of the linear system. This is called the characteristic series.

In the case of a plane curve of order m, the canonical series is given by the intersections of the curve with a variable adjoint curve of order $m-3$. The question arises whether it may not be possible, in the case of a linear system, $|C|$, of curves upon a surface, to define in connexion therewith, another system of curves whose intersections with any curve C of $|C|$ shall constitute, or be contained in, the canonical series of $|C|$. The answer is that this is generally possible. For instance, when $|C|$ consists of all possible plane sections of the surface, the new system of curves which is desired is given by the system of adjoint surfaces of order $N-3$, if N be the order of the fundamental surface. This system of curves is called the system adjoint to the original system, and denoted by $|C'|$. It is not, however, generally the case, though $|C'|$ gives upon any curve C sets of the canonical series, that all sets of the canonical series upon C are obtainable in this way. The series obtained upon C by means of the curves of $|C'|$ is not the complete canonical series of freedom $\pi-1$, but a series of only freedom $\pi-1-p$ contained therein, p being the irregularity of the surface. This fact has only recently been obtained. The original result was that the deficiency in freedom of the series cut upon C by $|C'|$ was never greater than p. The revised result was first proved by Picard, in a transcendental way, as an *a posteriori* result of a fundamental theorem, in regard to the number of integrals existing upon a surface, obtained in Italy in 1905, and the geometrical proof was then given in 1908 by Severi, upon the basis of a theorem due to Enriques (1904–5). This result, in regard to the deficiency of the series cut by $|C'|$ on C, may be looked upon as furnishing a definition of the irregularity of a surface. Considering the linear system of curves on the surface constituted by all plane sections, the adjoint surfaces of order $N-3$, where N is the order of the surface, give upon an arbitrary plane, not the complete system of curves of order $N-3$ adjoint to the plane sections of order N and genus π, which is of freedom $\pi-1$, but give a system of plane curves of freedom $\pi-1-p$. This defines p and shews that it is unaltered by birational transformation. As $p=p_g-p_a$, and p_g is evidently unaltered by birational transformation, we can thus infer that so also is p_a. The original proof of this fact, given by Zeuthen, depending upon a consideration of the possible multiplicities of the surface, may thus be dispensed with, save for the necessary rearrangement in the order of development of the theory.

Another result in which the irregularity of the surface enters relates to the characteristic series. Any curve of a linear system of freedom r cuts upon an individual curve of the system a series of freedom $r-1$. But it does not follow from the fact that the linear system of curves is complete that this linear series of sets of points is also complete. We will give an example in a moment in which this is not so. The proposition to which we are referring is that if the complete series on the individual curve of the system, which contains the sets cut thereon by the other curves of the system, be of freedom $r-1+k$, then k is not greater than the irregularity p of the surface. And to this there is the complementary proposition that upon any surface there exist linear systems for which k is exactly p.

To give an example of the theorem that the characteristic series of a complete linear system is not necessarily complete, we may consider a particular ruled surface. Consider two plane cubic curves of the same modulus, between which a (1, 1) correspondence has been defined; let them be placed in space in two different planes, so that two corresponding points coincide. If, then, pairs of corresponding points, one on each curve, be joined by straight lines, these lines will give rise to a ruled surface of order 5, having a double curve of order 5 and genus 1. On this surface there is a linear system, of freedom 3, of curves defined by all the plane sections of the surface. This system is not complete, however. For if we take 5 arbitrarily chosen fixed generators of the surface, and construct quartic surfaces passing through the double curve of the surface, and also through these 5 generators, we shall obtain 5 linearly independent quartic surfaces, whose variable intersections with the ruled surface constitute a linear system, of freedom 4, of curves of order 5 and genus 1. This linear system contains that of the plane sections of the surface; for when the 5 fixed generators are properly chosen, as is possible with arbitrary positions for 4 of them and a suitable position for the remaining one, a cubic surface can be drawn through the double curve of the surface and these 5 generators; and then the quartic surfaces spoken of contain as part of themselves the product of this cubic surface and an arbitrary plane. Considering now this complete system, of freedom 4, of curves of order 5, obtained by the variable quartic surfaces, any two curves of the system intersect in 5 points, and there is upon any separate curve of the system a characteristic series of freedom 3 of sets of 5 points. As the separate curve is of genus 1, the complete series of sets of 5 points has freedom 4, not 3, and is not determined upon this curve by the other curves of the system. There is thus a deficiency of freedom in the characteristic series equal to 1. This is, in fact, equal to the irregularity of the ruled surface. For

there are no adjoint surfaces of order $N-4$, the double curve not being plane; so that $p_g = 0$. The number of conditions that a surface of order m, sufficiently high, should contain the double curve, of genus 1, is $5m$; such a surface will thus contain

$$\tfrac{1}{6}(m+1)(m+2)(m+3)-5m,$$

homogeneously entering arbitrary parameters. When $m = 1$, this gives

$$p_a = \tfrac{1}{6}\,2.3.4-5 = -1,$$

which is equal, as previously remarked, to the negative of the genus of the plane section of the ruled surface. And the irregularity

$$p = p_g - p_a = 1.$$

Any ruled surface might similarly have been taken to furnish an example of a complete linear system with an incomplete characteristic series, as follows from the researches of Segre. Another example is furnished by the surface investigated by Picard and Humbert, of which the coordinates are quadruply periodic functions, of a certain kind, of two independent variables. These examples are referred to as having been among the earliest to suggest themselves; on any surface, however, which is not regular an example is obtained by considering the system of curves adjoint to an arbitrary linear system (without base points).

The theorem of the possible incompleteness of the characteristic series of a linear system of curves undergoes a modification when we consider systems of curves depending on variable parameters which enter algebraically but not linearly, in the equations of the surfaces determining the curves of the system. A curve in space can be determined completely as the common part of the intersection of at most four surfaces. Thus a curve on a given surface can be completely specified by at most three other surfaces, or, if we are considering a family of curves and neglect isolated points, can be determined by at most two other surfaces. If the equations of these surfaces depend algebraically upon a certain number of parameters, we obtain an algebraic system of curves. For instance, if we wished, on a given surface, to obtain all the curves of a specified order, we might begin with a variable intersecting surface, and should be led, in specifying the order, to a certain number of algebraic conditions for its coefficients. It is known that any system of algebraic relations breaks up into a finite number of irreducible systems, each representable by the points of an algebraic construct such as $f(z, x_1, \ldots, x_R) = 0$, where f is a polynomial in $z, x_1, \ldots, x_R$. We may thus say that an algebraic system of curves consists of a finite number of irreducible systems, each curve of such an

irreducible system corresponding without ambiguity to one point of such an algebraic construct as is represented by an equation $f(z, x_1, \ldots, x_R) = 0$. A linear system of curves is the particular case when the representative construct may be taken to be all the points of a linear space. The number R is called the freedom, or dimension, or multiplicity of the system. A complete algebraic system contains all the curves of the same order represented by all the points of the construct, and, in particular, contains the curves of the linear systems to which individual curves of the algebraic system belong. If we consider an algebraic system of one dimension, represented therefore by the points of an algebraic curve, the intersections of any curve of the system with a consecutive curve of the system have limiting positions as the consecutive curve tends to coincidence with the original. These limiting positions constitute what is meant by a characteristic set in this case. If we consider an algebraic system of R dimensions, the intersections of a particular curve of the system, with all possible consecutive curves, give rise, by their limiting positions, to a linear series of characteristic sets upon the original curve, represented, on the representative construct, by the pencil of directions which exist on the construct at any point of it, a series of freedom $R-1$. Any set of this characteristic series determines on the particular curve chosen, a complete series. Contrary now to what may happen in the case of a *linear* system of curves, this complete series coincides with the series determined by the consecutive curves of the algebraic system, being of freedom $R-1$; namely, every set of the complete characteristic series is given by the limiting position of a consecutive curve of the algebraic system. This theorem is due to Enriques and is of the most fundamental importance in the theory. In order to understand it, we may recur to the example of the ruled quintic surface previously discussed. We considered a linear system of quintic curves, of freedom 4, determined by variable quartic surfaces passing through the double curve and 5 fixed generators. Take an arbitrary plane section of the ruled surface, which is a quintic curve with 5 double points, and hence of genus 1. Any of the quartic surfaces will meet this plane quintic curve in points lying upon a quartic curve, which passes through the double points and through 5 fixed points of the plane quintic curve; there will thus be 5 residual intersections, which vary, when the quartic surface varies, in a linear series. The sum of the elliptic arguments at the 5 points of a set of this series is the same whatever be the quartic surface, and is a constant belonging to the linear system of curves, on the ruled surface, determined by taking all the possible quartic surfaces passing through the same set of 5 fixed generators. By taking a different set of 5 fixed generators through which to pass the

adjoint quartic surfaces we obtain another linear system of curves on the ruled surface, in general associated with another value for the sum of the 5 elliptic arguments of the points where a curve of the system cuts a plane section. The various possible linear systems of quartic surfaces thus obtained are therefore all included in an algebraic system of quartic surfaces, involving, beside the original 4 parameters which enter linearly (not homogeneously), a further elliptic parameter. The equation of a general quartic surface of this algebraic system will contain, beside the 4 linear parameters, rational functions of two independent elliptic functions of this elliptic argument [say $\wp(u)$ and $\wp'(u)$]. Thus the algebraic system is of dimension 5, and the characteristic series it determines on any one of its curves is of freedom 4, and is complete, the series consisting of sets of each 5 points on a curve of genus 1. The circumstance we have illustrated by this case is general. Namely, when in a linear system of curves of freedom r, the characteristic series on any curve, which is of freedom $r-1$, belongs to a complete series of freedom $r-1+k$, it is because the linear system of curves belongs to an algebraic system of freedom $r+k$, of which the general curve, beside depending linearly on r parameters, depends further on parameters, effectively k in number, which are rational functions of Abelian functions of p independent arguments, where p is the irregularity of the surface. This result, developed by Castelnuovo from previous theorems of Picard and Enriques, shows how intimately connected are the problems of the theory of surfaces with the theory of Abelian functions.

A particular corollary from this result may be stated, as it is an example of the fact which has been remarked, that the irregularity of a surface is analogous in many respects with the genus of a curve. As we may have on a surface an algebraic system of curves, so we may have on a curve an algebraic series of sets of points, each set of the series being then represented uniquely by a point of another representative construct, which, when it is of dimension 1, is also a curve. Suppose now the fundamental curve is of genus zero. We may then suppose it to be a straight line. Upon this an algebraic series will be given by an equation of the form

$$x^n+u_1x^{n-1}+\ldots+u_n=0,$$

wherein $u_1, \ldots, u_n$ are rational functions of the coordinates of a point of the representative construct. These functions $u_1, \ldots, u_n$ must be such that when their values are given it is possible, with the help of the equation of the construct, to express rationally the coordinates, say $\xi_1 \ldots, \xi_r, \eta$, of the representative point which corresponds to the set of points of the straight line given by this equation. The series on the straight line

is therefore evidently contained in the linear series represented by

$$\lambda x^n + \lambda_1 x^{n-1} + \ldots + \lambda_n = 0,$$

wherein $\lambda, \lambda_1 \ldots, \lambda_n$ are arbitrary parameters. In other words, on a rational curve any algebraic series of sets of points is contained in a linear series. What we are pointing out is that this theorem for rational curves is analogous to a theorem for regular surfaces. We have been remarking that the variable parameters in an algebraic system of curves on a surface consist of a certain number entering linearly and a certain number, depending on the irregularity of the surface, entering as Abelian functions of variable parameters. When the surface is regular, parameters of this kind cannot enter. Wherefore on a regular surface every algebraical system of curves consists wholly of curves belonging to a certain linear system. This is analogous to the theorem established above for rational curves.

The results so far stated have dealt for the most part with a single linear system of curves on a surface. Consider now *two* linear systems, say $|A|$ and $|B|$, both existing on the surface. There is then a linear system, denoted by $|A+B|$, containing every composite curve obtained by combining a curve of $|A|$ with a curve of $B|$, which, completed, will in general contain also other curves which are not reducible. This system $|A+B|$ may then be defined as the complete linear system containing the curve A_1+B_1, where A_1 is any chosen curve of $|A|$, and B_1 any chosen curve of $|B|$, it being understood that if $|A|$ and $|B|$ have assigned base points, such a point, i-ple for curves of $A|$, and j-ple for curves of $|B|$, is $(i+j)$-ple for curves of $|A+B|$. In order to understand this completely it is necessary to be clear about the fact that any curve $|A|$ determines a definite complete linear system to which it belongs. For take any definite curve A of the surface. Either there is no linear system of curves of which A is a member; this is possible; then A is called an isolated curve, and may be said to define a linear system of freedom zero. Or there is such a linear system with assigned base points. In this case the system is of limited freedom r; for if n be the number of free intersections of any two curves of the linear system it is evident that $r-1$, the freedom of the characteristic series, cannot be greater than n. Take now r as great as possible, so that A belongs to a complete system of freedom r. The only doubt then remaining is whether A can belong to two complete systems of the same assigned fixed points. If this is possible, suppose it belongs to both the systems $|A|$ and $|B|$ given, suppose, by

$$\lambda\phi + \lambda_1\phi_1 + \ldots + \lambda_r\phi_r = 0, \quad \mu\psi + \mu_1\psi_1 + \ldots + \mu_s\psi_s = 0,$$

being given by the part of $\phi = 0$ which is not common to all the curves

$\phi = 0$, $\phi_1 = 0$, ..., $\phi_r = 0$, and also given by the part of $\psi = 0$ which is not common to all the curves $\psi = 0$, $\psi_1 = 0$, $\psi_s = 0$. Then consider the linear system given by

$$\phi(\mu\psi + \mu_1\psi_1 + \ldots + \mu_s\psi_s) + \psi(\lambda\phi + \lambda_1\phi_1 + \ldots + \lambda_r\phi_r) = 0.$$

The fixed intersection, independent of the parameters, of the surfaces here represented consists of the fixed common intersection of all the surfaces of both the two original surfaces, every multiple point assigned to be i-ple for all the curves of the first system and j-ple for all the curves of the second system, being $(i+j)$-ple for the curves of the new system; evidently both of i and j must be at most equal to an actually occurring multiplicity for the original curve given, A. Another curve common to all the surfaces of the new system is the curve A itself, as satisfying both $\phi = 0$ and $\psi = 0$. These fixed intersections being taken account of, the new system is a linear system whose variable intersections contain all the curves of $|A|$ and also all the curves of $|B|$. Moreover, the fixed points, simple or multiple, supposed to be assigned for $|A|$ and for $|B|$ being taken the same, part of those actually belonging to the curve A, will also be common to the new system and of the same multiplicities. But the new system is of greater freedom than either of the systems $|A|, |B|$. And this is impossible, since each of these is supposed complete with the assigned fixed points. Wherefore the curve A can only belong to one complete system of assigned fixed points, and determines this system.

Thus we are justified in defining the system $|A+B|$ as that complete system determined by one composite curve A_1+B_1 of assigned fixed points.

Conversely, it may happen, if $|P|$ and $|Q|$ be two linear systems, that the general surface defining the curves of $|P|$ can, by proper limitation of its parameters, be made to contain a curve, Q_1 of $|Q|$, and still retain linearly arbitrary parameters. The residual intersection will then be a linear system of curves defining a complete linear system, which we may denote by $|P-Q_1|$. For definiteness suppose that neither $|P|$ nor $|Q|$ has any assigned fixed points. If Q_2 be another curve of $|Q|$ we can, in that case, similarly obtain a complete linear system $|P-Q_2|$. These two systems $|P-Q_1|$, $|P-Q_2|$ must, however, coincide. For if R_1 be a particular curve of $|P-Q_1|$, the composite curve $|R_1+Q_1|$ is a curve of $|P|$, and the complete system $|P-R_1|$, which thus contains Q_1, must coincide with $|Q|$ by what has been proved, and must hence contain Q_2. Thus there is a curve of $|P|$ formed by the composite curve R_1+Q_2, and so the complete system $|P-Q_2|$ must contain R_1. As $|P-Q_1|$, $|P-Q_2|$

thus contain a common curve R_1 they must coincide. For this reason we denote the system so obtained by $|P-Q|$.

One of the most important applications of these ideas may now be referred to. If $|C|$ be any linear system, and $|C'|$ its adjoint system, it may happen that the latter system contains the former, so that we can form a system $|C'-C|$. It can be proved that when this is the case for one linear system $|C|$, it is equally so for any other linear system $|D|$, and that the curves $|D'-D|$ are the same as the curves $|C'-C|$. Some qualifications of this statement are noticed below. For instance, if $|C|$ be the system of the plane sections of the surface of order N, then $|C'|$ is determined by adjoint surfaces of order $N-3$. Thence $|C'-C|$ is determined by adjoint surfaces of order $N-4$. These are, in fact, the surfaces which arise in the Clebsch everywhere finite double integrals whose number is p_g. The curves on the surface determined by them are called the canonical curves.

We have indicated that some qualification may be necessary in the above definition of the canonical system of curves. This arises owing to two circumstances: (*a*) the linear system $|C|$ may have multiple points assigned as fixed base points; (*b*) the system $|C'-C|$ may have fixed components. As regards the former, if $|C|$ have a certain point of the surface as a fixed point of multiplicity i, it is contained in a more ample system $|C_1|$, having this point only as an $(i-1)$-ple point; if $|\Gamma|$ be the system formed from $|C|$ in this way, by reducing the multiplicity of every assigned fixed base point by unity, it is the system $|C'-\Gamma|$ which gives the canonical system. As regards (*b*), there may be upon the surface curves which are capable, by a properly taken birational transformation, of being transformed to simple points. Such curves are called exceptional curves, and they arise as fixed components in the curves of the canonical system. Moreover, in what we have said throughout, we have made no allusion to the question of multiple points belonging to the fundamental surface itself. In a general survey such as the present it is impossible to enter into the consideration of such qualifications with any clearness without unduly increasing the length of what is said. But it may be stated that the advance of the theory shews how to deal with the restrictions referred to.

From the statement which has been made, it follows, when the surface possesses a canonical system of curves, that the series cut upon any curve of a linear system by the curves of the adjoint system is the sum of the characteristic series of the curve and the series cut upon the curve by the canonical system of the surface.

Hence, when there exists a canonical system of curves on the surface, the characteristic series of any linear system is special, namely, contained

in the canonical series of the curve of the system upon which it is considered. Now, upon a curve of genus π, a special series of sets of m points is of freedom $r = m-s$, where s is the strength of a set of the series as conditions for a canonical set. And, if j be the number of linearly independent canonical sets which contain a set of the given series, the so-called index of specialty of the series, and r' the freedom of the complementary series, obtained by subtracting the given series from the canonical series of the curve, we have

$$j = \pi - s, \qquad r' + 1 = \pi - s\,;$$

and

$$r = m - \pi + j = m - \pi + r' + 1.$$

We apply this to the case of the characteristic series of a linear system of curves upon the surface, of freedom r. We have already stated that the characteristic series is of freedom

$$r - 1 + p_g - p_a - h,$$

where $0 \leqslant h \leqslant p_g - p_a$; if i be the number of linearly independent canonical curves of the surface which contain any curve of the linear system, so that $p_g - i$ is the number of linearly independent sets cut by the canonical system upon the curve of the system under consideration, and if $p_g - i + \delta - 1$ be the freedom of the complete series to which these sets belong, we obtain

$$r - 1 + p_g - p_a - h = n - \pi + p_g - i + \delta,$$

where n, π are the grade and genus of the linear system considered. Thus we have

$$r = n - \pi + 1 + p_a + s - i,$$

where i, defined above, is called the index of specialty of the linear system of curves, and s, which is called the superabundance of the linear system, is the sum of the "algebraic deficiency" of the characteristic series and of the deficiency of the series cut upon a curve of the given linear system by the canonical curves of the surface. Neither of the numbers s, i can be negative. When they are both zero, the linear system is said to be regular. It can be shewn that on any surface there exist linear systems of curves which are regular—and, indeed, by a recent theorem, the adjoint system of any linear system of curves is a regular system. If the surface possess no singularities beside an ordinary double curve and triple points

triple for the curve, all adjoint surfaces of sufficiently high order give rise to regular systems.

The above formula, known as the Riemann-Roch theorem for surfaces, is obtained on the assumption that the adjoint system of a linear system of curves is the sum of the system and the canonical system of the surface. This statement requires modification when the original system of curves possesses multiple base points. Then the number s in the formula must be increased by a number μ, where μ is the number of conditions that a curve of the adjoint system passing through the points of a characteristic set, should pass through each multiple base point, or μ is the index of specialty of the characteristic series less the index of specialty of the series cut upon a curve of the given linear system by curves differing therefrom only in the fact of having an $(i-1)$-ple base point at each i-ple base point of the original system.

A regular system of curves has the properties (1) of not being special, (2) of having its completed characteristic series "algebraically complete," that is, of freedom $r-1+p_g-p_a$; (3) of having for the index of specialty of its characteristic series the number p_g.

The Riemann-Roch theorem, and a certain converse of it, can be extended to reducible curves, provided we define the genus and grade of a curve consisting of several components. If $|A|$, $|B|$ be two linear systems, without base points, the general curve of the sum system $|A+B|$ will, most often, not be reducible. If n, π be the grade and genus for $|A|$, n' and π' those for $|B|$, and N, Π those for $|A+B|$, and i the number of common points of a curve of $|A|$ with a curve of $|B|$, it can be shewn that

$$N = n+n'+2i, \qquad \Pi = \pi+\pi'+i-1.$$

If now we have any single curve whatever, composite or not, say B, it is possible to take a system $|A|$, of irreducible curves, of order so great that the sum system $|A+B|$ consists of irreducible curves. Let n, π be the grade and genus for $|A|$, and N, Π those for $|A+B|$. Then the grade and genus for the curve B are defined by the relations above given. And this is legitimate, provided it be shewn that the result so obtained is the same whatever system $|A|$ be taken. This is easily established by choosing another system $|A_1|$ and considering the sum system $|A+A_1+B|$, first as the sum of $|A_1|$ and $|A+B|$, and then as the sum of $|A|$ and $|A_1+B|$.

It is at once clear that the equations above give rise to

$$2\Pi-2-N = (2\pi-2-n)+(2\pi'-2-n').$$

The number $2\pi-2-n$, for the system $|A|$, may be called the canonical

number; when the surface possesses a canonical system of curves, it is equal to the number of intersections of any curve of $|A|$ with a canonical curve. The canonical number of the sum of any number of linear systems, without base points, is the sum of their canonical numbers. Denoting by n, π, γ, respectively, the grade, genus, and canonical number for any linear system $|C|$, and by n', π', γ', the corresponding numbers for the adjoint system $|C'|$, the canonical number for $|C'+C|$ is $\gamma'+\gamma$. This, however, when a canonical system of curves exists, is equal to the number of intersections of $|C'-C|$ with $|C'+C|$, which is $n'-n$. Equating these, we find

$$n' = \pi' + \pi - 2,$$

which can be put also into the form

$$\pi' - \pi = \gamma',$$

and is easily established independently of the hypothesis of the existence of a canonical system.

Since the system $|C'-C|$ is the same whatever system $|C|$ may be, so also is $\gamma'-\gamma$. This number is denoted by $\omega-1$, and is thus an invariant of the surface. Various forms for it are

$$\omega-1\,(=\gamma'-\gamma) = \pi'-\pi-(2\pi-2-n), \quad \text{or} \quad \pi'-1+n-3(\pi-1),$$

$$(=\gamma''-\gamma') = \pi''-\pi'-(\pi'-\pi), \quad \text{or} \quad \pi''-2\pi'+\pi,$$

where π'' and γ'' refer to the adjoint system, $|C''|$, of $|C'|$.

As a simple illustration of the formula

$$\pi-1 = n'-\pi'+1,$$

we may make the following remark. We have previously stated that $|C'|$ cuts on $|C|$ a series of freedom $\pi-1-(p_g-p_a)$. There are p_g curves $|C'|$ which contain C; hence, if r' be the freedom of $|C'|$, we have

$$r'-p_g = \pi-1-(p_g-p_a),$$

or

$$r' = \pi-1+p_a.$$

By the formula in question this is the same as

$$r' = n'-\pi'+1+p_a,$$

which is in accordance with a statement above made that $|C'|$ is regular.

We turn aside now from these geometrical investigations, for the most

part elaborated in Italy, to the question of the integrals of the form

$$\int (A\,dx + B\,dy),$$

that can exist on the surface, A and B being rational functions of the coordinates x, y, z. Such integrals may be of three kinds. First, integrals which are everywhere finite. Second, integrals which become algebraically infinite along curves, so that on an arbitrary curve, say a plane section of the surface, they give rise to integrals having points of algebraic infinity or poles. Third, integrals which are logarithmically infinite along curves, so that, on a plane section say, they give rise to Abelian integrals with points of logarithmic infinity. It is now known that the number of linearly independent everywhere finite integrals is p, or $p_g - p_a$, the irregularity of the surface, and the number of linearly independent integrals of the second kind, independent of these and of rational functions, is also p, the number of periods for any such integral of the first or second kind, being $2p$. But this result was not obtained until many partial results had been first obtained by different workers. Humbert proved, quite early, in 1894, in a paper which contains many ideas since utilised, that on a surface not possessing everywhere finite integrals of perfect differentials, every algebraic system of curves must consist of curves all belonging to the same linear system, so that a surface containing an algebraic system of curves not contained in a linear system must possess integrals of the first kind. Severi proved that any integral of the second kind can be reduced by subtraction of rational functions, to an integral with only a single curve of infinity of the first order, and then, that this curve, if belonging to a linear system of curves, must have an incomplete characteristic series. As, by Castelnuovo's theorem, the deficiency of the characteristic series of a linear system is not greater than p, it follows hence that a surface with integrals of the second kind must have a value of p greater than zero. Denoting the number of linearly independent integrals of the first kind by q, and the number of linearly independent integrals of the second kind, including those of the first kind, but excluding rational functions, by r, Severi deduced that $r - q \leqslant p$. This was in 1904, about eight years ago. Picard had already proved (in 1885), by consideration of the group of the linear differential equation whose solutions are the periods of an integral of the second kind, that such an integral could be constructed with arbitrary periods. From this it can be inferred that $r \geqslant 2q$. Finally, on the basis of a theorem due to Enriques, which we have quoted, that the characteristic series of a complete algebraic system of curves is necessarily complete, Castelnuovo and

Severi, early in 1905, established that $q \geqslant p$. These inequalities necessitate

$$r = 2q = 2p.$$

This enumeration is not intended to be complete. It makes no reference, for instance, to Picard's paper in the *Comptes Rendus* early in 1905.

The various investigations that have been referred to are all of permanent interest even apart from the fact that together they established the fundamental theorem $q = p$.

Picard's use of the linear differential equation satisfied by the periods of an integral is extremely remarkable, and his investigation, in connexion therewith, of the Riemann surface of four dimensions which represents the original algebraic surface, is striking especially from the disproportion between the apparent meagreness of the method and the evident difficulty of the problems considered. Perhaps it is not fanciful also to note that Riemann's development, for the case of complex variables, of Gauss's consideration of the linear differential equation satisfied by the hypergeometric series, was subsequent to his preoccupation with complex variables for his theory of Abelian functions; and that it was Fuch's application of Riemann's ideas to linear differential equations in general which led to Poincaré's and Picard's investigations for the group of such an equation.

Severi's investigation of the formula $r-q \leqslant p$ is not only remarkable as giving incidentally a practical method of finding the algebraical part of an integral $\int A\,dx+B\,dy$, but also by the great light it throws on the characteristic series of a curve. In association with an integral of the second kind having a given curve for its sole polar curve, of the first order, he builds a rational function which, in a sense, serves to identify the integral. Finding the poles and zeros of this rational function upon the polar curve, he obtains the result that its poles are the same for every integral of the second kind having this curve as sole polar curve of the first order, while its zeros are partly the same. The remaining zeros are the points of the curve where the integral is finite, and are different for different integrals with this same polar curve. Severi show that these variable zeros are a set of the characteristic series of the curve; and proves that, if the polar curve be a curve of a linear system, these points cannot be a set of the characteristic series determined on the polar curve thereby, unless the integral is reducible to a rational function. This explains then why the characteristic series of a linear system may be incomplete. Moreover, the matter is thus in a certain sense analogous to what occurs in the theory of Abelian integrals on a curve, where the rational functions are found to be members

of a larger class of functions, namely, of the integrals of the second kind. Another striking part of the same investigation is the reduction, already incidentally referred to under another point of view, of an integral of the second kind with any set of polar curves, to an integral with a single polar curve of the first order for the integral.

Severi's proof that the number q, of everywhere finite integrals is as great as the irregularity p, of the surface, is associated with a theorem which is an extension of the converse of Abel's theorem for a curve. If we have two sets, of the same number of points, upon a curve, the converse of Abel's theorem gives, as a sufficient and necessary condition that the two sets should belong to the same linear series, or, in other words, as a sufficient and necessary condition that the two sets should be the poles and zeros, respectively, of a rational function on the curve, the condition that the sums of the values of the everywhere finite integrals at the points of the first set should be equal, save for periods, to the corresponding sums at the points of the second set. Severi sets out to obtain conditions, expressed in terms of the everywhere finite Picard integrals of the surface, in order that an algebraic system of curves on the surface should consist of curves all belonging to the same linear system. Of the various forms of the result which he obtains, the first may be quoted: "If we have an algebraic system of curves on a surface, a sufficient and necessary condition that its curves should belong to the same linear system is that the sums of the everywhere finite integrals of the surface should be the same at the points of intersection of every two curves of the system." Another interesting result, contained as a lemma in this investigation, has subsequently led to an important theorem, proved by Castelnuovo, for algebraic series of points upon a curve. If we have an algebraic series, of freedom unity, of sets of m points upon a curve of genus π, with the property that, containing an arbitrary point of the curve, there are m' of its sets, then a sufficient and necessary condition that the sets should all belong to the same linear series is that the number of united points, at which two points of a set coincide, should be $m'(2m+2\pi-2)$; when the sets do not belong to the same linear series the number of united points is less than this. To read the descriptive proof of this given by Castelnuovo, and to follow out the proof given by Segre, of a formula, originally due to Schubert, upon which it depends, is to realize how deep are the roots of the present brilliant school of geometry in Italy. Further, I would remark, in this connexion, the following intimately connected problem. If we have on a surface of irregularity p a curve of genus π, where $\pi > p$, the p integrals of the surface which are everywhere finite will give upon this curve, generally, just p integrals of the first kind forming a defective system; and

there will be, in addition, upon this curve, $\pi-p$ integrals of the first kind, forming the complementary defective system. If now we consider a set of m points upon the curve, what relation exists between the sums of the p integrals of the first defective system at these m points and the sums of the $\pi-p$ integrals of the second defective system at the same points?

The fundamental theorem upon which we have been dwelling, that the number of everywhere finite Picard integrals upon a surface is equal to the irregularity of the surface, may be regarded as being to a large extent a corollary from Picard's theory of the integrals (of total differentials) which possess curves of algebraic infinity. The theory elaborated by him of the integrals which possess curves of logarithmic infinity has led to another geometrical theory of a singularly interesting kind. We can illustrate the character of the results by referring first of all to the cases of the quadric and cubic surfaces, though the theory referred to explicitly leaves aside all rational surfaces. Two curves, A and B, upon a surface are coresidual when there exists a third curve C, such that C and A together form a complete intersection, of the fundamental surface with another surface ϕ, and also C and B form a complete intersection, of the fundamental surface with another surface ψ, of the same order as ϕ. In such case we may also speak of A and B as equivalent, and express this result by writing $A \equiv B$. In this sense, it is a statement of a familiar theorem to say that upon a quadric surface every curve is equivalent with a curve expressible in terms of multiples of two arbitrarily chosen intersecting generators u, v—every curve C can be written in the form

$$C \equiv mu+nv,$$

where m, n are integers. In the same way upon a cubic surface every curve C can be written in the form

$$C \equiv mu+m_1a_1+\ldots+m_6a_6,$$

where a_1, a_2, ..., a_6 are six lines of the surface forming the upper, or the lower, row of a double six of lines upon the surface, and u is a certain associated twisted cubic curve of the surface, while m, m_1, ..., m_6 are integers appropriate to the particular curve C. In a similar, but much more general way, upon any irregular surface, any curve can be expressed linearly in terms of a certain finite number of curves, with integral coefficients, in such a form as

$$C \equiv \lambda v+\lambda_1v_1+\ldots+\lambda_kv_k,$$

where v, v_1, ..., v_k are the base curves, λ, λ_1, ..., λ_k are integers appropriate to the curve C, but the sign $\equiv$, instead of meaning that the curves on

the two sides of the sign belong to the same linear system, means that they belong to the same algebraic system. For instance, it can be shewn that on a Kummer surface, every curve of the surface can be expressed in terms of 17 elements. We explained at the commencement that we are dealing with a theory of birational transformation, and it is sufficiently clear, though for simplicity's sake we have avoided discussion of this, that a point may transform into a curve or conversely. Thus in the expression of an arbitrary curve in terms of elements, it may be convenient to take, upon a particular representative of a surface, points, instead of actual curves, as elements. And indeed upon the Kummer surface, 16 of the 17 elements may be taken to be the double points of the surface.

It has been said that this theory is in intimate connexion with the theory of Picard's integrals of the third kind, having curves of logarithmic infinity. To explain this, we must premise that curves $C_1, C_2, \ldots, C_k$ upon a surface are said to be algebraically connected when there exist integers $\lambda_1, \ldots, \lambda_k$, such that the two curves

$$\lambda_1 C_1 + \ldots + \lambda_m C_m, \qquad \lambda_{m+1} C_{m+1} + \ldots + \lambda_k C_k$$

belong to the same algebraic system, for some value of m, where $1 \leqslant m < k$. This being clear, there is the theorem that, for any integral of a total differential which becomes logarithmically infinite along curves of the surface, these curves of logarithmic infinity are algebraically connected; and conversely that, given any set of curves which are algebraically connected, an integral of the third kind exists having these for its curves of logarithmic infinity, with, in fact, logarithmic coefficients (or periods) directly deducible from the coefficients occurring in the expression of the algebraic connexion of the curves. Further, a necessary and sufficient condition for the curves $C_1, \ldots, C_k$ to be algebraically connected, is that every determinant of the matrix, of $(k+1)$ rows and k columns,

$$\begin{vmatrix} n_{11} & n_{12} & \ldots & n_{1k} \\ n_{21} & n_{22} & \ldots & n_{2k} \\ \ldots & \ldots & \ldots & \ldots \\ n_{k1} & n_{k2} & \ldots & n_{kk} \\ m_1 & m_2 & \ldots & m_k \end{vmatrix}$$

should vanish, where $m_1, \ldots, m_k$ are the orders of these curves, n_{ij} is the number of intersections of C_i and C_j and n_{ii} is the grade of C_i.

We have remarked, as an example of the analogy between the genus of a curve and the irregularity of a surface, that on a regular surface every

algebraic system of curves consists of curves belonging to the same linear system. The curves of logarithmic infinity of a Picard integral of the third kind, being in general algebraically connected, are thus, on a regular surface, linearly connected. This enables us to show that, on a regular surface, every Picard integral reduces to rational functions and logarithms of rational functions. And the converse statement, that this only occurs on a regular surface, is deducible from Enriques' theorem in regard to the characteristic series of an algebraic system of curves. Surely this is an extremely interesting, if only a particular, result.

We have now attempted to give in rough outline some account of two of the main divisions of the new theory. First of the method of defining invariants of birational transformation by means of systems of curves lying upon the surface, due largely to the Italian geometers; second of the algebraic integrals, due to Picard, and of the influence which either of these has exerted upon the other. One of the practical outcomes of the combined theory is a set of theorems, still, of course, far from complete, enabling us to classify surfaces according to the values of their invariants. Some of these results may now be referred to.

One striking theorem may be mentioned at once, because it can be stated very simply. We see very soon that ruled surfaces, which may be looked upon as aggregates of straight lines, differ essentially, from the point of view we are considering, from surfaces in general. They possess no adjoint surfaces of order $N-4$, where N is their order, and their plane sections possess no adjoint system of curves. The geometrical genus p_g is in all cases zero, and the numerical genus p_a equal to the negative of the genus of a plane section. Thus the irregularity is equal to this genus, and the everywhere finite integrals of the surface are the integrals associated with the curve of any plane section, all of which are functions of one of them. It is thus of interest to have a criterion for the birational transformation of any surface into a ruled surface. It can be shown that any surface possessing a linear system of curves, of grade n and genus π, such that the canonical number γ, $= 2\pi-2-n$, is negative, can be transformed birationally into a ruled surface. The proof of this which has been given is certainly of a very delicate character; it requires a careful examination of the fact, to which we have adverted, which makes so much of the difficulty of the whole theory, that curves can exist on a surface, called exceptional curves, which can be transformed birationally into simple points of a new surface; but beside this, a proof was impossible until a sufficient number of special cases of transformation into ruled or rational surfaces, had been examined in detail. One of the first of such detailed examinations was Noether's proof that a surface possessing a

linear system of rational curves of freedom 1 is a rational surface. A consequence of the general theory is that any surface which is not birationally transformable into a ruled surface possesses only a finite number of exceptional curves, which do not intersect, and can be transformed birationally into a surface with no exceptional curves. A particular theorem obtained is that a surface possessing a linear system of elliptic curves with one common point at least is rational or transformable into a ruled surface; and the same may be said of a surface of which all the plane sections are hyperelliptic curves of any genus. Or, again, a surface of order greater than 4 which has plane sections of genus 3 is transformable into a ruled surface (or is rational).

Another class of surfaces which, though very special from the present point of view, are interesting from their comparative frequency, are the rational surfaces. We have already referred to them at the outset. Evidently they must have $p_g = 0$, $p_a = 0$. But these two conditions are not sufficient to ensure that a surface be rational. For instance, we may consider (after Enriques) a sextic surface having six edges of a finite tetrahedron as double lines. Such a surface is represented by an equation

$$\phi(x, y, z, t) + xyzt\psi\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}, \frac{1}{t}\right) = 0,$$

where ϕ, ψ are both homogeneous quadratic functions of their four arguments. For this surface evidently $p_g = 0$, there being no adjoint surfaces of order $N-4$, that is, quadric surfaces, passing through the six edges of the tetrahedron. Moreover also $p_a = 0$; for a surface of order m containing the 6 edges of the tetrahedron would have

$$\tfrac{1}{6}(m+1)(m+2)(m+3) - 4 - 6(m-1)$$

homogeneous coefficients, which when $m = 2$ is $10-10$, or zero. Nevertheless, this surface is not rational. The proof of this, from general principles, we may give some account of, because it offers an opportunity of explaining an invariant of birational transformation to which we have not as yet referred. The property $p_g = 0$ of a surface is an expression of the fact that the adjoint system $|C'|$ of a linear system $|C|$ of curves upon the surface does not contain the system $|C|$. It may, however, happen, even when this is so, that the system $|2C'|$ contains the system $|2C|$. Sufficient conditions for a surface to be rational are (1) that the arithmetical genus p_a be zero; (2) that for some linear system $|C|$ of curves upon the surface the system $|2C'|$ does not contain the system $|2C|$; and these conditions are necessary. The possibility of curves $|2C' - 2C|$ for one system $|C|$ involves the possibility for every other system, and the impossibility of

curves $|2C'-2C|$ involves the impossibility of curves $|C'-C|$, or $p_g = 0$. Hence, in order to show that the sextic surface under consideration is not rational, it is sufficient to find a linear system $|C|$ for which the system $|2C'-2C|$ is possible, and any linear system $|C|$ may be taken. Taking, then, the plane sections, the adjoint system is given by cubic surfaces passing through the edges of the tetrahedron. Such a surface cannot contain a plane section, which is a curve of order 6, without breaking up into the plane of this section and a quadric surface. As there exists no quadric surface passing through the edges of the tetrahedron, we see that the system $|C'-C|$ is impossible. The system $|2C|$ is given by the intersection of the sextic surface with all possible quadric surfaces. The system $|2C'|$ is given by the intersections of the sextic surface with all possible sextic surfaces having the edges of the tetrahedron as double lines; for evidently such a surface, when it breaks up into two cubic surfaces passing each through the edges of the tetrahedron, has for its free intersection with the fundamental sextic surface the sum of two curves of $|C'|$. In order to shew that the system $|2C'-2C|$ is possible we have then merely to shew that a sextic surface can be given, one at least, which passes through the intersection of the fundamental sextic surface with an arbitrary quadric surface, and has, moreover, the six edges of the tetrahedron as double lines. Such a sextic surface is evidently given by the product of the quadric and the four faces of the tetrahedron. Thus the system $|2C'-2C|$ exists, though it consists of a curve of order zero. Another example may be given in which there is an infinity of curves $|2C'-2C|$, though both $p_g = 0$ and $p_a = 0$. This second example, due to Castelnuovo, was suggested by the former, which we have thought should be mentioned for its historical importance as well as because it will be referred to below in another connexion. Consider a surface of the seventh order having a double conic, a triple straight line not intersecting the conic, and three tacnodes whose tangent planes intersect in the triple line. That such a surface exists is evident thus. From a line lying on a cubic surface C five tangent planes can be drawn to touch the surface at points not on the line, and the surface possesses conics not intersecting the line. Let α, β, γ be three of the five tangent planes, P an arbitrary plane through their line of intersection; and let Q be a quartic surface having one of the conics of the cubic surface which do not intersect the line, as a double conic, and, further, touching the planes α, β, γ at their points of contact with C. Then the equation

$$C^2P + Q\alpha\beta\gamma = 0,$$

represents such a septimic as described. The curves adjoint to the plane

sections of this surface will be obtained by quartic surfaces, having the triple line of the surface as a double line, passing through the double conic, and passing through the tacnodes. A quartic surface cannot contain a plane section of order 7, unless it break up into the plane and a cubic surface, which will then have the triple line as a double line, pass through the double conic, and also pass through the tacnodes. Such a cubic surface, being intersected in four points by every line meeting the triple line and lying in the plane of the conic, would contain this plane. The residual would be two planes through the triple line; these could not contain the three tacnodes. Hence it is clear that if $|D|$ be the system of the plane sections of the septimic, the system $|D'-D|$ is impossible, or $p_g = 0$. But also $p_a = 0$; for the conditions that a surface of order m should have a given line as double line are $3m+1$ in number, and so a surface of order m having the proper adjoint behaviour would in this case contain, homogeneously,

$$\tfrac{1}{6}(m+1)(m+2)(m+3)-(3m+1)-(2m+1)-3$$

terms, which, when $m = N-4 = 3$, gives

$$20-10-7-3 = 0.$$

Consider now the formation of a system $|2D'-2D|$. The system $|2D|$ is given by arbitrary quadric surfaces. The system $|2D'|$ will be given by octavic surfaces, having the conic as a double conic, passing through the triple line so as to give on an arbitrary plane section twice the number of intersections with the section of the fundamental septimic which are given by a surface of the system $|D'|$, and passing through the tacnodes in such a way as would be given by two surfaces $|D'|$. The octavic must then give twelve intersections at an arbitrary point of the triple line, and, as we shall assume, must have at the tacnodes the planes α, β, γ as tangent planes. These conditions are satisfied by a surface of the form

$$H\alpha\beta\gamma k^2\phi,$$

wherein H is an arbitrary plane through the triple line, k is the plane of the double conic, and ϕ is an arbitrary quadric surface. Thus we see that the system $|2D'-2D|$ is possible, and contains, beside fixed intersections, the intersection of the septimic with any arbitrary plane through the triple line. It can be shewn that this is its most general form. The number of linearly independent curves $|2D'-2D|$ is, in general, for any fundamental surface, denoted by P_2, or indeed of linearly independent curves $|sD'-sD|$ is denoted by P_s. If $P_s = 0$, then $P_1 = 0$, that is $p_g = 0$, since else we could obtain curves $|sD'-sD|$ by repeating, s times,

curves $|D'-D|$. But conversely when $p_g = 0$, we may have $P_2 \geqslant 0$, or, in general, $P_s \geqslant 0$. In the two examples we have considered, for both of which $p_g = 0$, $p_a = 0$, we have respectively $P_2 = 1$, $P_2 = 2$. For a surface to be rational, it is necessary and sufficient that $p_a = 0$ and $P_2 = 0$, these involving $p_g = 0$. If the surface have no multiple curve other than a double curve with triple points which are triple also for the surface, these conditions, for a surface of order N, with a double curve of order m, of genus π, with t triple points, can be stated as follows

(1) $\frac{1}{6}(N-3)(N-2)(N-1)-m(N-4)+2t+\pi-1=0.$

(2) No surface, other than those containing the original as part of themselves exists, of order $2(N-4)$, having the double curve as double curve for it.

In regard to the genera P_s, another remark may be made in passing. Enriques has found that a necessary and sufficient condition that a surface should be reducible to a ruled surface by birational transformation, is that $P_4 = 0$, $P_6 = 0$. And in connexion with this and results previously given, we may also state that not only is it true that for every ruled surface is $p_a \leqslant 0$, but conversely, every surface for which $p_a < -1$ can be reduced to a ruled surface. The cases $p_a = 0$, $p_a = -1$ have been examined.

Another class of surfaces to which reference may be made is that of which the points represent the couples of points of one or two curves. Taking first two curves, if we associate every point of the first curve with every point of the second curve, we obtain an ∞^2 of couples. If X, Y, Z be three functions, each rational in the coordinates of each of the two associated points, these will be connected by a rational equation, which we may interpret as the equation of a surface. If these functions X, Y, Z can be chosen, so that the coordinates of each of the associated points, one on each curve, are determinable uniquely, and therefore rationally, in terms of X, Y, Z, we shall have such a surface as is under consideration. Similar remarks apply when we begin with a single curve and consider all possible pairs of points upon this. Here, if X, Y, Z are not symmetrical in regard to the coordinates of the two associated points, we merely have a particular case of the former state of affairs, in which the two curves are repetitions of one another. When X, Y, Z are each symmetrical functions of the coordinates of the two associated points, the case is practically distinguishable from the former, in that, when X, Y, Z are given, we cannot evidently obtain the two points from which these arose, but only the symmetrical functions of them. Consider these cases more in detail, and first when we start from two curves, say of genera π and π'. Then it is easy to see that each of the everywhere finite integrals of the two curves gives

rise to an everywhere finite integral on the surface, and conversely, that every such integral arises in this way. Thus, if p_g, p_a be the geometrical and arithmetical genera of the surface, we have

$$p_g - p_a = \pi + \pi'.$$

Further, there are everywhere finite double integrals on the surface, of the form

$$\iint du\,dv,$$

where u is an everywhere finite integral of the first curve and v of the second, and it can be shewn that all everywhere finite double integrals of the surface arise in this way. Hence we also have

$$p_g = \pi\pi'.$$

By associating a definite point of one curve C with all the points of the other curve C' in turn, we evidently obtain a curve on the surface birationally equivalent with C'. There are thus two systems of curves on the surface, each of freedom 1, those of one system being birationally equivalent each with the curve C, and those of the other system each birationally equivalent with C'. Through an arbitrary point of the surface there clearly passes one curve of each system. The composite curve on the surface consisting of $2\pi-2$ curves of the system $\{C'\}$, each birationally equivalent with the curve C', of genus π', taken with $2\pi'-2$ curves of the system $\{C\}$, each birationally equivalent with C, defines a linear complete system of curves on the surface, which is seen without much difficulty to constitute the canonical system of curves on the surface. This may be symbolically denoted by $(2\pi-2)\,C'+(2\pi'-2)\,C$. As two curves C do not intersect, and two curves C' do not intersect, it is thus clear that the grade of the canonical system is $2(2\pi-2)(2\pi'-2)$ or $8(\pi-1)(\pi'-1)$. The surface does not in general possess any rational curve, because through any point of such a curve would pass a single curve of each of the systems $\{C\}$ and $\{C'\}$, and the intersections of these with the rational curve would give on this curve involutions of respective genera π' and π, whereas such cannot exist on a rational curve, unless both π and π' be zero. Wherefore the surface does not possess any exceptional curves, and the invariant $\omega-1$, previously explained in this sketch, is thence easily seen to be equal to the grade of the canonical system, or

$$\omega-1 = 8(\pi-1)(\pi'-1).$$

For a surface with e exceptional curves, as we may remark, unless it is

transformable to a ruled, or rational, surface, the grade of the canonical system is $\omega-1+e$.

In this account of the surface we have utilised the fact that the number of everywhere finite integrals is p_g-p_a, and our formulæ then enable us to calculate both p_g and p_a. But, in fact, these values were obtained for the surface under consideration before it was definitely known that the number of everywhere finite integrals is in general equal to p_g-p_a, thus forming some presumptive evidence for the general result. It is interesting to see how this was done, especially as this furnishes an excuse for explaining an invariant of birational transformation to which no reference has so far been made. For this, it should be premised that there may exist upon a surface a system of curves of freedom 1, of which a number i, greater than 1, pass through an arbitrary point of the surface. When $i = 1$, and only one curve of the system passes through an arbitrary point, it is very convenient to have a name for such a system. We may call it a *pencil* of curves, though the adoption of the Italian word *fascio* would seem a preferable course. Such a pencil may either be linear, or it may be algebraical. In the latter case, the curves of the system can be associated with the points of an algebraic curve of a certain genus. The linear system arises when this curve is rational. Now first suppose we have upon the fundamental surface a linear pencil of curves, each of genus π, having n points common to all; no two curves can intersect except in points common to all the curves of the pencil. Let δ be the number of curves of the pencil which have a double point, the case of several double points upon one curve, or of higher singularities, being neglected for our explanatory purposes. For instance, the curves may be the intersections of the surface with planes passing through a fixed axis; then π will be the genus of a plane section of the surface, n the order of the surface, and δ the number of tangent planes of the surface passing through the axis. With these definitions it can be shewn that the number

$$I = \delta-n-4\pi$$

is the same as for any other linear pencil of curves upon the surface, and, moreover, if e be the number of exceptional curves upon the surface, which is, in fact, finite, unless the surface can be transformed into a ruled (or rational) surface, then $I-e$ is unaltered by birational transformation. The number I is called, from its origin, the Zeuthen-Segre invariant. The same number can be defined by an irrational pencil of curves on the surface. Let, as before, π be the genus of each curve of the pencil, π' the genus of the algebraic curve which represents the pencil, and δ the

number of curves of the pencil which have a double point. Then

$$I = \delta + 4(\pi-1)(\pi'-1) - 4.$$

And it is remarkable, the result being ultimately due to Noether, that the invariants ω and I are connected by the relation

$$I + \omega = 12p_a + 9.$$

Returning now to the case of the surface which represents the pairs of points of two curves of genera π and π', recalling that it possesses a pencil of curves, each of genus π, with genus π' (as well as another such pencil), we have for this surface

$$I = 4(\pi-1)(\pi'-1) - 4.$$

Further, we shewed how to prove that

$$\omega - 1 = 8(\pi-1)(\pi'-1).$$

Hence, from the formula above,

$$p_a = \tfrac{1}{12}[12(\pi-1)(\pi'-1) - 3 - 9] = \pi\pi' - \pi - \pi'.$$

We saw, however, that $p_g = \pi\pi'$. From these we derive

$$p_g - p_a = \pi + \pi',$$

shewing that the irregularity of the surface is equal to the number of everywhere finite integrals.

Consider now briefly the case of the surface representing pairs of points upon the same curve C of genus π, without regard to the order of these. For this surface we find, as before, the equations

$$p_g - p_a = \pi, \quad p_g = \tfrac{1}{2}\pi(\pi-1).$$

Combining any point of the curve C with all other points of this curve, we have upon the surface, corresponding to the couples so obtained, a curve which is birationally equivalent with C. These curves form a system of freedom 1 upon the surface, with the property, however, that through an arbitrary point of the surface 2 of these curves pass. This is clear because any point of the surface corresponds to two points of the curve C, and either of these may be kept fixed, while the other describes the curve. The surface is thus essentially different from that before considered, being in a correspondence (1, 2) with it. We do not enter further into its properties. Evidently the hyperelliptic surface, of which the co-

ordinates are quadruply periodic functions of two arguments, of a certain kind, is a particular case of it, in which $\pi = 2$.

Other kinds of surfaces which it may be interesting to refer to, are those known as double planes, of which the equation can be written in the form

$$t^2 = \phi(x, y),$$

wherein $\phi(x, y)$ is a polynomial in x and y. Of these the cases which are rational surfaces have been ascertained. By a birational transformation, in x and y only, all the rational surfaces of this form may be supposed to arise from the three cases in which $\phi(x, y) = 0$ represents (1) an ordinary plane quartic curve, (2) a sextic curve with a triple point of which the three tangents coincide, (3) a curve of order $2m$ with a $(2m-2)$-fold triple point.

The verification that these represent rational surfaces may be given as follows. In case (1),

$$t^2 = y^4+y^3a+y^2b+yc+d,$$

where a, b, c, d are integral polynomials in x, of respective dimensions 1, 2, 3, 4, if we replace y by y_0, $= Ax+B$, and determine the two constants A, B, so that the right side is the square of a quadratic polynomial t_0, $= Px^2+Qx+R$, and then, in the original equation, write

$$t = t_0+\tau(y-y_0),$$

we shall obtain a relation connecting x, y and τ representing a cubic surface. This is known to be rational. In case (2),

$$t^2 = y^3+y^2a+yb+c,$$

where a, b, c are integral polynomials in x, of respective dimensions 2, 4, 6, if we replace y by y_0, $= Ax^2+Bx+C$, and determine the constants A, B, C, so that the right side is the square of a cubic polynomial

$$t_0, = Px^3+Qx^2+Rx+S,$$

and then, in the original equation, write

$$t = t_0+\tau(y-y_0),$$

we obtain the square of $y+\frac{1}{2}(y_0+a-\tau^2)$ expressed as a quartic polynomial in x and τ, whereby this case is reduced to the former. The algebraic problem which is to be solved in each of these cases is a particular case of the standard problem of finding, for a plane curve of order n, the

adjoint polynomials of order $n-3$ whose free intersections are all contacts, and arises in the theory of the vanishing of the theta functions. The number of solutions is known. In case (3),

$$t^2 = y^2a+yb+c,$$

where a, b, c are polynomials in x of respective orders $2n-2$, $2n-1$, $2n$, we may similarly determine

$$y_0 = u/w, \qquad t_0 = v/w,$$

where u, v, w are polynomials in x of respective dimensions $n-1$, $2n-2$, $n-2$, and then, putting

$$t = t_0+\tau(y-y_0),$$

we find y rationally in terms of x and τ.*

And, as double planes which are rational are interesting to consider, so double planes of genera other than zero are also worthy of investigation. A very general and interesting result may be mentioned. Every double plane whose branch curve ($\phi = 0$) is irreducible is rational or regular, having $p_g = p_a$. More precisely, if the surface $z^2 = \phi(x, y)$ possesses q integrals of the first kind, the curve $\phi(x, y) = 0$ is formed by $2q+2$ or $2q+1$ separate curves belonging to the same pencil. Every curve of this pencil is obtainable by projection from two curves of the surface which vary in a hyperelliptic pencil of genus q. We recall, in passing, the comparison with the case of a Kummer surface, which possesses no integrals of the first kind, but is in (1, 2) correspondence with a surface which possesses two such integrals. It is known that the Kummer surface is reducible to the form $z^2 = P_1P_2 \dots P_5$, where $P_1, \dots, P_5$ are five

* The most general surface of order n having the line at infinity of $x = 0$ for an $(n-2)$-fold line can similarly be satisfied by writing for the other coordinates y, t respectively $y_0 = u/w$, $t_0 = v/w$, where u, v, w are polynomials in x of respective orders $n-2$, $n-2$, $n-3$. Then the substitution $t = t_0+\tau(y-y_0)$ enables us to express y rationally in terms of τ and x. The cones $y = y_0$, $t = t_0$ intersect in a (rational) curve of order $n-2$, meeting the multiple line in $n-3$ points. Such a curve of the surface may also be obtained by considering the $3n-4$ planes through the multiple line whose remaining intersection is a pair of lines a_r, b_r. Through the $3n-6$ lines $(a_1, \dots, a_{3n-7})$ and a_{3n-6}, can be drawn a surface of order $2n-4$ having the multiple line of the given surface as $(2n-5)$-ple line. Another such surface can be drawn through $(a_1, \dots, a_{3n-7})$ and a_{3n-5}. The intersection of these surfaces determines such a rational curve of order $n-2$ as has been described; and there is only one such curve meeting the $3n-5$ lines $a_1, \dots, a_{3n-7}, a_{3n-6}, a_{3n-5}$.

straight lines. The general result in regard to irregular double planes has been obtained in connexion with the following theorems :—(*a*) The necessary and sufficient condition in order that a surface should possess an irrational fascio of curves is the existence of two Picard integrals which are functions one of the other ; (*b*) if $p_g > 2p_a+3$ the surface possesses an irrational fascio of curves which is of genus 2 at least. There may, however, be an irrational fascio when this condition is not satisfied, as is shown by the case of surfaces for which $p_g = 0$, $p_a = -1$. Such surfaces arise by considering the surfaces representing the pairs of points of an elliptic curve and a rational curve. In general, when a surface possesses an irrational fascio of curves, the genus of the fascio can never exceed the irregularity of the surface.

Next we refer in brief terms to another general question of the theory of surfaces—that, namely, of surfaces which allow birational transformations into themselves.

Such transformations may be possible which are expressible by formulæ wherein an arbitrary parameter enters, so that, by taking different values of this parameter, the same point may be capable of being transformed into an infinite number of other points of the surface. When this is the case the combination of two such transformations, with the same general formula and different values of the parameter, may lead to a transformation expressible by the same general formula with a parameter depending on the parameters of the component transformations. More generally the surface may be transformable into itself by the formulæ of a continuous group of birational transformations depending upon a finite number of parameters. Or, while depending upon variable parameters, the formulæ may not be included in those of a continuous group with a finite number of parameters. But a surface may be capable of birational transformations in which no variable parameter enters, these being incapable of being regarded as particular transformations of a continuous group. And, when this is the case, the number of transformations may be only finite or it may be infinite. It is probably unnecessary to remark that for a surface a transformation may be uniquely reversible without being birational, a circumstance which does not arise for the transformation of a curve into itself.

A surface allowing continuous transformations into itself (that is, birational transformations containing arbitrary parameters) which do not form a continuous group of a finite number of parameters is reducible to a ruled (or rational) surface.

Surfaces allowing the birational transformations of a continuous group of finite order are either (1) surfaces with a rational group depending on

one parameter (these are reducible to ruled surfaces) or (2) what are called elliptic surfaces, allowing an elliptic group with one parameter. The surfaces contain two fasci of curves—first, a fascio of elliptic curves, trajectories of the group, not necessarily a linear fascio ; second, an elliptic fascio of curves of any genus ; or (3) the surfaces may be hyperelliptic surfaces, of which the coordinates are quadruply periodic functions of two variables. The necessary and sufficient condition for the existence of a birational group of finite order of self-transformations of a surface is $p_a < 0$. When $p_a < -1$ the surface is reducible to a ruled surface. When $p_a = -1$ the surfaces are elliptic, unless $p_g = 1$, the fascio of elliptic trajectories of the group being of genus p_g. Even when $p_g = 1$, if beside $p_a = -1$, we have $P_4 > 1$, the surfaces are still elliptic. But when $p_a = -1$, $p_g = 1$, $P_4 = 1$ the surfaces are hyperelliptic, with a continuous group of two parameters, which may have two elliptic subgroups.

As regards transformations in which no variable parameter enters, these may be of finite period, and, in that case, may belong to a finite discontinuous group. Any quartic surface with a double point evidently possesses such a periodic self-transformation, and the case of a Weddle or Kummer surface is a familiar example of a finite group of such transformations ; while, since the Cremona transformations of a plane do not form a finite group, any rational surface furnishes an example of an infinite group of discontinuous birational transformations. A surface possessing a continuous group may also possess discontinuous transformations which are not of finite period, as has been shewn by Humbert and Painlevé for particular classes of hyperelliptic or elliptic surfaces. Two other cases of surfaces possessing discontinuous transformations of infinite period, which have been remarked more recently, may also be referred to. First, the sextic surface, previouly considered, having each of the edges of a finite tetrahedron as a double line ; second, the quartic surface of the most general kind described through a sextic curve in space of genus 2. This is not a completely general quartic surface, since the curve in question is not a complete intersection, being the partial intersection of two cubic surfaces whose remaining intersection is a conic and a straight line not intersecting this. This line cuts the sextic in four points. In general a surface which allows a birational transformation of infinite period, and does not allow a continuous group of finite order of birational self-transformations, must have $p_a \geqslant 0$ as well as $P_2 > 0$ and $p^{(1)} = 1$. Here $p^{(1)}$ is the invariant equal to the previously defined ω, when no exceptional curves exist, but otherwise equal to $\omega + e$, where e is the number of exceptional curves, which, under the hypotheses $p_a \geqslant 0$, $P_2 > 0$, must be finite. The surface must in general have a fascio of

elliptic curves. The exceptional case is when $p_a = P_2 = 1$; these are satisfied in the case of the quartic surface just instanced, which is therefore a particularly interesting example. Conversely, a surface with $p_a \geqslant 0$, $P_2 > 0$, $p^{(1)} = 1$ always has a rational or irrational fascio of elliptic curves except when $p_a = P_2 = 1$. Whether the existence of this fascio is in general sufficient to prove the existence of such a transformation seems as yet not quite clearly decided.

As a particular result, if a surface have a non-periodic projective transformation, it allows a continuous group of projective transformations and is ruled or rational.

Notwithstanding the length to which these remarks have extended, we have by no means referred to all the topics which seem to be of capital importance for the theory. One more matter at least, which well illustrates the intimate relation between the geometrical and analytical aspects of the theory, must be mentioned in brief terms, though a competent description would need to be of some length. Our references to the algebraical integrals existing upon the surface, though beginning with Clebsch's everywhere finite double integral, have been, for the most part, to integrals of total differentials. But Picard has also investigated a theory of double integrals analogous in many ways to that of the Abelian integrals upon a curve which are called of the second kind, possessing algebraical infinities, calling them double integrals of the second kind. In rough terms we may say that on the Riemann surface of four dimensions which represents the fundamental surface, they are related to the two-dimensional barriers, which arise in considering the connectivity of that surface, very much as are the Abelian integrals of the second kind, for a Riemann surface representing a curve, related to the curve barriers or period loops described thereon. It was natural then to seek the number of such double integrals of the second kind which are independent of rational functions. Picard has found a formula for this number, working through the connexion of single and double integrals to the linear differential equation whose group is fundamental in his theory of the connectivity of the Riemann surface. Such a formula is evidently an invariant of birational transformation. Now it is particularly interesting to see that he is thus led to consider an expression, involving the invariants previously considered in this sketch, which had already offered itself in the early algebraic investigations of Zeuthen and Noether. This is surely very remarkable, and offers a fresh encouragement to develop the theory of the connectivity of the four-dimensioned Riemann surface in further detail.

With this cursory reference to a large and fascinating theory, I will

conclude these remarks. They will have served a purpose if they should help towards a wider study of the subject.*

* The address is limited to developments of which I have myself been able to study the proofs given by their several authors. Picard's investigations are collected in the volumes entitled *Théorie des Fonctions Algébriques de deux variables indépendantes*, t. I, 1897 ; t. II, 1900 and 1906. On pp. 485–522 of the second volume is an exhaustive account of the Italian work to 1906, by MM. Castelnuovo et Enriques, under the title *Sur quelques résultats nouveaux dans la théorie des surfaces algébriques*. An earlier paper by the same authors, with a similar title, is on pp. 241–316 of the *Math. Ann.*, Bd. XLVIII, 1897. These will furnish a guide to the demonstrations of the results, which cannot, however, in many cases, be at present developed in the order, or with the simplicity, with which they can be explained.

ON THE FOURIER SERIES OF BOUNDED FUNCTIONS

By W. H. YOUNG, Sc.D., F.R.S.

[Received June 10th, 1912.—Read June 13th, 1912.]

1. Considerable progress has been made in the theory of Fourier series of bounded functions. It is only bounded functions whose Fourier series necessarily converge when summed in the Cesàro manner, index unity, and it is only the Fourier series of bounded functions of which we can assert that they can be integrated term-by-term in the same Cesàro manner after previous multiplication by any summable function. The particular class of bounded functions known as functions of bounded variation has still simpler properties in this connexion, while properties peculiar to the Fourier series of other special types of bounded functions, such as continuous, or simply discontinuous, functions, not necessarily of bounded variation, have not been, as far as I know, noted.

The theory is, none the less, much less complete than might be expected from a comparison with the known results in the case of the larger class of functions whose squares are summable. And in the present paper I propose to give certain theorems with regard to the effect on the Fourier series of the three classes of bounded functions above referred to, and on their allied series, of the introduction of what have in another connexion been called convergence factors, those more especially kept in view belonging to the logarithmic scale. The results may also be stated in terms of the coefficients of the series. When this has been done they take, in the main, the following form :—

I. If a_n and b_n are respectively the typical Fourier cosine and sine constants of the bounded function considered,

(i) The series whose general term is $a_n q_n$ converges, if q_n is such that $q_n \log n$ has the unique limit zero, while the succession $q_1, q_2, \ldots$ is any member of a large class of monotone sequences; in particular the series $\Sigma a_n q_n$ converges if $q_n = 1 \Big/ \prod_{r=1}^{r=m} l_r^{\rho_r} n$, where $\rho_1 \geqslant 1$, and $\rho_r \geqslant 0$, and $\Sigma \rho_r > 1$, the notation being that used in Hardy's *Orders of Infinity*, p. 17.*

(ii) The series whose general term is $b_n q_n$ converges, if $\Sigma q_n/n$ converges, while the succession $q_1, q_2, \ldots$ is any monotone descending sequence with zero as limit.

In particular the series $\Sigma b_n q_n$ converges, if $q_n = 1 \Big/ \prod_{r=1}^{r=m} l_r^{\rho_r} n$, where $\rho_r \geqslant 1$, and $\sum_{r=1}^{r=m} \rho_r > m$.

II. If a_n is the typical Fourier cosine constant of a function which is continuous, or whose discontinuities, if any, are of the first kind, then $\Sigma a_n/\log n$ converges.

III. If a_n and b_n are respectively the typical Fourier cosine and sine constants of a function of bounded variation, then

(i) $na_n q_n$ and $nb_n q_n$ are respectively the Fourier sine and cosine constants of a summable function, if the succession $q_1, q_2 \ldots$, belongs to a large class of monotone descending sequences with zero as limit, and whose decrements also form a monotone descending sequence;

(ii) $nb_n q_n$ and $na_n q_n$ are respectively the Fourier sine and cosine constants of a summable function, if the succession $q_1, q_2, \ldots$ belongs to a large class of monotone descending sequences with zero as limit and possesses the property that $\Sigma q_n/n$ converges.

From the last of these three general theorems a number of necessary conditions, which must be satisfied by the coefficients of the Fourier series of a function of bounded variation, may be deduced. Thus we see that the series $\Sigma n q_n a_n b_n$ must converge, if $q_n = 1 \Big/ \prod_{r=1}^{r=m} l_r^{\rho_r} n$, where $\rho_r \geqslant 0$, and $\Sigma \rho_r > 0$. Further, if the function associated with the allied series of the

---

* See below, § 7.

Fourier series of a function of bounded variation is bounded, the series $\Sigma n q_n (a_n^2 + b_n^2)$ must converge.

If we express our results in terms of the Fourier series and its allied series, instead of making statements about the coefficients, we are struck by finding how extremely narrowly the Fourier series of a bounded function, still more that of a continuous function, escapes convergence, and, what is more surprising, that the same is true of the allied series. Narrow, however, as this escape is, the derived series of a Fourier series of a function of bounded variation, and of its allied series, escape still more narrowly from being Fourier series.

On the other hand, from the point of view of the general theory of the coefficients of Fourier series, the results obtained seem of considerable importance. Taken in conjunction with other results of a similar nature, they show how, with the almost obvious theorem of Parseval—obvious, at least, in its formal statement—there have come to be associated a whole series of much less obvious relations satisfied by the Fourier series of an extensive and graduated group of classes of functions.

It should be noted, however, that the results above stated do not, like the Theorem of Parseval, postulate a, so to speak, homogeneous character of the function in the interval of periodicity. Though, for convenience of statement, we have spoken of the function as being bounded, or continuous, or of bounded variation, all that is required for the truth of our theorems is that, in the neighbourhood of the origin, or other point considered, the function should have the character in question.

As regards the method employed, it will be seen that it consists in the use of a particular family of Fourier series as an instrument of research. Such properties of this family of series as are required for the purpose in hand are obtained *ad hoc*. The results then follow, either by the application of a general theory, which I have employed on more than one occasion elsewhere, to a pair of functions, one belonging to the particular family and the other being of the most general type consistent with its belonging to the class complementary † to that to which the particular member of the family employed belongs or by the use of other general processes of a similar character.

In all cases we are enabled to obtain not only a proof of the convergence of the series considered; our method furnishes us at the same time with an expression for the sum of each such series, at least as soon as the particular series of the auxiliary family of Fourier series concerned

---

† For the notion of complementary classes of functions here employed, see my paper, recently presented to the Royal Society, on "Summable Functions and their Fourier Series."

has been itself summed. I have not, however, thought it necessary to give on the present occasion more than a single example of the formulæ so obtained. It is one which has a special interest of its own in consequence of its comparative simplicity, as well as for the peculiar place it occupies among these formulæ. It gives us the sum of the series whose general term is $a_n/\log n$, where a_n is the typical Fourier cosine constant of any continuous or simply discontinuous function in terms of the sum-function of the auxiliary Fourier series $\sum_{n=2} \cos nx/\log n$. Although its allied series has a non-integrable sum, having at the origin an infinity of the order $1/x \log x$, this Fourier series has, as I have shown, a sum which has at the origin an infinity of the order $1/x(\log x)^2$ at most, though its order of infinity is greater than that of $1/x(l_1 x)^2 l_2 x \ldots (l_n x)^{1+e}$ for any value of e. In a note at the end of the paper I have actually calculated the expression for the sum-function.

Both in the note and in the paper I have avoided the use of the complex variable, though in one instance a little space might have been saved by referring a subsidiary result I required to a known theorem in analytical continuation.

2. In the proof of the following theorems we shall require the result that the partial summations of the series $\sum_{n=1} q_{2n-1} \sin(2n-1)x$, where $q_1, q_2, \ldots$ form a monotone descending sequence with zero as limit, are, for positive values of x, all positive, while those of the series $\sum_{n=1} q_n \sin nx$ are bounded below, being, in fact, certainly $\geqslant -\frac{1}{2}q_1$. These facts follow at once from the two equations

$$2s_m(x)\sin x = -q_2 t(x) + q_{m-1} t(mx) + q_m t\{(m+1)x\} + \sum_{n=2}^{m-1} (q_{n-1} - q_{n+1}) t(nx),$$

$$2s_m(x)\sin \tfrac{1}{2}x = -q_1 t(\tfrac{1}{2}x) + q_m t\{(m+\tfrac{1}{2})x\} + \sum_{n=2}^{m} (q_{n-1} - q_n) t\{(n-\tfrac{1}{2})x\},$$

where $s_m(x)$ denotes the m-th partial summation of the series $\sum_{n=1} q_n \sin nx$, and

$$t(nx) = 1 - \cos nx.$$

3. We shall now proceed to examine the summability of the sum-function of certain trigonometrical series (§§ 3–7).

THEOREM.—*If* $q_0 \geqslant q_1 \geqslant \ldots$ *form a monotone descending sequence* *

* This monotony, here and elsewhere, follows, of course, from the remaining conditions.

with zero as limit, and their decrements also form a monotone descending sequence, viz.,

$$q_0 - q_1 \geqslant q_1 - q_2 \geqslant \ldots,$$

then the trigonometrical series

$$\tfrac{1}{2} q_0 + \sum_{n=1} q_n \cos nx \qquad (1)$$

is the Fourier series of a positive summable function.

In any interval (e, X) not containing the origin, the series (1) converges uniformly, and may therefore be integrated term-by-term. Thus, denoting the sum-function by $f(x)$, $\int_e^X f(x)\,dx$ is the sum of the integrated series

$$\tfrac{1}{2} q_0 (X - e) + \sum_{n=1} n^{-1} q_n (\sin nX - \sin ne).$$

But, since $\sum_{n=1} n^{-1} \sin nx$ oscillates boundedly,* the series $\sum_{n=1} n^{-1} q_n \sin nx$ converges uniformly throughout the closed interval $(-\pi, \pi)$, and therefore converges to a continuous function. Hence, letting $e \to 0$, we see that $\int_0^X f(x)\,dx$ exists, at least as a non-absolutely convergent integral, and is the sum of the integrated series for $e = 0$.

Now $2 \sin \tfrac{1}{2}x \left(\tfrac{1}{2} q_0 + \sum_{r=1}^{n} q_r \cos rx \right)$

$$= (q_0 - q_1) \sin \tfrac{1}{2}x + (q_1 - q_2) \sin \tfrac{3}{2}x + \ldots$$

$$+ (q_{n-1} - q_n) \sin \frac{2n-1}{2} x + q_n \sin \frac{2n+1}{2} x \geqslant - q_n,$$

using the result above referred to. Since, by hypothesis, q_n has the unique limit zero, when $n \to \infty$, this shows that the sum-function $f(x)$ is a positive function.

Hence the integral, already proved to exist, is absolutely convergent, that is, $f(x)$ is a summable function.

4. *If we omit the condition as to the monotony of the decrements, all that the method of proof employed shows is that the trigonometrical series* (1) *is a generalised Fourier series,* the integrals which occur as ex-

---

* For a general theorem, including this well-known result as a special case, see my paper on "The Integration of Fourier Series," § 3, 1911, *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 452.

pressions for the coefficients being at least non-absolutely convergent integrals. Indeed, it follows from the proof that

$$q_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx,$$

where $f(x)$ is no longer known to be a positive function, but is continuous except, perhaps, at the origin. Similarly,

$$q_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx,$$

since $$2\cos nx\left(\tfrac{1}{2}q_0+\sum_{r=1}^{m} q_r\cos rx\right)$$

$$= q_0\cos nx+\sum_{r=1}^{m} q_r\{\cos(r+n)x+\cos(r-n)x\}$$

is a series of the same kind as that with which we started, so that the expression of q_n as an integral follows as before by term-by-term integration. Or we may see the same thing by the use of the results of my paper "On the Condition that a Trigonometric Series should have the Fourier Form."*

5. We have now to consider a theorem for a trigonometric sine series which corresponds to some extent to that proved in § 3 for the cosine series.

THEOREM.—*If* $q_1 \geqslant q_2 \geqslant \ldots$ *form a monotone descending sequence of constants with zero as limit and* $\sum_{n=1} n^{-1}q_n$ *converges, then*

$$\sum_{n=1} q_n \sin nx$$

is the Fourier series of a summable function which, for positive values of x, *is bounded below, and, for negative values of* x, *is bounded above.*

For we have seen (§ 2) that the partial summation S_n of the trigonometrical series in question is, for all values of n and all values of $x \geqslant 0$, greater than, or equal to, $-\frac{1}{2}q_1$. Thus, the sum-function $f(x)$ is certainly bounded below for positive values of x.

Again, since $\sum_{n=1} n^{-1}q_n$ converges, the integrated series converges uniformly. Therefore, as in § 3, the theorem now follows.

---

* *Proc. London Math. Soc.*, Ser. 2, Vol. 9, pp. 425–427.

6. *If the series whose general term is $n^{-1}q_n$ diverges,* the integrated series does not converge at the origin at all; nevertheless, *the series $\sum_{n=1} q_n \sin nx$ is still, in a certain sense, a Fourier series, the coefficients being expressible in the usual form of the Fourier sine constants as the integrals of continuous functions, although the function $f(x)$ which figures in them is not summable, nor even integrable in the Harnack-Lebesgue manner. Moreover, this is true whether the limit of the monotone sequence $q_1, q_2, \ldots$, is zero or any other finite quantity.* To prove this we may proceed as follows:—

If all the q_n's are equal to unity, the Fourier function $f(x)$ is $\frac{1}{2}\cot\frac{1}{2}x$. Thus, if all the q_n's are equal, the theorem is true. In the general case the q_n's will have a definite limit, which is by hypothesis, finite, say q. Subtracting q from each q_n we get a sequence of constants descending monotonely to its limit zero. If, therefore, the theorem is true for the latter sequence, and the Fourier function is $f(x)$, the theorem will be true for the original sequence, and the Fourier function will be $f(x)+\frac{1}{2}q\cot\frac{1}{2}x$. Thus, we have only to prove the theorem when the limit of the sequence is zero.

Now

$$q_1 2\sin^2 x + q_2 2\sin x \sin 2x + \ldots + q_n 2\sin x \sin nx$$
$$= q_1(1-\cos 2x) + q_2(\cos x - \cos 3x) + q_3(\cos 2x - \cos 4x) + \ldots$$
$$+ q_n[\cos(n-1)x - \cos(n+1)x]$$
$$= q_1 + q_2\cos x - q_{n-1}\cos nx - q_n\cos(n+1)x + \sum_{r=2}^{n-1}(q_{r+1}-q_{r-1})\cos rx.$$

Hence, by Weierstrass's M-test, the series $\sum_{n=1} 2q_n \sin x \sin nx$ converges uniformly to $q_1 + q_2\cos x + \sum_{r=2}(q_{r+1}-q_{r-1})\cos rx$, that is, to a continuous function, say $2g(x)$.

Thus, except at the origin, the series $\sum_{n=1} q_r \sin rx$ converges to $g(x)\operatorname{cosec} x$, or, say, $f(x)$.

Integrating term by term the uniformly convergent series above mentioned we get

$$\frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin x\,dx = q_1.$$

Similarly, multiplying the series by $\sin px$, instead of $\sin x$, we get

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin px\,dx = q\ ,$$

which proves the statement made at the beginning of this article.

7. It need hardly be remarked that the main results* of the preceding articles still hold if the trigonometrical series considered are deficient in a certain number of terms at the beginning, or if a finite number of the coefficients do not verify the conditions imposed on the succession $q_0, q_1, q_2, \ldots$.

In particular this is the case if

$$q_n = 1/\log n,$$

or

$$q_n = 1/\log\log n,$$

or, with the notation used in Hardy's *Orders of Infinity*, p. 17.†

$$q_n = 1/l_r n,$$

or, still more generally,

$$q_n = 1\Big/\prod_{r=1}^{r=m} l_r^{\rho} n,$$

where ρ is the index of a power depending on the integer r.‡

8. We have now to show that, under certain circumstances specified, the trigonometrical series of the type we are considering converge to functions which are integrals in every interval not containing the origin.

THEOREM.—*The trigonometrical series*

$$\tfrac{1}{2}q_0 + \sum_{n=1} q_n \cos nx, \tag{1}$$

is the Fourier series of a function which is an integral in any interval internal to the completely open interval $(0, \pi)$, *provided* q_n *has the unique limit zero as n increases indefinitely, and the series*

$$\sum_{n=1} Q_n \cos nx, \tag{2}$$

where

$$Q_n = n(q_{n-1} - q_{n+1}), \tag{3}$$

converges uniformly in the interval, and the series (1) *is known to converge at some one internal point of this interval.*

In fact, denoting the sum of the first $(m+1)$ terms of (1) by $s_m(x)$,

$$s_m(x)\, 2\sin x = q_{m-1}\sin mx + q_m \sin(m+1)x + \sum_{n=1}^{m-1} (q_{n-1} - q_{n+1}) \sin nx.$$

---

* The sum function in the theorem of § 3 is no longer necessarily positive.

† $l_r x = \log l_{r-1} x$, $l_{r-1} x = \log l_{r-2} x$, ..., $l_1 x = \log x$.

‡ See below, § 19.

Therefore

$$\frac{d}{dx}\left[s_m(x)\,2\sin x\right] = mq_{m-1}\cos mx+(m+1)\,q_m\cos(m+1)x+\sum_{n=1}^{m-1}Q_n\cos nx.$$

Since therefore the series (2) converges, it is equal to

$$\operatorname*{Lt}_{m\to\infty}\left(\frac{d}{dx}\left[s_m(x)\,2\sin x\right]-mq_{m-1}\cos mx-(m+1)\,q_m\cos(m+1)\,x\right),$$

and since the convergence is uniform, denoting the sum of the series (2) by $g(x)$,

$$\int_A^x g(x)\,dx = \operatorname*{Lt}_{m\to\infty}\{s_m(x)\,2\sin x-s_m(A)\,2\sin A-q_{m-1}(\sin mx-\sin mA) -q_m[\sin(m+1)x-\sin(m+1)A]\}.$$

Hence, assuming the point A to be the point at which the series (1) is known to converge, and writing C for the sum, we see, since q_n has the unique limit zero, that the series (1) converges, exception being made of the origin, to the function

$$\tfrac{1}{2}\operatorname{cosec}x\left(\int_A^x g(x)\,dx+C\right),$$

which, being the product of two integrals, is itself an integral.

This proves the theorem.

COR.—*The trigonometrical series*

$$\tfrac{1}{2}q_0+\sum_{n=1}q_n\cos nx \qquad (1)$$

converges and is the Fourier series of a function which is an integral in every interval internal to the completely open interval $(0, \pi)$, *provided the quantities* q_n, $(q_{n-1}-q_n)$, *and* $n(q_{n-1}-q_{n+1})$ *trace out monotone descending sequences with zero as limit.*

For in this case the convergence of the series (1) is ensured, and the auxiliary series converges uniformly.

9. THEOREM.—*The trigonometrical series*

$$\sum_{n=1}q_n\sin nx \qquad (1)$$

converges to a function which is an integral in any interval inside the completely open interval $(0, \pi)$, *provided* q_n *has the unique limit zero when n increases indefinitely, and the series*

$$\sum_{n=1}Q_n\sin nx, \qquad (2)$$

where, as in § 8, $Q_n = n(q_{n-1} - q_{n+1})$,

converges uniformly in the interval, and the series (1) *is known to converge at some one point internal to this interval.*

For, differentiating the second of the two identities given in § 2,

$$\frac{d}{dx}[s_m(x)\, 2\sin x] = -q_2 \sin x + q_{m-1}\, m \sin mx + q_m(m+1)\sin(m+1)x + \sum_{n=2}^{m-1} Q_n \sin nx.$$

Therefore, since the series (2) converges, it is equal to

$$\operatorname*{Lt}_{m\to\infty}\left(\frac{d}{dx}[s_m(x)\, 2\sin x] + q_2 \sin x - q_{m-1}\, m \sin mx - q_m(m+1)\sin(m+1)\, x\right) = g(x), \text{ say.}$$

Moreover, since the convergence is uniform,

$$\int_A^x g(x)\,dx = \operatorname*{Lt}_{m\to\infty}\{s_m(x)\, 2\sin x - s_m(A)\, 2\sin A - q_2(\cos x - \cos A) + q_{m-1}(\cos mx - \cos mA) + q_m[\cos(m+1)x - \cos(m+1)A]\}.$$

Hence, assuming the point A to be the point at which the series (1) is known to converge, and writing C for its sum, we see, since q_n has the unique limit zero, that the series (1) converges, exception being made of the origin and the point π, to the function

$$\tfrac{1}{2}\operatorname{cosec} x\left(\int_A^x g(x)\,dx + C + q_2(\cos x - \cos A)\right),$$

which, being the product of two integrals, is itself an integral in the interval considered, which does not contain the origin or the point π. This proves the theorem.

COR.—*The trigonometrical series*

$$\sum_{n=1} q_n \sin nx \qquad (1)$$

converges to a function which is an integral in every interval not containing the origin or the point π, *provided* q_n *and* $n(q_{n-1} - q_{n+1})$ *form monotone descending sequences with zero as limit.*

For in this case the convergence of the series (1) is ensured, and the auxiliary series converges uniformly.

10. One difference in the nature of our results with regard to the family of sine-series and cosine series considered will have been remarked. If we are given a monotone descending sequence of constants with zero as limit

$$q_0, q_1, \ldots, q_n, \ldots,$$

the necessary and sufficient condition that the sine series should be a Fourier series is that

$$\Sigma n^{-1} q_n$$

should converge, while this condition may even be omitted if we regard a trigonometrical series as being a Fourier series when its coefficients are capable of being expressed in the Fourier form, equally whether the function which appears in the expressions for the coefficients is itself integrable in some sense or in no sense.

In the case of the cosine-series, on the other hand, we have merely obtained a sufficient condition, namely, that the decrements themselves form a monotone descending sequence. It is not clear, however, that such an additional condition is required, except by the exigencies of the method of proof adopted. All that our reasoning enables us to state is that, if we omit this additional condition, the series is certainly a Fourier series in the generalized sense in which non-absolutely convergent integration is allowable, we are not able to assert that the sum-function of the series is summable.

If it should be true that a trigonometrical series of the form $\Sigma q_n \cos nx$ need not be a Fourier series when the succession q_n is monotone descending to zero, it would follow that the series $\Sigma n^{-1} \sin nx$, when multiplied by a monotone descending sequence of constants, might cease to have a function of bounded variation for its sum, though this sum becomes continuous.

To see this we remark that $\Sigma n^{-1} q_n \sin nx$ cannot have a function of bounded variation for its sum, unless that sum be an absolutely convergent indefinite integral. It is, in fact, the integrated series of the series $\Sigma q_n \cos nx$, and therefore is certainly, as it represents a continuous function, at least a non-absolutely convergent integral. Hence, since it is a function of bounded variation, and therefore its integrand is summable, it is an absolutely convergent integral.

In this connection another remark may be made. It also follows from our results that, if the coefficients of a Fourier series and of its allied series are multiplied by a monotone descending sequence of constants having zero as limit, the Fourier series will remain a Fourier series, and the allied series will become a Fourier series, always provided that, in the case of the Fourier series, these constants satisfy the condition for the cosine-

series above referred to, and, in the case of the allied series, that for the sine series. Here again the result appears incomplete—at least, in the case of the Fourier series.

If it should be true that, when our process is performed, the Fourier series ceases to be a Fourier series, it would, of course, follow that the Fourier series of an integral might cease to be the Fourier series of an integral, when this same process is performed.

I shall not discuss this question here, as the object of the paper is rather to show how the more obvious properties of our particular family of Fourier series may be applied to the theory of the coefficients of the general Fourier series, or rather of the Fourier series of certain general types of functions.

11. We now pass to the application of the preceding results to the theory of the Fourier constants of functions of bounded variation.

THEOREM.—*The result of multiplying the Fourier cosine and sine constants* a_n *and* b_n *of a function of bounded variation by* q_n, *for all values of* n, *is to change them into the Fourier cosine and sine constants of a Lebesgue integral, provided the quantities* $q_n, (q_{n-1}-q_n)$ *and* $n(q_{n-1}-q_{n+1})$ *have zero as limit and form monotone descending sequences.*

For in this case, by what has been already proved, if we write

$$g(x) = \tfrac{1}{2}q_0 + \sum_{n=1} q_n \cos nx,$$

$g(x)$ has the following properties :—

(1) $g(x)$ is a positive summable function.

(2) $g(x)$ is an integral in every interval not containing the origin.

But writing $$f(x) \sim \tfrac{1}{2}a_0 + \sum_{n=1} a_n \cos nx,$$

we have $$\tfrac{1}{2}[f(x+t)+f(x-t)] \sim \tfrac{1}{2}a_0 + \sum_{n=1} (a_n \cos nx + b_n \sin nx)\cos nt.$$

Assuming $f(x)$ to have bounded variation, we may multiply by the summable function $g(t)$ and integrate from $-\pi$ to π. Dividing by π, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\,[f(x+t)+f(x-t)]\,dt = \tfrac{1}{2}a_0 q_0 + \sum_{n=1} q_n (a_n \cos nx + b_n \sin nx).$$

Thus the proof of our theorem depends solely on proving that the function of x on the left is a Lebesgue integral.

Now this has been already done* when the function $g(t)$ is t^{q-1}, and the only properties of the function used in the proof are precisely the properties (1) and (2) above referred to; thus the proof applies in our case and serves to complete the proof of the present theorem.

COR.—*The Fourier series of a function of bounded variation becomes the Fourier series of an integral, if the coefficients a_n and b_n are multiplied by q_n for all values of n, where the quantities q_n satisfy the conditions of the above theorem.*

12. As the corollary of § 11 shews, the theorem of that article really relates to the change of the Fourier series of a function of bounded variation into that of an integral. The next theorem virtually deals with the allied series of such a Fourier series.

THEOREM.—*The result of multiplying the Fourier cosine and sine constants a_n and b_n of a function of bounded variation by q_n, for all values of n, is to change them into the sine and cosine constants of a Lebesgue integral, provided*

(i) $\sum_{n=1} n^{-1} q_n$ *converges;*

(ii) *the quantities q_n and $n(q_{n-1}-q_{n+1})$ form monotone descending sequences with zero as limit.*

Indeed, the conditions of § 5 and § 9 are fulfilled; therefore if we write

$$g(x) = \sum_{n=1} q_n \sin nx,$$

$g(x)+q_1$ is positive for positive values of x, and $g(x)-q_1$ is negative for negative values of x, and $g(x) \pm q_1$ is a summable function which is a Lebesgue integral in every interval not containing the origin or the point π or $-\pi$. Hence, as in the preceding article,

$$\int_0^{\pi} [g(t)+q_1][f(x+t)-f(x-t)]\,dt$$

and

$$\int_{-\pi}^{0} [g(t)-q_1][f(x+t)-f(x-t)]\,dt$$

are both functions of x which are Lebesgue integrals, and therefore the

* W. H. Young, "On a Mode of Generating Fourier Series," § 7, 1911, *Proc. Roy. Soc.*, Ser. A, Vol. 85, p. 424. The fact that the upper limit of integration is unity instead of π makes no difference.

same is true of

$$\int_{-\pi}^{\pi} g(t)[f(x+t)-f(x-t)]\,dt.$$

Hence, multiplying the Fourier series of the function $[f(x+t)-f(x-t)]$ of bounded variation by the summable function $g(t)$, and integrating term-by-term from $-\pi$ to π, we see that the series

$$\sum_{n=1} q_n(-a_n \sin nx+b_n \cos nx)$$

converges to a function of x which is a Lebesgue integral. This proves the theorem, since we have only to change the sign of x in order to get $q_n a_n$ in place of $-q_n a_n$.

COR.—*The allied series of the Fourier series of a function of bounded variation becomes the Fourier series of a Lebesgue integral if the coefficients a_n and b_n are multiplied by q_n for all values of n, where the quantities q_n satisfy the conditions of the above theorem.*

13. We now require to consider certain additional properties of our auxiliary family of Fourier series which hold good when we postulate suitable supplementary conditions.

THEOREM. — *If $q_0, q_1, \ldots$ form a monotone descending sequence, whose decrements form a monotone descending sequence, and such that $q_n \log n$ has the unique limit zero, when $n \to \infty$, and s_n denote the n-th partial summation of the series*

$$\tfrac{1}{2}q_0+\sum_{n=1} q_n \cos nx,$$

then

$$\operatorname*{Lt}_{E\to 0,\ n\to\infty} \int_E s_n(x)\,dx = 0.$$

For, as in § 2,

$$2 \sin \tfrac{1}{2}x \,.\, s_n(x)$$

$$= (q_0-q_1)\sin \tfrac{1}{2}x+\ldots+(q_{n-1}-q_n)\sin\frac{2n-1}{2}x+q_n \sin\frac{2n+1}{2}x,$$

where the partial summation on the right, omitting the final term, is positive. Hence we may write

$$s_n(x) = T_n(x)+q_n \frac{\sin(n+\frac{1}{2})x}{\sin \frac{1}{2}x}, \tag{1}$$

where $T_n(x)$ is positive for $0 < x < 2\pi$, and, like $s_n(x)$, converges for these values of x to $f(x)$.

Now, by a theorem of Lebesgue's,* the integral of $\frac{|\sin(n+\frac{1}{2})x|}{\sin\frac{1}{2}x}$ is of the order of $\log n$, integration being over the whole interval, and therefore the integral of the last term on the right of (1) vanishes in the limit, by the hypothesis made, integration being over the whole interval, or over any set of points in that interval.

Also since, by the theorem of § 3, the series of which $s_n(x)$ is the partial summation is, under the hypothesis made, a Fourier series, the integral of $s_n(x)$ over any interval converges to the integral of $f(x)$

Hence, by (1), the integral of $T_n(x)$ over any interval converges to the integral of $f(x)$, whence it follows by a theorem of Vitali's,† since $T_n(x)$ is positive and converges to $f(x)$, that the succession $T_n(x)$ is absolutely integrable, so that

$$\operatorname*{Lt}_{E\to 0,\, n\to\infty} \int_E T_n(x)\,dx = 0. \tag{2}$$

Hence again, by (1), the required result at once follows.

From the argument of the preceding proof we can at once deduce the following corollary to the theorem :—

COR.—*If $q_n \log n$, without approaching zero, remains bounded, the remaining conditions of the preceding theorem being satisfied, then*

$$\int_a^x s_n(x)\,dx$$

is a bounded function of (n, x).

For, in consequence of (2), the integral of $T_n(x)$ is a bounded function of (n, x). Hence, by (1), the result follows, since the integral of the modulus of the second term on the right of (1) is, in consequence of the hypothesis now made, less than $B(b-a)$, where (a, b) is the interval considered.

---

* H. Lebesgue, "Sur la représentation trigonométrique approchée des fonctions satisfaisant à une condition de Lipschitz," 1910, *Bull. de la Soc. Math. de France*, Vol. XXXVIII, pp. 116, 117; see also Dunham Jackson, *Ueber die Genauigkeit der Annäherung stetiger Funktionen durch ganze rationale Funktionen gegebenen Grades und trigonometrische Summen gegebener Ordnung*, 1911, pp. 50, 51.

† G. Vitali, "Sull' Integrazione per Serie," 1907, *Rend. di Palermo*, Vol. XXIII, p. 149. See also W. H. Young, "On Semi-Integrals and Oscillating Successions of Functions," §§ 26 and 24, 1910, *Proc. London Math. Soc.*, Ser. 2, Vol. 9, pp. 311 and 309, and "Successions of Integrals and Fourier Series," § 9, 1912, *ibid.*, Vol. 11, p. 60, where the theorem is extended.

14. The corresponding theorem for sine series is the following. There is no corresponding corollary :—

THEOREM.—*If* $q_1, q_2, \ldots$, *form a monotone descending sequence with zero as limit and such that* $\Sigma n^{-1} q_n$ *converges, and* s_n *denote the n-th partial summation of the series*

$$\sum_{n=1} q_n \sin nx,$$

then

$$\operatorname*{Lt}_{E \to 0,\ n \to \infty} \int_E s_n(x)\,dx = 0.$$

For, by § 5, $s_n(x)$ is bounded below for positive values of x, and bounded above for negative values of x. Also $s_n(x)$ converges to $f(x)$, and, since it is the partial summation of a Fourier series, its integral over any interval converges to that of $f(x)$. Hence, by the theorem of Vitali, quoted in the preceding article, the required result follows.

15. We have now to apply the results of §§ 13 and 14 to bounded functions, and in particular to simply discontinuous functions.

THEOREM.—*If* $f(x)$ *is any bounded function whose typical Fourier cosine constant is* a_n, *the series*

$$\tfrac{1}{2} a_0 q_0 + \sum_{n=1} q_n a_n \tag{1}$$

converges to

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, g(x)\,dx,$$

where $g(x)$ *is the sum-function of the Fourier series*

$$\tfrac{1}{2} q_0 + \sum_{n=1} q_n \cos nx, \tag{2}$$

provided $q_0, q_1, \ldots$, *form a monotone descending sequence, whose decrements form a monotone descending sequence, and* $q_n \log n$ *has the unique limit zero, when* $n \to \infty$.

Moreover, the series (2), *after multiplying each term by* $f(x)$, *may be integrated over any interval or set of points.*

It is evidently only necessary to prove the final statement, from which the rest follows, choosing for the interval of integration the whole interval $(-\pi, \pi)$.

Now by § 3, under the given circumstances, the series (2) is a convergent Fourier series, and therefore capable of term-by-term integration over any interval. Also, by § 13, the partial summation $s_n(x)$ of the

series (2) satisfies the condition

$$\operatorname*{Lt}_{E\to 0,\ n\to\infty} \int_E s_n(x)\,dx = 0.$$

Therefore also, $f(x)$ being a bounded function,

$$\operatorname*{Lt}_{E\to 0,\ n\to\infty} \int_E f(x)\,s_n(x)\,dx = 0.$$

Hence, by a theorem of Vitali's,* since $f(x)s_n(x)$ converges to $f(x)g(x)$, the final statement of the enunciation is true. This proves the theorem.

Similarly, using the corollary of § 13 and Theorem 5 of § 8 of my paper on "The Application of Expansions to Definite Integrals,"† we have the following theorem :—

THEOREM.—*If $q_n \log n$, without approaching zero, remains bounded, the remaining conditions of the preceding theorem being satisfied, and $g(x)$ being, not merely bounded, but continuous, or possessing only discontinuities of the first kind, then $\Sigma q_n a_n$ converges to the same sum as before.*

16. In a precisely similar manner, using §§ 5 and 14 instead of §§ 3 and 13, we can prove the following theorem :—

THEOREM.—*If $f(x)$ is any bounded function, whose typical Fourier sine constant is b_n, the series*

$$\sum_{n=1} q_n b_n \tag{1}$$

converges to

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,g(x)\,dx,$$

where $g(x)$ is the sum-function of the Fourier series

$$\sum_{n=1} q_n \sin nx, \tag{2}$$

provided $q_1, q_2, \ldots$, form a monotone descending sequence with zero as limit and such that $\Sigma n^{-1} q_n$ converges.

Moreover, the series (2), after multiplying each term by $f(x)$, may be integrated over any interval or set of points.

17. In practice it will frequently happen that q_n is the value, when $x = n$, of a function $f(x)$ which is capable of differentiation. When this

* *Loc. cit.*, p. 147. "Semi-Integrals," p. 308.

† *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 472.

is the case it will usually be more convenient to make use, instead of the theorems of §§ 8 and 9, of a theorem due to Lerch.* For the convenience of the reader a proof of this theorem is appended. It may be noticed that the enunciation has been in more than one particular altered from its original form.

THEOREM.—*If the series*

$$\sum_{n=0} \{[(n-1)q_{n-1}-(n+1)q_{n+1}]\cos nx+[(n-1)p_{n-1}-(n+1)p_{n+1}]\sin nx\}, \tag{1}$$

where $$q_{-1}=p_{-1}=q_0=p_0=0,$$

converge uniformly in an interval (A, B), *not containing the origin or the end-points of the interval* $(-\pi, \pi)$, *to a function* $g(x)$, *then the series*

$$\sum_{n=1} \{q_n(\cos nx-\cos nA)+p_n(\sin nx-\sin nA)\}$$

converges at every point of the interval (A, B) *to a function* $f(x)$, *whose differential coefficient is given by the equation*

$$g(x) = 2f'(x)\sin x.$$

Since the series (1) converges throughout an interval, its coefficients must tend towards zero as n increases indefinitely. Hence we may write, for example,

$$|nq_n-(n+2m)q_{n+2m}| < me,$$

where e is as small as we please and n depends on e only. Hence also

$$|q_{n+2m}| < \left|\frac{me+|nq_n|}{n+2m}\right|;$$

and therefore has limits which, as n increases, are all not greater than e. As e is at our choice, it follows that q_n has the unique limit zero.

Similarly p_n has the unique limit zero.

---

* Lerch, *Annales sc. de l'école normale*, Sér. 3, Vol. XI, p. 351. See also Bôcher's "Introduction to the Theory of Fourier Series," *Annals of Math.*, Vol. VII, p. 120.

A modified statement of this theorem, without proof, is given in Hobson's *Theory of Functions of a Real Variable*, p. 728. It contains the redundant condition that p_n and q_n should have the unique limit zero. It also expressly postulates the convergence of the series under discussion at some point internal to the interval of uniform convergence of the auxiliary series.

We next remark that

$$\frac{\cos nx - \cos(n+2)x}{2\sin x} = \sin(n+1)x$$

and

$$\frac{\sin(n+2)x - \sin nx}{2\sin x} = \cos(n+1)x.$$

Hence, rearranging the first m terms of (1), we get

$$g(x) = \operatorname*{Lt}_{m\to\infty} \Big\{ \sum_{n=1}^{m} [(np_n \cos nx - nq_n \sin nx)\, 2\sin x] - (m+1)\, q_{m+1} \cos mx$$
$$-(m+2)\, q_{m+2} \cos(m+1)x - (m+1)p_{m+1} \sin mx$$
$$-(m+2)\, p_{m+2} \sin(m+1)x \Big\}.$$

Therefore, since the convergence is uniform,

$$\int_A^x \frac{g(x)}{2\sin x}\, dx = \operatorname*{Lt}_{m\to\infty} \Big\{ \sum_{n=1}^{m} \big([q_n \cos nx + p_n \sin nx]_A^x\big)$$
$$-\frac{m+1}{m} q_{m+1}(\sin mx - \sin mA) - \frac{m+2}{m+1} q_{m+2}[\sin(m+1)x - \sin(m+1)A]$$
$$+\frac{m+1}{m} p_{m+1}(\cos mx - \cos mA) + \frac{m+2}{m+1} p_{m+2}[\cos(m+1)x - \cos(m+1)A] \Big\}.$$

Hence, proceeding to the limit,

$$\int_A^x \frac{g(x)}{2\sin x}\, dx = \sum_{n=1} [q_n(\cos nx - \cos nA) + p_n(\sin nx - \sin nA)]. \qquad (2)$$

Thus the series on the right-hand side converges to a continuous function $f(x)$ whose differential coefficient is given by

$$2\sin x \,.\, f'(x) = g(x).$$

COR. — *If* $\{(n+1)q_{n+1} - (n-1)q_{n-1}\}$ *and* $\{(n+1)p_{n+1} - (n-1)p_{n-1}\}$ *are the general terms of two monotone descending sequences, whose limits are both zero, and if the series*

$$\sum_{n=1} (q_n \cos nx + p_n \sin nx)$$

converges at some point other than the origin, then it converges everywhere, except possibly at the origin, to a function which is the integral of a continuous function in every interval not containing the origin.

In fact, under the conditions stated, the uniform convergence of (1) is assured in every interval not containing the origin. Moreover, if we take A to be the point at which the series under consideration converges,

by hypothesis, we may in (2) write on the right-hand side zero for $\cos nA$ and for $\sin nA$, provided only we add a suitable constant to the left-hand side. Thus, the corollary is true.

Hence we get two theorems exactly corresponding to those of §§ 11 and 12 with their corollaries:—

THEOREM.—*The result of multiplying the Fourier cosine and sine constants a_n and b_n of a function of bounded variation by q_n, for all values of n, is to change them into the Fourier cosine and sine constants of a Lebesgue integral, provided q_n, $(q_{n-1}-q_n)$ and $[(n+1)q_{n+1}-(n-1)q_{n-1}]$ have zero as limit and form monotone descending sequences.*

COR.—*The Fourier series of the function of bounded variation then becomes the Fourier series of an integral.*

THEOREM.—*The result of multiplying the Fourier cosine and sine constants a_n and b_n of a function of bounded variation by q_n, for all values of n, is to change them into the Fourier sine and cosine constants of a Lebesgue integral, provided*

(i) $\sum_{n=1} n^{-1}q_n$ *converges;*

(ii) *the quantities q_n and $[(n+1)q_{n+1}-(n-1)q_{n-1}]$ form monotone descending sequences with zero as limit.*

COR.—*The allied series of the Fourier series of the function of bounded variation then becomes the Fourier series of an integral.*

18. We now shew that if q_n is any one of the logarithmic scale of coefficients it possesses the property required in the first corollary of § 17. That this is so is evident for the rest to those acquainted with the theory of analytic functions.

Let ρ denote the index of a power depending on the integer r, and

$$f(x) = \frac{1}{\prod_{r=1}^{r=n} l_r^{\rho}},$$

using the notation of Hardy's *Orders of Infinity*, p. 17, so that

$$l_r = \log l_{r-1}$$

and

$$l_1 = \log x.$$

Then $$f'(x) = -f(x) \sum_{r=1}^{r=n} \frac{\frac{d}{dx}(l_r^\rho)}{l_r^\rho} = -f(x) \sum_{r=1}^{r=n} \frac{\rho}{x l_1 l_2 \dots l_r}. \qquad (1)$$

Therefore, denoting by N the greatest of the n indices ρ, we have for all values of x greater than N_1, where N_1 is the number which makes $l_r(N_1) = 1$,

$$xf'(x) > -f(x)\, nN/l_1.$$

Hence $$\frac{d}{dx}\{xf(x)\} > f(x)(1-nN/l_1) \quad (N_1 < x),$$

which shows that for all values of x greater than a certain number, depending only on n, $\frac{d}{dx}[xf(x)]$ is positive, and therefore $xf(x)$ is monotone increasing. Thus, for sufficiently large values of the integer m,

$$(m+1)f(m+1)-(m-1)f(m-1) > 0. \qquad (2)$$

Again, from (1),

$$x\frac{d^2}{dx^2}\{xf(x)\} = x\frac{d}{dx}\{f(x)+xf'(x)\} = xf'(x)+x\frac{d}{dx}\{xf'(x)\}$$

$$= xf'(x)-xf'(x)\sum_{r=1}^{r=n}\frac{\rho}{l_1 l_2 \dots l_r} + xf(x)\sum_{r=1}^{r=n}\frac{\rho}{l_1 l_2 \dots l_r}\sum_{t=1}^{t=r}\frac{l_t'}{l_t}$$

$$= -f(x)\sum_{r=1}^{r=n}\frac{\rho}{l_1 l_2 \dots l_r}\left\{1-\sum_{r=1}^{r=n}\frac{\rho}{l_1 l_2 \dots l_r}-\sum_{t=1}^{t=r}\frac{1}{l_1 l_2 \dots l_t}\right\}$$

$$< -f(x)\sum_{r=1}^{r=n}\frac{\rho}{l_1 l_2 \dots l_r}\{1-n(N+1)/l_1\}.$$

Thus for all values of x greater than a certain number depending only on n, $\frac{d^2}{dx^2}\{xf(x)\}$ is negative, and therefore $\frac{d}{dx}\{xf(x)\}$ for such values of x is monotone decreasing, and, by (1), tends towards zero as x increases indefinitely. Thus also the integral of $\frac{d}{dx}\{xf(x)\}$ from $x = n-1$ to $x = n+1$ is greater than its integral from $x = n$ to $x = n+2$, so that

$$(n+1)f(n+1)-(n-1)f(n-1) > (n+2)f(n+2)-nf(n) > \dots \to 0. \quad (3)$$

19. From the results of the preceding article it follows that, if

$$f(x) = 1\Big/\prod_{r=1}^{r=n} l_r^\rho,$$

where $$l_r^\rho = (\log l_{r-1})^\rho, \qquad l_1 = \log x,$$

the series whose general terms are respectively

$$\{(m+1)f(m+1)-(m-1)f(m-1)\}\begin{matrix}\cos mx\\ \sin mx\end{matrix},$$

converge uniformly in any interval not containing the origin.

Again, since $f'(x)$ is clearly negative and monotone increasing for all values of x greater than a certain quantity depending only on n, the succession $f(m), f(m+1), \ldots$, is monotone decreasing, with zero as limit, and with decrements which form a monotone descending sequence. Therefore the series whose general term is $f(m)\cos mx$ is certainly a Fourier series and converges, except at the origin, to a summable function by the result of § 3.

Hence, by the first theorem proved in § 17, *the Fourier series whose general term is* $f(m)\cos mx$ *converges uniformly to an integral throughout any interval in the completely open interval* $(0, \pi)$.

Also, by the second theorem of § 17, we see that, *if* a_n *and* b_n *are respectively the typical Fourier cosine and sine constants of a function of bounded variation, then* $f(n)\,a_n$ *and* $f(n)\,b_n$ *are respectively the typical Fourier cosine and sine constants of a Lebesgue integral.*

Similarly, for all values of the indices ρ which make $\sum_{m=2} m^{-1}f(m)$ a convergent series, it follows, by § 5 and the first theorem of § 17, that *the series whose general term is* $f(m)\sin mx$ *is a Fourier series, and converges uniformly to an integral throughout any interval in the completely open interval* $(0, \pi)$.

Also, by the third theorem of § 17, *if* a_n *and* b_n *are respectively the typical Fourier cosine and sine constants of a function of bounded variation, then* $f(n)\,b_n$ *and* $f(n)\,a_n$ *are respectively the typical Fourier cosine and sine constants of a Lebesgue integral, provided* $\sum_{n=2} n^{-1}f(n)$ *converges.*

20. From the results of the preceding article we can also deduce the particular cases of the Theorem I, signalised in the introduction. In fact, from the first theorem of § 15 and the theorem of § 16, we see that the results there enunciated hold when q_m has the form $f(m)$ of the preceding article, provided in the case of the cosine series $f(m)\log m$ has the unique limit zero, when $m \to \infty$, and, in the case of the sine series, $\Sigma m^{-1}q_m$ converges. Hence we easily obtain the following theorems :—

THEOREM.—*If* $f(x)$ *is any bounded function whose typical Fourier cosine constant is* a_n, *the series*

$$\sum_{n=2} a_n \Big/ \prod_{r=1}^{r=n} l_r^{\rho_r}$$

converges to $$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,g(x)\,dx,$$

where $g(x)$ *is the sum-function of the Fourier series*

$$\sum_{n=2} \cos nx \Big/ \prod_{r=1}^{r=n} l_r^{\rho_r},$$

provided $\rho_1 \geqslant 1$, *and* $\rho_r \geqslant 0$, *while* $\Sigma\rho_r > 1$.

THEOREM.—*If* $f(x)$ *is any bounded function, whose typical Fourier sine constant is* b_n, *the series*

$$\sum_{n=2} b_n \Big/ \prod_{r=1}^{r=n} l_r,$$

converges to $$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,g(x)\,dx,$$

where $g(x)$ *is the sum-function of the Fourier series*

$$\sum_{n=2} \sin nx \Big/ \prod_{r=1}^{r=n} l_r^{\rho},$$

provided $\rho_r \geqslant 1$, *and* $\sum_{r=1}^{r=n} \rho_r > n$.

Again, by the theorem at the end of § 15, we have the following additional theorem, also mentioned in the introduction :—

THEOREM.—*If* a_n *is the typical Fourier cosine constant of a function which is continuous, or whose discontinuities are only of the first kind, then* $\Sigma a_n/\log n$ *converges to*

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,g(x)\,dx,$$

where $g(x)$ *is the sum-function of the Fourier series* $\sum_{n=2} \cos nx/\log n$.

21. We now proceed to deduce, from § 11 and a theorem in the paper on "The Integration of Fourier Series," two other results given in the introduction. Let q_n describe one of the sequences of constants which change the Fourier series of a function of bounded variation into that of an integral. Then, if a_n and b_n be the typical cosine and sine constants of a function of bounded variation, $q_n a_n$ and $q_n b_n$ are the typical constants of an integral, and therefore $nq_n b_n$ and $-nq_n a_n$ are respectively the cosine and sine constants of a summable function. But evidently,

changing x into $-x$, a_n and $-b_n$ are the Fourier cosine and sine constants of a function of bounded variation. Hence, by § 4 of the paper cited, the series whose general term is $nq_n a_n b_n$ converges.

Further, if the allied series of the Fourier series of a function of bounded variation is itself the Fourier series of a bounded function, b_n and $-a_n$ are respectively the cosine and sine constants of the Fourier series of a bounded function. Hence, by the use of the theorem of § 7 of the same paper, we can assert that in this case the series whose general term is $nq_n(a_n^2+b_n^2)$ certainly converges in the Cesàro manner, and therefore also in the ordinary manner, since every term is positive.

22. It will have been noted that, when the coefficients belong to what we have called the logarithmic scale, all the cosine series are Fourier series, using the term in the ordinary sense of the word, but that the sine series are only Fourier series when a certain condition holds. If we start with sine and cosine series which are undeniably Fourier series, and gradually reduce the degree of summability of the sum-function of the cosine series, the corresponding sine series will at a certain point cease to converge to a summable function.

If the coefficients are inverse powers of the logarithm of the index n, this will happen when $q_n = 1/\log n$. The theory of the complex variable shows that the sum-function of the corresponding complex power series is of the order $1/x \log x$, which is not summable. It does not, however, immediately indicate that this failure of integrability is due to the imaginary part, and not to the real part, of the sum. This appears most simply not by the use of the complex variable, but by means of considerations such as those here exposed. It may be added that it follows from the fact that the series whose general term is $1/n \log n$ diverges, and a result I have recently given,* that if u be the sum function, $u \log |u|$ is not summable. It thus follows in particular, that, at the origin, u has a higher order of infinity than

$$\frac{1}{x(\log x)^{2+\epsilon}},$$

and a higher order even than $\dfrac{1}{x l_1^2 l_2 \ldots l_m^{1+\epsilon}}$, however small ϵ may be. By the actual examination of the expression for the sum function, found in

* "On a Certain Series of Fourier," 1912, *Proc. London Math. Soc.*, Ser. 2, Vol. 11, p. 358.

the Appendix, it follows that the order of infinity is not higher than

$$\frac{1}{x(\log x)^2}.$$

In fact the expression there obtained may be broken up into two integrals, whose limits of integration are respectively (A, ∞) and $(0, A)$, where A is any convenient number greater, say, than unity. The latter of these two integrals clearly represents a bounded function of x, since its integrand is comparable with

$$\frac{1}{y\{1+(\log y)^2\}}.$$

The former integral is comparable, if A be big enough, with

$$\int_A^\infty \frac{e^{-xy}}{(\log y)^2}\,dy,$$

or, writing $xy = At$, with

$$\frac{A}{x}\int_x^\infty \frac{e^{-At}}{(\log t+\log A-\log x)^2}\,dt,$$

which, for values of x less than unity, is less than $e^{-Ax}/(x\log x)^2$. This proves the truth of the statement made above.

23. If we require the actual formulæ for the sums of the series which we have shown to converge, we shall need to know the sum-functions of the series of the auxiliary family. In the case in which the coefficients belong to the logarithmic scale, or are characterised in some other simple manner, the complex variable will afford the most powerful instrument of dealing with the matter. In certain cases we may proceed still more simply by the use of the real variable alone. I terminate this paper by obtaining by this method the sum of the cosine-series whose general term is $\cos nx/\log n$, and by applying it to obtain the expression for the sum of the convergent series $\sum_{n=2} a_n/\log n$, when a_n is the typical Fourier cosine constant of a bounded function whose discontinuities are all of the first kind.

Appendix on the Summation of the Series $\sum_{n=2}^{\infty} \frac{\cos nx}{\log n}$, $\sum_{n=2} \frac{a_n}{\log n}$.

Since, by well-known formulæ,

$$\int_0^\infty t^{q-1}\cos t\,dt = \Gamma(q)\cos\tfrac{1}{2}q\pi,$$

and

$$\Gamma(q)\Gamma(1-q) = \pi\operatorname{cosec} q\pi \quad (0<q<1),$$

we have, by a simple transformation, the identity

$$2 \sin \tfrac{1}{2}q\pi . \Gamma(1-q) \int_0^\infty t^{q-1} \cos nt\, dt = n^{-q}\pi.$$

Integrating with respect to q between the limits 0 and 1, we get

$$\frac{\pi}{2} \frac{1-1/n}{\log n} = \int_0^1 dq \sin \tfrac{1}{2}q\pi \int_0^\infty dt . t^{q-1} \cos nt \int_0^\infty e^{-y} y^{-q} dy$$

$$= \int_0^1 dq \sin \tfrac{1}{2}q\pi \int_0^\infty dt \cos nt \int_0^\infty e^{-ty} y^{-q} dy.$$

We have first to reverse the order of integration with respect to y and t.

To see that this is allowable, we remark first that

$$\int_0^A dt \cos nt \int_0^\infty e^{-ty} y^{-q} dy = \int_0^\infty dy . y^{-q} \int_0^A \cos nt . e^{-ty} dt.$$

In fact, the interval $(0, A)$ consists of a finite number of partial intervals throughout which $\cos nt$ has the same sign. Hence, by de la Vallée Poussin's theorem,* the change of order is permissible.

Next to show that

$$\int_A^\infty dt \cos nt \int_B^\infty e^{-ty} y^{-q} dy = \int_B^\infty dy . y^{-q} \int_A^\infty \cos nt . e^{-ty} dt,$$

we have merely to remark that, if we omit the factor $\cos nt$, the integrands are positive, and the repeated integral on the right becomes

$$\int_B^\infty y^{-q-1} e^{-Ay} dy,$$

and thus certainly exists. Hence the change of order is then allowable by de la Vallée Poussin's theorem. Hence also it is still permissible when we re-insert the factor $\cos nt$.†

It remains therefore merely to show that

$$\int_A^\infty dt \cos nt \int_0^B e^{-ty} y^{-q} dy = \int_0^B dy . y^{-q} \int_A^\infty e^{-ty} \cos nt\, dt.$$

---

* For the general theorem see W. H. Young, "On the Change of Order in a Repeated Integral," 1910, *Trans. Camb. Phil. Soc.*, Vol. XXI, p. 373.

† W. H. Young, "On Semi-Integrals," § 37, 1910, *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 321.

To see this, we note that the left-hand side may be written

$$\operatorname*{Lt}_{X\to\infty} \int_A^X dt \cos nt \int_0^B e^{-ty} y^{-q} dy.$$

Now $e^{-ty}y^{-q}$ is positive, and the interval (A, X) breaks up into a finite number of intervals throughout which $\cos nt$ has the same sign; therefore change of order in $(A \leqslant t \leqslant X,\ 0 \leqslant y \leqslant B)$ is allowable. Hence the left-hand side of the equation to be proved is

$$\operatorname*{Lt}_{X\to\infty} \int_0^B dy . y^{-q} \int_A^X \cos nt . e^{-ty}\, dt.$$

Here the inside integral is a bounded function of (y, X), and therefore approaches its limit $\int_A^\infty \cos nt . e^{-ty} dy$, which certainly exists, boundedly; also y^{-q} is a summable function of y. Therefore, by a known theorem, we may introduce the limit inside the integral with respect to y. Hence the required result is proved.

We have thus obtained the equation

$$\frac{\pi}{2}\,\frac{1-1/n}{\log n} = \int_0^1 dq \sin \tfrac{1}{2}q\pi \int_0^\infty dy . y^{-q} \int_0^\infty \cos nt . e^{-ty} dt$$

$$= \int_0^1 dq \sin \tfrac{1}{2}q\pi \int_0^\infty \frac{y^{1-q} dy}{n^2+y^2}$$

$$= \int_0^\infty dy\, \frac{y}{n^2+y^2} \int_0^1 y^{-q} \sin \tfrac{1}{2}q\pi\, dq$$

(by de la Vallée Poussin's theorem)

$$= \frac{2}{\pi} \int_0^\infty \frac{y - \dfrac{2}{\pi}\log y}{1+\left(\dfrac{2}{\pi}\log y\right)^2} \frac{dy}{n^2+y^2}.$$

Put $\frac{1}{n}$ for n, and $\frac{1}{y}$ for y, and we get, after dividing by n,

$$\frac{n^{-1}-n^{-2}}{\log n} = \frac{4}{\pi^2} \int_0^\infty \frac{\dfrac{1}{y} + \dfrac{2}{\pi}\log y}{1+\left(\dfrac{2}{\pi}\log y\right)^2} \frac{dy}{n^2+y^2}.$$

Hence also,

$$\frac{1-n^{-2}}{\log n} = \frac{4}{\pi^2}\int_0^\infty \frac{1}{1+\frac{4}{\pi^2}(\log y)^2}\,\frac{(1+y^2)\,dy}{y\,(n^2+y^2)},$$

whence
$$\frac{1}{\log n} = \int_0^\infty h(y)\left\{\frac{1}{n^2-1}\,\frac{1}{y}+\frac{y}{n^2+y^2}\right\}dy,$$

where
$$h(y) = \frac{4}{\pi^2}\,\frac{1}{1+\frac{4}{\pi^2}(\log y)^2}; \tag{1}$$

and therefore
$$\int_0^\infty \frac{h(y)}{y}\,dy = \int_{-\infty}^{+\infty}\frac{dt}{\frac{\pi^2}{4}+t^2} = 2.$$

Thus
$$\frac{1}{\log n}-\frac{2}{n^2-1} = \int_0^\infty \frac{y\,h(y)}{n^2+y^2}\,dy. \tag{2}$$

Now, by a known formula,*

$$2y\sum_{n=1}^{\infty}\frac{\cos nx}{n^2+y^2} = \pi\,\frac{\cosh y\,(\pi-x)}{\sinh \pi y}-\frac{1}{y}\quad(0\leqslant x\leqslant 2\pi),$$

and evidently

$$\begin{aligned}2\sum_{n=2}^{\infty}\frac{\cos nx}{n^2-1} &= \sum_{n=2}^{\infty}\left\{\frac{\cos nx}{n-1}-\frac{\cos nx}{n+1}\right\}\\ &= \sum_{n=2}^{\infty}\frac{\cos nx-\cos(n-2)\,x}{n-1}+1+\tfrac{1}{2}\cos x\\ &= -\sum_{n=2}^{\infty}2\sin x\,\frac{\sin(n-1)\,x}{n-1}+1+\tfrac{1}{2}\cos x\\ &= 1+\tfrac{1}{2}\cos x-\sin x\,(\pi-x)\quad(0\leqslant x\leqslant\pi),\end{aligned}$$

using the known expression for the sum of the sine-series $\Sigma n^{-1}\sin nx$.

Thus, for $0\leqslant x\leqslant\pi$, multiplying equation (2) by $\cos nx$ and summing

$$\sum_{n=2}^{\infty}\frac{\cos nx}{\log n} = 1+\tfrac{1}{2}\cos x-\sin x(\pi-x)+\tfrac{1}{2}\int_0^\infty h(y)\left\{\frac{\pi\cosh y(\pi-x)}{\sinh \pi y}-\frac{1}{y}\right\}dy$$
$$-\int_0^\infty \frac{y\cos x\,h(y)}{1+y^2}\,dy.$$

---

* See, for example, Bromwich's *Introduction to the Study of Infinite Series*, p. 257, Ex. 22.

But, by (1),

$$\int_0^\infty \frac{yh(y)}{1+y^2}dy = \int_0^\infty \frac{\frac{1}{y}dy}{(1+y^2)\left\{\frac{\pi^2}{4}+(\log y)^2\right\}} = \tfrac{1}{2}\int_0^\infty \frac{\left(y+\frac{1}{y}\right)h(y)}{1+y^2}dy,$$

and writing t for $\log y$, we see that the last member has the value unity.

Hence, when $0 \leqslant x \leqslant \pi$,

$$\sum_{n=2}^{\infty} \frac{\cos nx}{\log n} = -\tfrac{1}{2}\cos x - \sin x(\pi - x) + \frac{\pi}{2}\int_0^\infty \frac{\cosh y(\pi - x)\,dy}{\sinh \pi y\left\{\frac{\pi^2}{4}+(\log y)^2\right\}}. \quad (3)$$

Thus we have obtained an expression for the sum of the cosine series whose typical coefficient is $1/\log n$.

It follows in particular that

$$\sum_{n=2}^{\infty} \frac{(-)^n}{\log n} = \tfrac{1}{2} + \frac{\pi}{2}\int_0^\infty \frac{dy}{\sinh \pi y\left\{\frac{\pi^2}{4}+(\log y)^2\right\}}.$$

The formula (3) just obtained enables us at once to sum the series $\Sigma \frac{a_n}{\log n}$, where a_n is the typical cosine constant of a function $f(x)$ which is bounded, and is further simply discontinuous only should the summation be required to be made in the ordinary manner. Denote the expression on the right side of (3) by $g(x)$, then

$$\sum_{n=2}^{\infty} \frac{a_n}{\log n} = \frac{2}{\pi}\int_0^\pi \{f(x)+f(-x)\}\,g(x)\,dx. \quad (4)$$

We may also proceed somewhat differently as follows:—

We know that $$y/(n^2+y^2) = \int_0^\infty e^{-ty}\cos nt\,dt,$$

and therefore, by (2),

$$\frac{1}{\log n} - \frac{2}{n^2-1} = \int_0^\infty k(t)\cos nt\,dt, \quad (5)$$

where, by (1), $$k(t) = \int_0^\infty \frac{e^{-ty}}{\frac{\pi^2}{4}+(\log y)^2}dy, \quad (6)$$

and is accordingly a summable function of t.

Now $$\tfrac{1}{2}\{f(t)+f(-t)\} \sim \tfrac{1}{2}a_0+\sum_{n=1}^{\infty} a_n \cos nt.$$

Multiply by $k(t)$ and integrate, and we get, since $k(t)$ is summable and $f(t)$ is bounded,

$$\tfrac{1}{2}\int_0^{\infty} k(t)\{f(t)+f(-t)\}\,dt = a_0+a_1+\sum_{n=2}^{\infty} a_n \left\{\frac{1}{\log n}-\frac{2}{n^2-1}\right\},$$

the summation being made in the Cesàro manner. In particular, therefore, if $f(t)$ is simply discontinuous this formula is true when the summation is made in the ordinary manner. We may, of course, get rid of the terms $\sum_{n=2}^{\infty} \frac{2a_n}{n^2-1}$ as before.

ON THE DETERMINATION OF THE SUMMABILITY OF A FUNCTION BY MEANS OF ITS FOURIER CONSTANTS

By W. H. YOUNG, ScD., F.R.S.

[Received July 14th, 1912.—Read November 14th, 1912.]

1. The fact that, in the cases which ordinarily arise in practice, the series formed by the squares of the Fourier constants of a function converges, and has for its sum, to a numerical factor *près*, the integral of the square of the function over an interval of periodicity, seems to have been first remarked by Parseval, and the theorem in which this result has come to be embodied is accordingly usually called the Theorem of Parseval. In its most extended form this theorem states that the series in question converges, and has the sum stated, with the single limitation that the integral of the square of the function exists, the word integral being interpreted in the most general sense now attributed to it. It has only been by degrees, however, that this final result has been arrived at, and more than one writer has assisted in its evolution.*

The important part played in Hilbert's treatment of Fredholm's theory of integral equations by functions whose squares are summable, is well known, and it was inevitable that members of the Göttingen school of mathematics should be led to set themselves the task of proving the converse of Parseval's theorem, and to ask whether, when a trigonometrical series of the usual form possesses the property that the series formed by the squares of its coefficients converges, we can assert that the series is necessarily a Fourier series, and that the function associated with that series has its square summable. Almost simultaneously the question was

* Even in Hobson's *Treatise on the Theory of Functions of a Real Variable*, which appeared as recently as 1907, the proof is only given for bounded functions satisfying the condition of integrability of Riemann, reference being made to Liapounoff, de la Vallée Poussin, and Hurwitz. Two additional steps are necessary in order to get the complete generalisation; the restriction as to boundedness has to be removed and the concept of integration suitably generalised. Various distinct proofs have been given of the final theorem. P. Fatou, "Sur les séries trigonométriques et séries de Taylor," 1905, *Acta Math.*, Vol. xxx, pp. 373–379; H. Lebesgue, "Sur les Intégrales singulières," 1910, *Annales de Toulouse*, pp. 105–107; W. H. Young, "Successions of Integrals and Fourier Series," 1912, *Proc. London Math. Soc.*, Ser. 2, Vol. 11, pp. 87 and 91.

attacked and solved by Fischer and F. Riesz,* the answer being in the affirmative. The importance of the matter has led to a number of other proofs being devised in Göttingen dissertations and elsewhere.†

On the other hand, efforts made to extend these isolated results to embrace cases in which the known or unknown index of summability of the function is other than 2, appear to have failed. The sole published attempt in this direction is, as far as I know, the proof by F. Riesz,‡ of the following necessary and sufficient condition that the trigonometrical series should be the Fourier series of a function whose $(1+p)$-th power is summable, namely that, for all arbitrary sets of quantities m, and for every integer n,

$$\left| \sum_{i=1}^{n} m_i c_i \right|^{1+1/p} \Big/ \int_a^b \sum_{i=1}^{n} |m_i f_i(x)\,dx|^{1+1/p}\,dx \leqslant K,$$

where $c_i f_i(x)$ denotes the i-th term of the trigonometrical series, and K is a constant, independent of n and the m's.

This necessary and sufficient condition has, unlike its prototype, the very serious disadvantage of doing little more than express in other language the property in question. It is, at any rate, one of extreme difficulty in application, and no use has, I believe, so far been made of it.

It will be seen that Riesz's work left the question as to whether any simple generalisation of the Riesz-Fischer theorem exists entirely unanswered. It left equally unanswered the no less interesting question as to the possibility of generalising Parseval's theorem, of which the Riesz-Fischer theorem is the converse.

The latter of these two questions I have already answered in a short Note in the *Comptes Rendus*,§ and, somewhat more fully, in a paper to be published in the *Proceedings of the Royal Society*. I have, in fact, shown that for every odd integral value of p, the summability of the $(1+1/p)$-th power of $f(x)$ ensures the convergence of the series formed by

---

* F. Riesz, "Ueber orthogonale Funktionensysteme," 1907, *Gött. Nachr.*; E. Fischer, "Sur la convergence en moyenne," 1907, *Comptes Rendus*.

† For references see a forthcoming paper by W. H. Young and Grace Chisholm Young in the *Quarterly Journal of Mathematics*.

‡ F. Riesz, "Untersuchungen über Systeme integrierbarer Funktionen," 1910, *Math. Ann.*, Vol. 69, p. 474. The theorem is stated above for Fourier series; Riesz's result has a rather wider scope, as it is applicable to all orthogonal systems. The reader may also refer in this connection to § 11 below, where a somewhat more concise necessary and sufficient condition is given, suitable for Fourier series.

§ "Sur la généralisation du Théorème de Parseval," 1912, *Comptes Rendus*, Séance du 1r juillet. "On the Multiplication of Successions of Fourier Constants," 1912, *Roy. Soc. Proc.*, (A), Vol. 87, pp. 331–339.

the $(1+p)$-th powers of the coefficients of the Fourier series of $f(x)$. I have also given a corresponding result when p is an even integer. When p is unity this result, of course, coincides with Parseval's theorem; thus it constitutes a complete generalisation of that theorem. Basing myself on this generalisation, I have been able to answer the former of the two questions to which allusion has just been made; I have, in fact, succeeded in giving a simple and natural generalisation of the Riesz-Fischer theorem, which corresponds exactly to my generalisation of the Parseval theorem. It is that the convergence of the series formed by the $(1+1/p)$-th powers of the coefficients of a trigonometrical series ensures the summability of the $(1+p)$-th power of the function of which the trigonometrical series is necessarily the Fourier series. Here p is, as before, an odd integer.

It is the proof of this result which forms the object of the present note. I begin by giving the first incomplete attempt made by myself at obtaining the result, incomplete because the assumption has to be made that the function of which the trigonometrical series is the Fourier series is bounded above, or below. The nature of the reasoning seems, however, to be worth calling attention to, and it is possible that a few additional remarks might complete it. In any case, apart from its historical interest, it is certainly sufficiently powerful as a weapon to be capable of use in analogous matters.

I then give the main proof of the result. This is perfectly general in character, and does not indeed make use of the Riesz-Fischer theorem, of which the result is a generalisation; it constitutes therefore a new proof of this theorem. It should be remarked that the method employs an expression for the sum of a series known to be convergent. The comparative complication of this expression as the index p increases in magnitude constitutes the sole change in the proof for the various values of p.

Shortly after having obtained this proof, which involves considerations of a somewhat subtle character, I found myself able to give a purely algebraical demonstration for the case in which $p = 3$; the reasoning is clearly extendible to higher values of p, by induction or otherwise, the difficulties being solely of an algebraical nature. It requires us, however, to assume the Riesz-Fischer theorem.

It will be remarked that the theorem of this paper is not the direct converse of the result in the *Comptes Rendus*, though the two of them together constitute a complete generalisation of the theorem of Parseval and its converse. This is due to the fact that the *rôles* of p and $1/p$ are interchanged in the two results.

The question then remains so far open as to whether the direct converses of the two halves are true. The answer to this question must be

in the negative. The argument of the present paper shows indeed that the one converse cannot be true without the other; but, further it is not difficult to give an example confirming the negative answer. The series whose general term is $(\cos nx + \sin nx)/n^{\frac{1}{2}} |\log n|^{\frac{1}{2}}$ is the Fourier series of a function whose $\frac{4}{3}$-th power is not summable, though the series formed by the fourth powers of the coefficients converges.

For the rest this is what might be expected from other considerations. Both of my results are obtained directly, or indirectly, by the use of inequalities, and an argument based on inequalities is not, in general, reversible, even in the modified sense which would be here required. Taking, for example, the algebraical proof given below, the convergence of a certain series ΣC_n^2 is shown to follow from that of $\Sigma a_n^{\frac{1}{3}}$, using the fact that the sum of the former series is less than the cube of the sum of the latter series. Again, the proof of the corresponding result in the *Comptes Rendus* requires the theorem as to the summability of the function $h(x)$ defined by $\int f(x+t)\, g(t)\, dt$ when f and g have certain degrees of summability, which is obtained by means of an inequality. A knowledge of the degrees of summability of this function of x, and of one of the functions f and g, does not, however, lead easily to information as to the summability of the remaining function. The difficulty of the matter will be understood as soon as it is realised that the equation

$$h(x) = \int f(x+t)\, g(t)\, dt$$

is equivalent to a theorem in the multiplication of Fourier successions, and that $h(x)$ may be continuous, and $f(x)$ only summable, and yet there may be no function $g(t)$ satisfying the equation. Division of Fourier successions, unlike multiplication, is not always possible.

The results I have obtained must therefore be regarded as complete in themselves, though the problem of the determination of the summability of a function whose Fourier series is known is only partially solved by means of them. They do not enable us even to answer the question as to whether the allied series of a Fourier series has for its associated function one with the same degree of summability as the Fourier series, though in certain notable cases the answer to this question is certainly in the affirmative.

I hope, however, that the present paper may be regarded not only as interesting in itself, but as constituting a material contribution to the solution of the problem.*

---

* [In its original form as presented on July 14th to the Society, the paper consisted of a

2. We begin by stating the following fundamental theorem* in the integration of series :—

If the integrals of a succession of functions, each greater (less) than a certain summable function, nowhere diverge properly, then the original succession has a summable function for lower (upper) function, and the integral of this summable function is nowhere greater (less) than the lower (upper) limit of the integrals.

In the present instance we only need the following simple case of this general theorem :—

If the integrals of a succession of positive functions everywhere converge, and the original succession converges except at a set of content zero, then the limiting functions of the original succession are summable, and their common integral is nowhere greater than the limit of the integrals.

3. We can now prove the following lemma :—

LEMMA.—*If a_n, b_n are the typical Fourier constants of $f(x)$, and A_n, B_n those of $g(x)$, both functions being positive, and the series*

$$\tfrac{1}{2}a_0A_0+\sum_{n=1}(a_nA_n+b_nB_n)\cos nx-\sum_{n=1}(a_nB_n-b_nA_n)\sin nx \qquad (C)$$

is the Fourier series of a continuous function, e.g., if it converges uniformly, then

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\,g(t)\,dt$$

is a bounded function of x.

For, as pointed out in my note in the *Comptes Rendus*,† the latter function has the series (C) for Fourier series, and the integrated series of (C) is the Fourier series of

$$\frac{1}{\pi}\int_{-\pi}^{\pi} F(x+t)\,g(t)\,dt,$$

where $F(t)$ is the integral of $f(t)$. Therefore, if $q(x)$ denote the continuous

short introduction, and §§ 2–6 and § 11 as they stand at present. § 7 was added on August 13th, and §§ 8, 9, and 10 three days later. The introduction has been elaborated in accordance with what seemed to be the wish of a referee.—W. H. Y., Sept. 2nd, 1912.]

* This theorem is an immediate corollary from that in which the functions of the succession are all positive, or may be proved directly in the same way. See my paper on "Semi-Integrals and Oscillating Successions of Functions," 1910, *Proc. London Math. Soc.*, Ser. 2, Vol. 9, pp. 300. 301.

† Quoted in § 1.

function hypothecated which has the series (C) for Fourier series, the three functions

$$q(x), \quad \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\,g(t)\,dt, \quad \frac{d}{dx}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} F(x+t)\,g(t)\,dt\right),$$

agree except at a set of content zero, and therefore, by a familiar property of derivates, the first and last of these three functions agree everywhere. That is,

$$q(x) = \operatorname*{Lt}_{h\to 0}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{F(x+t+h)-F(x+t)}{h}\,g(t)\,dt\right).$$

Now the integrand of the integral, whose limit is here given, is positive, since, $f(t)$ being positive, $F(t)$ is a monotone increasing function. Hence this integrand, as $h\to 0$, describes a succession of positive functions whose integrals converge. Hence, by the preceding theorem, the function $f(x+t)\,g(t)$, which, except at a set of content zero, is the limit of the functions of the succession, is a summable function of t, and its integral, which is, of course, positive, is not greater than $\pi q(x)$, and is therefore a bounded function of x. This proves the theorem.

COR.—*The lemma is also true when $g(t)$ is positive, and $f(t)$ is bounded either below or above.*

For, suppose $f(t) > -A$. Then, since the Fourier constants of $f(t)+A$ are the same as those of $f(t)$, with the exception of the first, it follows from the theorem that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} [f(x+t)+A]\,g(t)$$

is a bounded function of x, whence the required result at once follows. If, however, $f(t) < A$, $A-f(t)$ is positive, and the same result follows.

[N.B.—*There may be an exceptional set of content zero, such that only at the remaining points is $f(t)$ known to be positive, or bounded below (above).* For $F(t)$ is unaltered by putting $f(t) = 0$ at the points of the exceptional set.]

4. We now require the following theorem:*—

THEOREM.—*If the succession of summable functions $f_1(x), f_2(x), \ldots$ is*

* This is a special case of the theorem of § 4 of my paper on "Successions of Integrals and Fourier Series," 1912, *Proc. London Math. Soc.*, Ser. 2, Vol. 11, pp. 50–53.

such that the succession of their integrals converges, and

$$\operatorname*{Lt}_{(E\to 0,\ n\to\infty)} \int_E f_n(x)\,dx = 0, \tag{1}$$

then the convergence of the succession of integrals is uniform, and the limiting function is itself an integral.

For $$\int_a^{x+h} f_n(x)\,dx = \int_a^x f_n(x)\,dx + \int_x^{x+h} f_n(x)\,dx.$$

By (1) the second integral on the right has the unique double limit zero, when $h \to 0$, $n \to \infty$. Hence the integral on the left has an unique double limit when $h \to 0$, $n \to \infty$, and this limit is the unique limit, which by hypothesis exists, of the first integral on the right. In other words, the sequence of integrals converges uniformly.

Let $F(x)$ be the limit to which the sequence of integrals converges uniformly, and write $F_n(x)$ for $\int_a^x f_n(x)\,dx$. Then taking the set E to be a set of intervals $(x_q,\ x_q+h_q)$, finite in number, we can write (1) in the form

$$\operatorname*{Lt}_{(E\to 0,\ n\to\infty)} \sum_{q=1}^{m} [F_n(x_q+h_q) - F_n(x_q)] = 0,$$

m being an integer depending on the chosen set of intervals of content E. Hence, taking any small positive quantity e, we can find E_0 and n_0 so that, provided only $E < E_0$ and $n \leqslant n_0$,

$$-e \leqslant \sum_{q=1}^{m} [F_n(x_q+h_q) - F_n(x_q)] \leqslant e.$$

Hence also, $$-e \leqslant \sum_{q=1}^{m} [F(x_q+h_q) - F(x_q)] \leqslant e.$$

Since this is true for every finite set of intervals of content $< E_0$, it is true for every infinite set of intervals of content $\leqslant E_0$. But this is Vitali's form of the condition that $F(x)$ should be the integral of a summable function. This, therefore, proves the theorem.

5. We shall also employ the following theorem due to F. Riesz.

THEOREM.—*If $f(x)$ is a given positive function, such that $f(x)\,g(x)$ is summable, whatever function $g(x)$ may be, provided only its $(1+1/p)$-th power is summable, then $f(x)$ has its $(1+p)$-th power summable.*

First, putting $g(x) = 1$, we see that $f(x)$ is summable.

Hence denoting by $f_m(x)$ the function which has the value k_n at the set

of points G_n, for which

$$k_n \leqslant f(x) < k_{n+1},$$

$$k_n = n e_m,$$

for each integer n, e_m being the m-th member of any convenient monotone descending sequence with zero as limit, we have

$$\int f_m(x)\,dx = \Sigma k_n G_n,$$

he series being convergent, and

$$\operatorname*{Lt}_{m\to\infty} \int f_m(x)\,dx = \int f(x)\,dx.$$

Similarly, if $[f(x)]^{1+p}$ were summable, we should have its integral as the limit of that of $[f_m(x)]^{1+p}$. Therefore, if we suppose that $[f(x)]^{1+p}$ is not summable, the series

$$\Sigma k_n^{1+p} G_n$$

must be divergent for some fixed sufficiently great value of m.

Hence, by theorems due to Abel,* the series

$$\Sigma \frac{k_n^{1+p} G_n}{\sum_{i=1}^{n} k_i^{1+p} G_i} = \Sigma k_n G_n \left(\frac{k_n^p}{\sum_{i=1}^{n} k_i^{1+p} G_i} \right) \equiv \Sigma k_n K_n G_n, \text{ say,}$$

is divergent, while the series

$$\Sigma \frac{k_n^{1+p} G_n}{\left(\sum_{i=1}^{n} k_i^{1+p} G_i \right)^{1+1/p}} = \Sigma G_n \left(\frac{k_n^p}{\sum_{i=1}^{n} k_i^{1+p} G_i} \right)^{1+1/p} = \Sigma K_n^{1+1/p} G_n$$

is convergent.

The convergence of this latter series shows that the function whose value is K_n at all points of the set G_n has its $(1+1/p)$-th power summable, which is in contradiction with the conditions of the theorem, since the divergence of the first series shows that the product of this function, say $K(x)$, by $f_m(x)$ is not summable, as it would have to be, since this product only differs from the summable function $K(x) f(x)$ by at most $e_m K(x)$.

Thus, by a *reductio ad absurdum*, the theorem is proved.†

* *Encycl. d. Math. Wiss.*, Vol. 1, p. 85, footnotes 180 and 182.

† This proof is due to F. Riesz, "Systeme Integrierbarer Funktionen," § 3, 1910, *Math. Ann.*, Vol. 69, pp. 457–459. Only the reference to Abel is substituted for his reference to Landau, who, for his part, had given quite properly the reference to Abel.

6. We can now prove the following theorem.

THEOREM.—*If a_n and b_n, the typical Fourier constants of a positive function $f(x)$, be such that*

$$\Sigma\{a_n^{1+1/p}+b_n^{1+1/p}\}$$

converges, then $\{f(x)\}^{1+p}$ is summable, p being an odd integer.

Let $g(x)$ be any positive function whose $(1+1/p)$-th power is summable, and A_n and B_n be its typical Fourier constants. Then, since p is an odd integer, by a theorem already proved,*

$$\Sigma\left[(A_n)^{1+p}+(B_n)^{1+p}\right]$$

converges. Hence, using the inequality

$$(p+1)\,uv \leqslant u^{p+1}+pv^{1+1/p} \quad (0 \leqslant u,\ 0 \leqslant v),$$

it follows at once that the series

$$\Sigma a_nA_n, \quad \Sigma a_nB_n, \quad \Sigma b_nB_n, \quad \text{and} \quad \Sigma b_nA_n$$

converge absolutely.

Hence the series

$$\Sigma\{(a_nA_n+b_nB_n)\cos nx-(a_nB_n-b_nA_n)\sin nx\}$$

converges uniformly.

Hence, by § 3, the function

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\,g(t)\,dt$$

is certainly bounded. In particular the integral representing it exists when x is zero.

But $g(t)$ is any positive function whatever whose $(1+1/p)$-th power is summable. Therefore, by § 5, $f(x)$ has its $(1+p)$-th power summable.

COR.—*If all that we know of $f(x)$ is that it is bounded, either above or below, the same is true.*

7.† The restriction has been made in §§ 3 and 6 that $f(x)$ should be bounded above or below. This restriction is, however, an accidental one, arising from the exigencies of the particular method employed in the details of the reasoning. In the present article the main theorem is

* W. H. Young, "On the Multiplication of Successions of Fourier Constants," *loc. cit.*, p. 336.

† Added August 13th, 1912.

proved by a different method and without this restriction. The importance of the original method, in which, as a close scrutiny shows, the hypothesis of § 6 is not exploited to the utmost, appears to me sufficient justification for the retention of the preceding articles as they originally stood. Moreover, the importance of the main theorem, and the difficulty with which it has been obtained, render it desirable to have an alternative proof.

The form of the enunciation in the present article is slightly different in another respect from that in § 6. That the modification is allowable is *a priori* obvious, since it is evident that if the series (1) is convergent, so is the series whose general term is $(a_n^2+b_n^2)$, so that by the Riesz-Fischer theorem the trigonometrical series is certainly the Fourier series of a function $f(x)$ whose square at least is summable.

THEOREM.—*If a trigonometrical series*

$$\tfrac{1}{2}a_0+\sum_{n=1}(a_n\cos nx+b_n\sin nx),.$$

is such that

$$\sum_{n=1}(a_n^{1+1/p}+b_n^{1+1/p}), \tag{1}$$

where p is an odd integer, is a convergent series, then the trigonometrical series is the Fourier series of a function $f(x)$ whose $(1+p)$-th power is summable.

The proof is here given for the case $p=3$. In the general case* the demonstration is precisely parallel.

The theorem will evidently be demonstrated,† if we can prove that, whatever positive function $g(x)$ is, provided only $[g(x)]^{\frac{3}{2}}$ is summable,

$$\int_{-\pi}^{\pi} f(x)\,g(x)\,dx \text{ exists.}$$

Now, if A_n and B_n are the Fourier constants of $g(x)$,‡

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(t+x)\,g(t)\,dt \sim \tfrac{1}{2}a_0A_0+\sum_{n=1}(a_nA_n+b_nB_n)\cos nx - \sum_{n=1}(a_nB_n-b_nA_n)\sin nx. \tag{2}$$

---

* The proof of the general theorem stated at the end of the note quoted from the *Comptes Rendus* will be found in my paper, "On the Multiplication of Successions of Fourier Constants," § 6. The expression corresponding to that on the right of (3) becomes, of course, more complicated.

† See above, § 5.

‡ See my note in the *Comptes Rendus*, quoted above, § 3.

Moreover, if $g(x)$ is bounded, and we sum in the Cesàro way, we may replace this relation by an equality.*

Again, if in (2) we replace $f(t)$ by $g(t)$, since $[g(t)]^{\frac{1}{2}}$ is summable, we know that the function of x on the left has its square summable,† so that, by Parseval's theorem,

$$\tfrac{1}{2}A_0^4+\sum_{n=1}(A_n^2+B_n^2)^2=\frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{\pi}\int_{-\pi}^{\pi}g(t+x)\,g(t)\,dt\right)^2, \tag{3}$$

the series on the left of (3) being convergent.

Using the inequality

$$(p+1)\,|uv|\leqslant p\,|u|^{1+1/p}+|v|^{1+p},$$

we have

$$\left|\sum_{n=1}a_nA_n\right|\leqslant\tfrac{3}{4}\sum_{n=1}a_n^4+\tfrac{1}{4}\sum_{n=1}A_n^4,$$

and a similar inequality with the b's. Accordingly

$$\left|\sum_{n=1}(a_nA_n+b_nB_n)\right|\leqslant\tfrac{3}{4}\sum_{n=1}(a_n^4+b_n^4)+\tfrac{1}{4}\sum_{n=1}(A_n^2+B_n^2)^2. \tag{4}$$

Hence it follows that, if $g(x)$ is a bounded function, (2) is an equality when $x=0$, since the series on the right then converges in the ordinary way.

Now, whatever function $g(x)$ is, we can approximate to it by means of a bounded function $g_r(x)$ having the same values as $g(x)$ except at the points of the set E where $g(x)\geqslant r$. Let us denote all the quantities already defined with respect to $g(x)$, when they refer to $g_r(x)$, by the same symbol as before with another subscript r. Then, by what has been pointed out,

$$\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\,g_r(t)\,dt=\tfrac{1}{2}a_0A_{r,0}+\sum_{n=1}(a_nA_{r,n}+b_nB_{r,n}). \tag{5}$$

Also, since the right-hand side of (3) is not increased, if we change $g(x)$ into $g_r(x)$,

$$\tfrac{1}{2}A_{r,0}^4+\sum_{n=1}(A_{r,n}^2+B_{r,n}^2)^2\leqslant\tfrac{1}{2}A_0^4+\sum_{n=1}(A_n^2+B_n^2)^2. \tag{6}$$

Denoting by S_k and $S_{r,k}$ the sum of the first k terms of these two series, and by R_k and $R_{r,k}$ the remainders, this may be written

$$S_{r,k}+R_{r,k}\leqslant S_k+R_k. \tag{6a}$$

---

* W. H. Young, "On the Integration of Fourier Series," 1910, § 7, *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 456. Another proof is given in "Successions of Integrals and Fourier Series," § 25.

† § 5 of the *Comptes Rendus* paper.

Now

$$\left| \frac{1}{\pi} \int_{-\pi}^{\pi} [g(x) - g_r(x)] \begin{matrix} \cos nx \\ \sin nx \end{matrix} dx \right| = \left| \frac{1}{\pi} \int_E g(x) \begin{matrix} \cos nx \\ \sin nx \end{matrix} dx \right| \leqslant \frac{1}{\pi} \int_E |g(x)| dx,$$

which is independent of n, and, as $r \to \infty$, has zero as limit, since $g(x)$ is a summable function; denoting it by e_r we have, therefore,

$$|A_n - A_{r,n}| \leqslant e_r, \quad |B_n - B_{r,n}| \leqslant e_r. \tag{7}$$

Hence

$$|S_k - S_{r,k}| \leqslant \left| \tfrac{1}{2}(A_0^4 - A_{r,0}^4) + \sum_{n=1}^{k} [(A_n^2 + B_n^2)^2 - \{(A_n - e_r)^2 + (B_n - e_r)^2\}^2] \right|,$$

from which it follows that, given k, we can find r_k, so that, for all values $r \geqslant r_k$, e_r is so small that

$$|S_k - S_{r,k}| \leqslant e, \tag{8}$$

where e is any chosen small positive quantity.

Since $R_{r,k}$ is positive, we have from (6a) and (8),

$$0 \leqslant R_{r,k} \leqslant R_k + e.$$

But the series (3) is convergent; therefore we can choose k so large that

$$\left.\begin{matrix} R_k \leqslant e \\ \text{and therefore also} \quad R_{r,k} \leqslant 2e \end{matrix}\right\} \quad (r \geqslant r_k). \tag{9}$$

Hence, using the inequality (4), but commencing the summations at $n = k$, and choosing k so that, in addition to the former condition,

$$\sum_{n=k} (a_n^4 + b_n^4) \leqslant \tfrac{1}{2}e,$$

we get $$\left|\sum_{n=k} (a_n A_n + b_n B_n)\right| < e; \quad \left|\sum_{n=k} (a_n A_{r,n} + b_n B_{r,n})\right| < e. \tag{10}$$

Thus it appears that if, in comparing the series whose general terms are $(a_n A_n + b_n B_n)$ and $(a_n A_{r,n} + b_n B_{r,n})$ respectively, we omit the remainders (10), the error will be less than $2e$.

Using (7) therefore, we have

$$\left|\tfrac{1}{2}a_0(A_0 - A_{r,0}) + \sum_{n=1} (a_n A_n + b_n B_n) - \sum_{n=1} (a_n A_{n,r} + b_n B_{n,r})\right|$$

$$\leqslant 2e + e_r \sum_{n=1}^{k} (|a_n| + |b_n|).$$

We can therefore find $r_k' \geqslant r_k$, so that, for all values of $r \geqslant r_k'$, the right-hand side of this inequality is less than $3e$.

Accordingly, by (5), we have

$$\left| \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) g_r(t)\, dt - \tfrac{1}{2} a_0 A_0 - \sum_{n=1} (a_n A_n + b_n B_n) \right| \leqslant 3e. \qquad (12)$$

Now let us write

$$g_r(t) = g_r'(t) + g_r''(t), \qquad g(t) = g'(t) + g''(t),$$

where in each case the first of the two functions on the right is zero, wherever $f(t)$ is negative, and the second wherever $f(t) \geqslant 0$.

Then everything that has been said so far applies, *mutatis mutandis*, when we replace $g(x)$ by $g'(x)$, and $g_r(x)$ by $g_r'(x)$, or when we replace $g(x)$ by $g''(x)$, and $g_r(x)$ by $g_r''(x)$.

But when we make the former substitution, the integrand in (12) is positive, since $g_r'(t)$ is positive, and is zero wherever $f(t)$ is negative.

Also, as $r \to \infty$, the functions $g_r'(t)$, and therefore also $f(t)\, g_r'(t)$, form a monotone succession. Therefore the integral on the left of (12) has, in this case a unique limit, which, if finite, is, by the theory of monotone successions, the integral of the limiting function $f(t)\, g'(t)$. Since, by (12), this limit differs from the sum of a convergent series by at most $3e$, the limit is certainly finite. Thus, proceeding to the limit with r, $f(t)\, g'(t)$ is summable and

$$\left| \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\, g'(t)\, dt - \tfrac{1}{2} a_0 A_0' - \sum_{n=1} (a_n A_n' + b_n B_n') \right| \leqslant 3e, \qquad (13)$$

where the dashes denote that the constants to which they are attached are the Fourier constants of $g'(t)$.

Since e may be chosen as small as we please, this shows that the left-hand side of (13) is zero. Similarly it is zero when we change the dashes into double dashes. Adding the two equations so obtained, we have, finally,

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\, g(t)\, dt = \tfrac{1}{2} a_0 A_0 + \sum_{n=1} (a_n A_n + b_n B_n). \qquad (15)$$

Thus $f(t)\, g(t)$ is summable for all functions $g(t)$, which proves the theorem.

8. In the preceding proof we make no use of the Riesz-Fischer theorem, that if $\Sigma(a_n^2 + b_n^2)$ converges, the trigonometrical series is the Fourier series of a function whose square is summable, except in so far as it assumes that the trigonometrical series is a Fourier series, a fact which,

for the rest, can be easily proved directly. Besides the advantage of a subsidiary nature of not employing the Riesz-Fischer theorem, the proof of § 8 is interesting as indicating naturally the reciprocal character of our theorem to that given by myself in the *Comptes Rendus*. Analogous reasoning evidently shows that the proper converses of the two theorems are either both true or both false.

If we employ the result of Riesz-Fischer, we can prove our theorem by means of formal algebraic reasoning alone. I give such a proof in the next article, confining my attention as before to the case in which $f(x)$ is to be proved to have its fourth power summable. It will be noticed that the extension to the general case is somewhat less immediate than in the proof of § 8.

9. We shall require the following inequality

$$\Sigma uv^{(1+p)(1+q)(1-pq)} \leqslant (\Sigma u^{1+p} v^{1+q})(\Sigma u^{1+p})^{q(1+p)(1-pq)} (\Sigma v^{1+q})^{p(1+q)/(1-pq)}$$

$$(0 \leqslant u,\ 0 \leqslant v,\ pq < 1).$$

This may be proved by the artifice already employed by me to prove the corresponding inequality for integrals. We take the known inequality

$$\Sigma UV \leqslant (\Sigma U^{1+p})^{1/(1+p)} (\Sigma V^{1+1/p})^{1/(1+1/p)},$$

and write $U = uv^{1/(1+p)}, \quad V = v^{p(1+p)}.$

We thus get $$(\Sigma uv)^{1+p} \leqslant (\Sigma u^{1+p} v)(\Sigma v)^p. \tag{1}$$

Again, writing $U = uv^{(1-pq)/(1+p)}, \quad V = v^{p(1+q)/(1+p)},$

where $pq < 1$, we get

$$(\Sigma uv)^{1+p} \leqslant (\Sigma u^{1+p} v^{1-pq})(\Sigma v^{1+q})^p. \tag{2}$$

Hence, writing $(1+q)/(1-pq)$ for $(1+p)$, u^{1+p} for v, and v^{1-pq} for u in (1), and substituting for the first summation in (2), the required inequality follows.

10. THEOREM.—*If the series*

$$\sum_{n=1} (a_n^{\frac{4}{3}} + b_n^{\frac{4}{3}})$$

converges, then the trigonometrical series

$$\tfrac{1}{2}a_0 + \sum_{n=1} (a_n \cos nx + b_n \sin nx)$$

is the Fourier series of a function whose fourth power is summable.

Let $$\sum_{n=1} a_n^{\frac{1}{3}} = K,$$

then $\sum_{n=1} a_n^2$ converges, and therefore $\sum_{n=1} a_n \cos nx$ is the Fourier series of an even function $f(x)$ whose square is summable. Let the Fourier cosine constants of f^2 be $C_1, C_2, \ldots$; then we shall have proved that f^4 is summable, if we show that $\sum_{n=1} C_n^2$ is convergent.

To do this, let $A_{r,n}$ be the typical Fourier cosine constant of the function $f(x)\cos nx$, so that

$$A_{r,n} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\cos rx\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\{\cos(n+r)x+\cos(n-r)x\}\,dx = \tfrac{1}{2}(a_{n+r}+a_{|n-r|});$$

and therefore

$$\sum_{r=1} A_{r,n}^{\frac{1}{3}} \leqslant \tfrac{1}{2}\left\{\sum_{r=1} a_{n+r}^{\frac{1}{3}} + \sum_{r=1} a_{|n-r|}^{\frac{1}{3}}\right\}$$

$$\leqslant \tfrac{1}{2}\left\{\sum_{r=1} a_{n+r}^{\frac{1}{3}} + \sum_{r=1}^{r=n} a_{n-r}^{\frac{1}{3}} + \sum_{r=n+1} a_{r-n}^{\frac{1}{3}}\right\} \leqslant K.$$

Now, since $f(x)$ and $f(x)\cos nx$ both have their squares summable, we may multiply the Fourier series of either by the other, and integrate term by term. Thus

$$C_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,f(x)\cos nx\,dx = \sum_{r=1} a_r A_{r,n}.$$

Hence, using the inequality

$$(\Sigma uv)^2 \leqslant \Sigma u^{\frac{4}{3}}v^{\frac{4}{3}}\sqrt{\Sigma u^{\frac{4}{3}}\Sigma v^{\frac{4}{3}}},$$

and putting $u = a_r$, $v = A_{r,n}$, we have

$$C_n^2 \leqslant K\sum_{r=1} a_r^{\frac{4}{3}} A_{r,n}^{\frac{4}{3}}.$$

Therefore

$$\sum_{n=1} C_n^2 \leqslant K\sum_{n=1}\sum_{r=1} a_r^{\frac{4}{3}} A_{r,n}^{\frac{4}{3}} \leqslant K\sum_{r=1}\sum_{n=1} a_r^{\frac{4}{3}} A_{r,n}^{\frac{4}{3}} \leqslant K^2\sum_{r=1} a_r^{\frac{1}{3}} < K^3.$$

This proves therefore that f^4 is summable.

Similarly, $\sum_{n=1} b_n \sin nx$ is the Fourier series of a function $g(x)$ whose fourth power is summable. Hence the trigonometrical series given in the

enunciation is the Fourier series of $f(x)+g(x)+\frac{1}{2}a_0$, which also has its fourth power summable. This proves the theorem.

11. As the mode in which the reasoning of § 10 is to be generalised may not be quite clear to some readers, I proceed to give some brief indications of the procedure to be adopted. We remark that we have to write down the necessary and sufficient condition that $\{f(x)\}^{\frac{1}{2}(1+p)}$ should have its square summable, expressing the coefficients of the Fourier series of this function in terms of the Fourier constants of $f(x)$ itself; these expressions will involve certain series which must themselves converge in order that the expressions found may have a meaning. We thus get a number of conditions which are both necessary and sufficient. These necessary and sufficient conditions become more and more numerous, and more and more complicated as the integer p increases. Our object is then to show, by inequalities similar to those in § 9, how, certain simpler conditions being satisfied, these necessary and sufficient conditions are verified.

In the case when $p = 5$, supposing $f(x)$ to be even, and E_r to be the typical Fourier constant of the even function $\{f(x)\}^3$, the necessary and sufficient condition that $f(x)$ should have its sixth power summable is that ΣE_r^2 should converge. Assuming then that $\{f(x)\}^6$ is summable, and therefore also all lower powers of $f(x)$ are summable, we may use § 10, and since

$$\{f(x)\}^2 \sim \sum_{r=1} C_r \cos rx,$$

and

$$f(x)\cos nx \sim \sum_{r=1} A_{r,n}\cos rx,$$

we get, as before, since each of these functions has its square summable,

$$E_n = \sum_{r=1} A_{r,n}\, C_r = \tfrac{1}{2}\sum_{r=1}(a_{n+r}+a_{|n-r|})\, C_r.$$

Thus the following conditions are *necessary* :—

Σa_r^2 is convergent, $\Sigma C_n^2 = \sum_{n=1}\sum_{r=1} \frac{1}{2}a_r(a_{n+r}+a_{|n-r|})$ is convergent,

and

$$\sum_{m=1} E_m^2 = \tfrac{1}{2}\sum_{m=1}\sum_{n=1}(a_{m+n}+a_{|m-n|})\, C_n$$

$$= \tfrac{1}{4}\sum_{m=1}\sum_{n=1}\sum_{r=1}(a_{m+n}+a_{|m-n|})(a_{n+r}+a_{|n-r|})\, a_r$$

is convergent.

These conditions are also *sufficient;* for, if they are satisfied, by the first of the three conditions, the quantities C_n exist, and $f(x)$ has its square summable; moreover, by the second condition, $f(x)$ has its fourth power summable, whence the expression found for E_n holds. Finally, therefore, the third condition ensures that $f(x)$ has its sixth power summable.

Thus the three given conditions are both necessary and sufficient. In the general case there will be $\frac{1}{2}(p+1)$ such series which have to be convergent.

In the algebraical method of proving our theorem in the general case, we have therefore to show, by means of extended forms of the inequalities of § 9, that all these series are convergent, when the series

$$\sum_{n=1} (a_n^{1+1/p} + b^{1+1/p})$$

converges. One may clearly employ the method of induction; the details of the work may safely be left to the reader.*

12. I take the present occasion of formulating and proving the following general condition.

THEOREM.—*A necessary and sufficient condition that a Fourier series should be the Fourier series of a function whose* $(1+p)$*-th power is summable, is that*

$$\operatorname*{Lt}_{E\to 0,\, n\to\infty} \int_E |f_n(x)|^{1+p}\, dx = 0, \tag{1}$$

$f_n(x)$ *denoting the Cesàro partial summation of the first* $(2n+1)$ *terms of the series.*

First to prove that the condition is sufficient.

Since the series is a Fourier series, $f_n(x)$ converges to the function $f(x)$ corresponding to the series, except at a set of content zero. Therefore also $|f_n(x)|^{1+p}$ converges to $|f(x)|^{1+p}$, except at a set of content zero. Since, by hypothesis, the double limit (1) is zero, the succession of functions $|f_n(x)|^{1+p}$ is absolutely integrable term-by-term,† the limiting function being summable. This proves the sufficiency of the condition.

The necessity of the condition has been already stated and proved.‡

---

* Added during the passage through the press, November, 1912.

† See, for example, "Successions of Integrals and Fourier Series," p. 54.

‡ *Loc. cit.*, § 27, p. 89.

In fact, denoting by $T_n(x)$ the corresponding Cesàro summation of the Fourier series of $|f(x)|^{1+p}$, we have*

$$|f_n(x)|^{1+p} \leqslant T_n(x).$$

Hence
$$\int_E |f_n(x)|^{1+p}\, dx \leqslant \int_E T_n(x).$$

But the latter integral is known* to have the unique limit zero, therefore the same is true of the former integral. This proves the necessity of the condition.

---

* *Loc. cit.*, § 25, p. 88.

ON GROUPS OF LINEAR SUBSTITUTIONS OF FINITE ORDER WHICH POSSESS QUADRATIC INVARIANTS

By W. BURNSIDE.

[Received October 8th, 1912.—Read November 14th, 1912.]

I.

A group of linear substitutions of finite order, the coefficients in which are all real, necessarily has one or more quadratic invariants. This fact has long been recognised. If σ denotes the sum of the squares of the variables operated on by the group, and if σ_s denotes what σ becomes under the operation of any substitution of the group, then $\sum_s \sigma_s$, where the sum extends to all the substitutions of the group, is such an invariant. In fact, the number of terms in the summation being finite and the coefficients of the substitutions being real, this sum is clearly a definite quadratic form in the variables operated on by the group, which is unaltered by all the substitutions of the group.

The principal object of this paper is to prove the converse of this theorem, in the following form :—

A group of linear substitutions of finite order, for which a quadratic form of non-zero determinant is invariant, can always be transformed into a group of real substitutions.

It may be noticed that, if the group is not of finite order, the theorem is not generally true. If Γ is any group of finite order of linear substitutions, and $\bar{\Gamma}$ the conjugate group (*i.e.*, the group in which the coefficients are the conjugate imaginaries of those of Γ), then there is always* a definite Hermitian form of non-zero determinant $\Sigma a_{ij} x_i \bar{x}_j$, which is invariant when the x's undergo any substitution of Γ and the $\bar{x}$'s, the corresponding substitution of $\bar{\Gamma}$. The theorem in question is proved here by considering the simultaneous reduction of a definite Hermitian form of non-zero determinant and any quadratic form of non-zero determinant.

* A. Loewy, *Comptes Rendus*, 1896, pp. 168–171.

LEMMA.—If $\sum_{ij} a_{ij}x_i\bar{x}_j$ is any definite Hermitian form of non-zero determinant [so that for all values of the x's, except simultaneous zero values the form takes a real positive (non-zero) value], then the equations

$$\sum_j a_{ij}x_j+\lambda\bar{x}_i = 0 \quad (i = 1, 2, \ldots, n)$$

always admit solution.

Consider the system of $2n$ simultaneous equations

$$\left.\begin{aligned}\sum_j a_{ij}y_j+\lambda z_i &= 0\\ \sum_j a_{ji}z_j+\lambda y_i &= 0\end{aligned}\right\} \quad (i = 1, 2, \ldots, n). \qquad \text{(i)}$$

Their determinantal equation is

$$\begin{vmatrix} \lambda & . & \ldots & . & a_{11} & a_{21} & \ldots & a_{n1} \\ . & \lambda & \ldots & . & a_{12} & a_{22} & \ldots & a_{n2} \\ \ldots & & \ldots & & \ldots & \ldots & & \ldots \\ . & . & \ldots & \lambda & a_{1n} & a_{2n} & \ldots & a_{nn} \\ a_{11} & a_{12} & \ldots & a_{1n} & \lambda & . & \ldots & . \\ a_{21} & a_{22} & \ldots & a_{2n} & . & \lambda & \ldots & . \\ \ldots & & \ldots & & \ldots & \ldots & & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} & . & . & \ldots & \lambda \end{vmatrix} = 0, \qquad \text{(ii)}$$

and to each solution of this equation there corresponds at least one solution of the equations (i) for which the y's and z's are not all zero.

If now, for any solution of (i), $\bar{y}_i$ and $\bar{z}_i$ are the conjugate imaginaries of y_i and z_i; then

$$\sum_{ij} a_{ij}y_j\bar{y}_i+\lambda\sum_i \bar{y}_i z_i = 0,$$

$$\sum_{ij} a_{ji}z_j\bar{z}_i+\lambda\sum_i y_i\bar{z}_i = 0.$$

Now $\sum_{ij} a_{ij}y_j\bar{y}_i$ and $\sum_{ij} a_{ji}z_j\bar{z}_i$ are real positive (non-zero) quantities; while $\sum_i \bar{y}_i z_i$ and $\sum_i y_i\bar{z}_i$ either are real or are conjugate imaginaries. It follows therefore that λ is necessarily real; that is, all the roots of (ii) are real.

Since a_{ij} and a_{ji} are conjugate imaginaries, and λ is real, there will correspond to any solution of the equations (i) the further relations

$$\left.\begin{aligned}\sum_j a_{ji}\bar{y}_j+\lambda\bar{z}_i &= 0\\ \sum_j a_{ij}\bar{z}_j+\lambda\bar{y}_i &= 0\end{aligned}\right\} \quad (i = 1, 2, \ldots, n). \qquad \text{(iii)}$$

Finally, adding the first equations of (i) to the corresponding last equations of (iii),

$$\sum_j \alpha_{ij}(y_j+\bar{z}_j)+\lambda(\bar{y}_i+z_i)=0 \quad (i=1, 2, \ldots, n),$$

which proves the lemma.

Suppose now that for a group of linear substitutions of finite order on $x_1, x_2, \ldots, x_n$ a quadratic form f, of non-zero determinant, is invariant. For such a group and its conjugate there is necessarily an invariant definite Hermitian form ϕ of non-zero determinant. By carrying out a suitable substitution (not necessarily real) on the x's, and the conjugate substitution on the $\bar{x}$'s, f may be reduced to the form $\sum_i x_i^2$, while ϕ remains a definite Hermitian form and may be denoted by

$$\sum_{ij} \alpha_{ij} x_i \bar{x}_j.$$

Carry out now on the x's the substitution C,

$$x_i' = \sum_j c_{ij} x_j \quad (i=1, 2, \ldots, n),$$

and on the $\bar{x}$'s the conjugate substitution $\bar{C}$,

$$\bar{x}_i' = \sum_j \bar{c}_{ij} \bar{x}_j \quad (i=1, 2, \ldots, n).$$

Then the quadratic invariant becomes

$$f' = \sum_i (\sum_j c_{ij} x_j)^2 = \sum_{uv} A_{uv} x_u x_v,$$

where

$$A_{uv} = \sum_i c_{iu} c_{iv} = A_{vu};$$

while the Hermitian form becomes

$$\phi' = \sum_{i,j,u,v} \alpha_{ij} c_{iu} \bar{c}_{jv} x_u \bar{x}_v = \sum_{u,v} \alpha'_{u,v} x_u \bar{x}_v,$$

where

$$\alpha'_{u,v} = \sum_{ij} \alpha_{ij} c_{iu} \bar{c}_{jv}.$$

Now

$$f' = \frac{1}{A_{11}} (A_{11}x_1+A_{12}x_2+\ldots+A_{1n}x_n)^2+f'',$$

where f'' is a form of non-zero determinant in $x_2, x_3, \ldots, x_n$. Also

$$\phi' = \frac{1}{\alpha'_{11}} (\alpha'_{11}x_1+\alpha'_{21}x_2+\ldots+\alpha'_{n1}x_n)(\alpha'_{11}\bar{x}_1+\alpha'_{12}\bar{x}_2+\ldots+\alpha'_{1n}\bar{x}_n)+\phi'',$$

where ϕ'' is a definite Hermitian form in

$$x_2, x_3, \ldots, x_n, \quad \bar{x}_2, \bar{x}_3, \ldots, \bar{x}_n.$$

If

$$\alpha'_{11}x_1+\alpha'_{21}x_2+\ldots+\alpha'_{n1}x_n+\lambda(A_{11}x_1+A_{12}x_2+\ldots+A_{1n}x_n) \equiv 0, \qquad \text{(iv)}$$

then $$a'_{u1}+\lambda A_{u1} = 0 \quad (u = 1, 2, \ldots, n),$$

or $$\sum_{ij} a_{ij} c_{iu} \bar{c}_{j1}+\lambda \sum_i c_{iu} c_{i1} = 0 \quad (u = 1, 2, \ldots, n),$$

or $$\sum_i c_{iu} \left(\sum_j a_{ij} \bar{c}_{j1}+\lambda c_{i1}\right) = 0 \quad (u = 1, 2, \ldots, n).$$

But it has been shewn in the Lemma that it is always possible to determine quantities c_{i1} $(i = 1, 2, \ldots, n)$, which are not all zero, so as to satisfy the equations

$$\sum_j a_{ij} \bar{c}_{j1}+\lambda c_{i1} = 0 \quad (i = 1, 2, \ldots, n).$$

Hence the substitution C can always be determined so that the relation (iv) holds. The quadratic and Hermitian forms f'' and ϕ'' may be similarly dealt with. It follows that, by a suitable substitution on the x's and the conjugate substitution on the $\bar{x}$'s, the forms f and ϕ can always be reduced to

$$\sum_i x_i^2 \quad \text{and} \quad \sum_i \beta_i x_i \bar{x}_i,$$

where the β's are real positive quantities.

If $$x'_i = \sum_j s_{ij} x_j \quad (i = 1, 2, \ldots, n)$$

is any substitution of the group, when transformed so that the invariants have the above forms, then

$$\sum_i s_{iu}^2 = 1, \qquad \sum_i \beta_i s_{iu} \bar{s}_{iu} = \beta_u,$$

$$\sum_i s_{iu} s_{iv} = 0 \ (u \neq v), \qquad \sum_i \beta_i s_{iu} \bar{s}_{iv} = 0 \ (u \neq v).$$

It follows that $$\sum_i s_{iu} \left(s_{iv} - \frac{\beta_i \bar{s}_{iv}}{\beta_v}\right) = 0$$

for all values of u and v, including $u = v$. Hence

$$s_{iv} - \frac{\beta_i \bar{s}_{iv}}{\beta_v} = 0$$

for all values of i and v. If $\beta_i \neq \beta_v$, $s_{iv} = 0$, and if $\beta_i = \beta_v$, s_{iv} is real. The group of linear substitutions therefore has real coefficients.

II.

In order that an irreducible group of linear substitutions of finite order may possess a quadratic invariant, the characteristics of all its substitutions must be real. In fact, if any characteristic is not real, the coefficients in the leading diagonal of the corresponding substitution cannot be real,

and by the preceding result the group cannot have a quadratic invariant. A necessary condition then that an irreducible group of linear substitutions should have a quadratic invariant is that it should be self-inverse.*

Now, if A, B, C, ... are the substitutions of an irreducible group, and if S and S_t denote transposed substitutions, then the relation

$$AB = C$$

involves the relation $$A_t^{-1} B_t^{-1} = C_t^{-1}.$$

If the group is self-inverse, so that the characteristics of A and A^{-1} are the same, then the set of substitutions

$$A,\ B,\ C,\ \ldots,$$

and the set $$A_t^{-1},\ B_t^{-1},\ C_t^{-1},\ \ldots,$$

give equivalent representations of the group, and therefore there must be a substitution T of the variables operated on by the group, such that

$$T^{-1} S T = S_t^{-1}$$

holds for each substitution of the group. Let T be

$$y_i = \sum_j t_{ij} x_j \quad (i = 1, 2, \ldots, n),$$

the x's being the variables operated on by the group. When the x's undergo any substitution of the group the y's undergo the inverse of the transposed substitution, and therefore

$$\sum_i x_i y_i \quad \text{or} \quad \sum_i x_i (t_{i1} x_1 + t_{i2} x_2 + \ldots + t_{in} x_n)$$

remains invariant. The group therefore has a quadratic invariant, unless the preceding quadratic form is identically zero. This obviously can only be the case if T is a skew symmetric substitution. Now a skew symmetric determinant of an odd number of rows and columns is identically zero. Hence, if the number of variables is odd, since T cannot then be skew symmetric, the group necessarily has a quadratic invariant, *i.e.*,

An irreducible group of linear substitutions of finite order in an odd number of variables, whose characteristics are all real, must have a quadratic invariant and can be expressed in real form.

---

* Compare my *Theory of Groups*, 2nd ed., p. 368.

SOME PROPERTIES OF SYMMETRIC AND ORTHOGONAL SUBSTITUTIONS

By HAROLD HILTON.

[Received June 2nd, 1912. — Read June 13th, 1912. — Revised October 17th, 1912.]

1. Suppose we have a substitution A,

$$x_t = a_{t1}x_1 + a_{t2}x_2 + \ldots + a_{tm}x_m \quad (t = 1, 2, \ldots, m),$$

in the m variables $x_1, x_2, \ldots, x_m$.

It is possible* in an infinity of ways to transform A by a choice of new variables into the canonical substitution

$$x'_t = \lambda_t x_t + \beta_t x_{t+1} \quad (t = 1, 2, \ldots, m), \qquad (\alpha)$$

where $\beta_t = 0$ or 1, and is certainly 0 if $\lambda_t \neq \lambda_{t+1}$.

The expression $x_1^2 + x_2^2 + \ldots + x_m^2$

will become a quadratic function $\Sigma c_{ij}x_i x_j$ in the new variables with non-zero determinant. We shall consider in what way the new variables must be chosen so as to transform A into the canonical substitution (α), and to make $\Sigma c_{ij}x_i x_j$ as simple as possible; in the cases in which A is symmetric or orthogonal.

The result is as follows:—

(1) Suppose A is *symmetric* and the canonical substitution N into which A is transformed is the direct product of substitutions of the form

$$x'_1 = \alpha x_1 + x_2, \ \ldots, \ x'_{s-1} = \alpha x_{s-1} + x_s, \ x'_s = \alpha x_s,$$

then the new variables can be chosen so that A becomes N, and $(x_1^2 + x_2^2 + \ldots + x_m^2)$ becomes the sum of functions of the type

$$(x_1 x_s + x_2 x_{s-1} + x_3 x_{s-2} + \ldots + x_s x_1).$$

(2) Suppose A is orthogonal, the canonical substitution N into which A

---

* *Messenger of Math.*, 1909, p. 24.

is transformed will be the direct product of substitutions of the form

$$\left.\begin{aligned} x_1' &= \alpha x_1 \quad + x_2, \ \ldots, \ x_{s-1}' = \alpha x_{s-1} \quad + x_s, \ x_s' = \alpha x_s \\ y_1' &= \alpha^{-1} y_1 + y_2, \ \ldots, \ y_{s-1}' = \alpha^{-1} y_{s-1} + y_s, \ y_s' = \alpha^{-1} y_s \end{aligned}\right\},$$

where $\alpha^2 \neq 1$, or else $\alpha^2 = 1$, and s is even, and of substitutions of the form

$$X_1' = \alpha X_1 + X_2, \ \ldots, \ X_{s-1}' = \alpha X_{s-1} + X_s, \ X_s' = \alpha X_s,$$

where $\alpha^2 = 1$, and s is odd.*

Then we can choose the variables x, y, X, so that A becomes N, and $(x_1^2 + x_2^2 + \ldots + x_m^2)$ becomes the sum of functions of the type $f_1(x, y)$ and $f_1(X, X)$ respectively, where $f_1(x, y)$ represents the function of x and y denoted by this symbol in *Proc. London Math. Soc.*, Ser. 2, Vol. 10, 1911, p. 426.

2. Suppose that in the substitution

$$\left.\begin{aligned} x_1' &= \alpha x_1 + x_2, \ \ldots, \ x_{s-1}' = \alpha x_{s-1} + x_s, \ x_s' = \alpha x_s \\ y_1' &= \beta y_1 + y_2, \ \ldots, \ y_{s-1}' = \beta y_{s-1} + y_s, \ y_s' = \beta y_s \end{aligned}\right\}, \qquad (\beta)$$

we replace

$$\left.\begin{aligned} x_1 &\text{ by } a_s x_1 + a_{s-1} x_2 + a_{s-2} x_3 + \ldots + a_1 x_s \\ x_2 &\text{ by } \quad a_s x_2 + a_{s-1} x_3 + \ldots + a_2 x_s \\ x_3 &\text{ by } \quad\quad a_s x_3 + \ldots + a_3 x_s \\ &\ldots \\ x_s &\text{ by } \quad\quad\quad\quad a_s x_s \\ y_1 &\text{ by } b_s y_1 + b_{s-1} y_2 + b_{s-2} y_3 + \ldots + b_1 y_s \\ y_2 &\text{ by } \quad b_s y_2 + b_{s-1} y_3 + \ldots + b_2 y_s \\ y_3 &\text{ by } \quad\quad b_s y_3 + \ldots + b_3 y_s \\ &\ldots \\ y_s &\text{ by } \quad\quad\quad\quad b_s y_s \end{aligned}\right\}. \qquad (\gamma)$$

This will not alter (β), for (γ) is a substitution permutable with (β).†

Suppose

$$\left.\begin{aligned} \phi_1(x, y) &\equiv x_1 y_s + x_2 y_{s-1} + x_3 y_{s-2} + \ldots + x_{s-2} y_3 + x_{s-1} y_2 + x_s y_1 \\ \phi_2(x, y) &\equiv \quad x_2 y_s + x_3 y_{s-1} + \ldots + x_{s-1} y_3 + x_s y_2 \\ \phi_3(x, y) &\equiv \quad\quad x_3 y_s + \ldots + x_s y_3 \\ &\ldots \\ \phi_s(x, y) &\equiv \quad\quad\quad\quad x_s y_s \end{aligned}\right\}.$$

* *Proc. London Math. Soc.*, Ser. 2, Vol. 10, 1911, pp. 429, 432.

† *Messenger of Math.*, 1911, p. 111.

The substitution (γ) replaces

$$k_1\phi_1(x, y)+k_2\phi_2(x, y)+k_3\phi_3(x, y)+\ldots+k_s\phi_s(x, y)$$

by

$$K_1\phi_1(x, y)+K_2\phi_2(x, y)+K_3\phi_3(x, y)+\ldots+K_s\phi_s(x, y),$$

where

$$\left.\begin{aligned} K_1 &= k_1\phi_s(a, b) \\ K_2 &= k_1\phi_{s-1}(a, b)+k_2\phi_s(a, b) \\ K_3 &= k_1\phi_{s-2}(a, b)+k_2\phi_{s-1}(a, b)+k_3\phi_s(a, b) \\ \ldots &\quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ K_s &= k_1\phi_1(a, b)+k_2\phi_2(a, b)+\ldots+k_s\phi_s(a, b) \end{aligned}\right\}.$$

The proof, which is easy, is left to the reader.

3. Again, suppose in (β) we take $\alpha\beta = 1$; and let $f_1(x, y), f_2(x, y), \ldots, f_s(x, y)$ be the invariants of (β) given in *Proc. London Math. Soc.*, Ser. 2, Vol. 10, 1911, pp. 426, 427.

Then the substitution (γ) replaces $f_1(x, y)$ by

$$\{f_s(a, b)f_1(x, y)+f_{s-1}(a, b)f_2(x, y)+f_{s-2}(a, b)f_3(x, y)+\ldots+f_1(a, b)f_s(x, y)\},$$

when s is even, and by

$$\{f_s(a, b)f_1(x, y)+f_{s-1}(a, b)f_2(x, y)+f_{s-2}(a, b)f_3(x, y)+\ldots+f_1(a, b)f_s(x, y)\}$$
$$+\tfrac{1}{4}\{f_{s-1}(a, b)f_4(x, y)+f_{s-3}(a, b)f_6(x, y)+\ldots+f_4(a, b)f_{s-1}(x, y)\},$$

when s is odd.

In fact, since $f_1(x, y)$ is an invariant of (β), and (γ) is permutable with (β), therefore (γ) replaces $f_1(x, y)$ by an invariant of (β), *e.g.*, by

$$k_1f_1(x, y)+k_2f_2(x, y)+k_3f_3(x, y)+\ldots+k_{s-1}f_{s-1}(x, y)+k_sf_s(x, y).$$

The above theorem may be verified by direct substitution when $s = 1, 2, 3, 4, 5$. Assume it true for all values of s up to the one considered. If we put x_s and y_s zero in $f_t(x, y)$, we get the result of changing $x_1, x_2, x_3, \ldots, x_s$ into $0, x_1, x_2, \ldots, x_{s-1}$ and $y_1, y_2, y_3, \ldots, y_s$ into $0, y_1, y_2, \ldots, y_{s-1}$ in $-f_{t+2}(x, y)$. Thence by our assumption $k_1, k_2, \ldots, k_{s-2}$ have the values required by the given theorem. It is at once seen by direct substitution that $k_s = f_1(a, b)$. We leave to the reader the proof that $k_{s-1} = f_2(a, b)$, when s is even, and

$$k_{s-1} = f_2(a, b)+\tfrac{1}{4}f_4(a, b),$$

when s is odd.

4. We now prove the theorem (1) of § 1 when A is *symmetric*.

Suppose N is the canonical substitution into which A can be transformed [cf. (α) of § 1], so that $P^{-1}NP = A$. Let $PP' = C$, where P' is the substitution transposed to P and C is

$$x' = c_{t1}x_1+c_{t2}x_2+\ldots+c_{tm}x_m \quad (t = 1, 2, \ldots, m).$$

When A is transformed into N, $(x_1^2+x_2^2+\ldots+x_m^2)$ is transformed into $\Sigma c_{ij}x_ix_j$.

Now, since A is symmetric, we prove, as in *Messenger of Math.*, 1912, p. 152, that new variables can be chosen so that N is unaltered, and $\Sigma c_{ij}x_ix_j$ is the sum of quadratic forms with matrices of the type

$$\begin{vmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & k_2 \\ 0 & 0 & \ldots & k_2 & k_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & \ldots & k_{s-2} & k_{s-1} \\ 1 & k_2 & \ldots & k_{s-1} & k_s \end{vmatrix},$$

each corresponding to one of the substitutions of the type

$$x_1' = \alpha x_1+x_2, \ \ldots, \ x_{s-1}' = \alpha x_{s-1}+x_s, \ x_s' = \alpha x_s,$$

of which N is the direct product.

Apply (γ) to this, putting in (γ) $x_t = y_t$ and $a_t = b_t$; this does not alter N. Then, by § 2, we get the sum of quadratic forms with matrices of the type

$$\begin{vmatrix} 0 & 0 & \ldots & 0 & K_1 \\ 0 & 0 & \ldots & K_1 & K_2 \\ 0 & 0 & \ldots & K_2 & K_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ K_1 & K_2 & \ldots & K_{s-1} & K_s \end{vmatrix},$$

where $$K_1 = a_s^2, \quad K_2 = 2a_sa_{s-1}+a_s^2k_2,$$

$$K_3 = (2a_{s-2}a_s+a_{s-1}^2)+2a_sa_{s-1}k_2+a_s^2k_3, \ \ldots.$$

Now choose in turn a_s, a_{s-1}, a_{s-2}, ..., so that

$$K_1 = 1, \quad K_2 = K_3 = \ldots = 0;$$

and the required transformation of A into N and $x_1^2+x_2^2+\ldots+x_m^2$ into the standard form of § 1 (1) is completed.

It follows from this theorem (1) of § 1 that any two symmetric substitutions with the same invariant-factors (and therefore with the same canonical substitution) can be transformed into one another by an orthogonal substitution.

For suppose $PAP^{-1} = N$, where $PP' = C$; and suppose that when we put

$$b_{t1}x_1 + b_{t2}x_2 + \ldots + b_{tm}x_m \quad \text{for} \quad x_t,$$

N remains unaltered while $\Sigma c_{ij}x_i x_j$ takes the standard form of § 1 (1), which we call $\Sigma l_{ij}x_i x_j$. Then, if L, B are

$$x_t' = l_{t1}x_1 + l_{t2}x_2 + \ldots + l_{tm}x_m,$$

and

$$x_t' = b_{t1}x_1 + b_{t2}x_2 + \ldots + b_{tm}x_m,$$

respectively, we have $BCB' = L$, and $BN = NB$.

Then, if $Q = BP$, $QAQ^{-1} = N$, and $QQ' = L$, *i.e.*, we can transform A into N, and $x_1^2 + x_2^2 + \ldots + x_m^2$ into $\Sigma l_{ij}x_i x_j$ simultaneously.

Take D so that $DD' = L$ (*Messenger of Math.*, 1912, p. 147).

Then we have $$(D^{-1}Q)\,A\,(D^{-1}Q)^{-1} = D^{-1}ND,$$

and

$$(D^{-1}Q)(D^{-1}Q)' = E,$$

i.e., $D^{-1}Q$ is orthogonal.

Similarly any symmetric substitution with the same invariant-factors as A is transformable into $D^{-1}ND$ by an orthogonal substitution, and is therefore transformable into A by an orthogonal substitution.

For another proof, see Bôcher's *Introduction to Higher Algebra*, p. 302.

5. The theorem (2) of § 1 when A is *orthogonal* is established in a somewhat similar manner by making use of § 3 instead of § 2, though the details of the proof are more troublesome.

In the *Messenger of Math.*, 1912, pp. 153, 154, it is shown that, without altering N, $(x_1^2 + x_2^2 + \ldots + x_m^2)$ can be expressed as the sum of invariants of the type

$$k_1 f_1(x, y) + k_2 f_2(x, y) + k_3 f_3(x, y) + \ldots, \quad \text{when} \quad \alpha^2 \neq 1,$$

or $$k_1 f_1(x, y) + k_2 f_2(x, y) + k_3 f_3(x, y) + \ldots + m_2 f_2(x, x) + m_4 f_4(x, x)$$
$$+ m_6 f_6(x, x) + \ldots + n_2 f_2(y, y) + n_4 f_4(y, y) + n_6 f_6(y, y) + \ldots,$$

when $\alpha^2 = 1$, and s is even, or

$$k_1 f_1(X, X) + k_3 f_3(X, X) + k_5 f_5(X, X) + \ldots,$$

when $\alpha^2 = 1$, and s is odd.*

Apply the substitution equivalent to (γ), and we get another invariant

* Remembering that $f_t(x, x)$, $f_t(y, y)$, and $f_t(X, X)$ are zero if $s - t$ is odd.

of N. Then we can choose a_s, a_{s-1}, a_{s-2}, ..., and b_s, b_{s-1}, b_{s-2}, ..., in turn to reduce this new invariant to $f_1(x, y)$, $f_1(x, y)$, or $f_1(X, X)$ respectively as in § 4.

It follows from this theorem (2) of § 1, just as in § 4, that any two orthogonal substitutions with the same invariant-factors can be transformed into one another by an orthogonal substitution.

More generally, whenever we have two substitutions which are such that either can be transformed into a given substitution at the same time that $x_1^2+x_2^2+\ldots+x_m^2$ is transformed into a given quadratic form, the two substitutions are transformable into one another by an orthogonal substitution.

Again :—

Any quadratic invariant with non-zero determinant of any substitution can be changed into any other quadratic invariant with non-zero determinant by a change of variables which does not alter the substitution.

If the substitution is the canonical substitution N, it has just been established that each of the two quadratic invariants can be transformed without altering N into a standard form [which is the sum of invariants such as $f_1(x, y)$ and $f_1(X, X)$]. Hence one quadratic invariant can be transformed into the other without altering N.

In the general case suppose A is any substitution with two quadratic invariants of non-zero determinant. Let $A = P^{-1}NP$, where N is the canonical form of A ; and let Q be a substitution permutable with N transforming one of the invariants of N corresponding to one of the invariants of A into the other.

Then $P^{-1}QP$ transforms one invariant of A into the other.

CERTAIN POTENTIAL FUNCTIONS AND A NEW SOLUTION OF LAPLACE'S EQUATION

By T. J. I'A. BROMWICH.

[Read June 13th, 1912.—Received September 5th, 1912.]

IT is a familiar fact that the potential of a uniform circular disc of radius a can be expressed by series of zonal harmonics of the form

$$\Sigma A_n P_n(\mu)/r^{n+1} \quad \text{or} \quad \Sigma B_n r^n P_n(\mu),$$

according as r is greater or less than a. And a similar statement holds for the potential of a uniform hemisphere (with a slight modification at internal points); the exact formulæ for both cases are given below (pp. 103 and 120).

There is no difficulty in shewing that, if $r > a$, a series of the type $\Sigma A_n P_n(\mu)/r^{n+1}$ will serve to represent the potential, whatever the law of density in the disc or hemisphere may be, provided that it is symmetrical.*

But when the density is not uniform, the series for points at which $r < a$ become more complicated: in particular, if the density is an odd power of the distance from the centre certain new types of harmonic (here denoted by X_n) will appear, containing terms of the type $r^n P_n(\mu) \log(a/r)$. The precise form of X_n is investigated in § II below.

The formulæ for the potentials in §§ I, VI and the formulæ for X_n were obtained four years ago, but were first published in the Abstracts of the *Proceedings* of the June Meeting of the Society; after these Abstracts had appeared, it was remarked to me by Mr. G. N. Watson that the function X_n is connected with the function

$$Y_n = \frac{\partial}{\partial n}\left\{\left(\frac{r}{a}\right)^n P_n(\mu)\right\},$$

which is a more obvious solution of Laplace's equation. A verification of this fact will be found in § V below; and some of the potential functions

* More generally, a series of this type can be found for any solid of revolution, if r is greater than the greatest radius of the solid.

have been now expressed in terms of Y_n rather than in terms of X_n, as this enables these formulæ to be written more compactly.

I. *Potential of a Disc whose Density Varies as s^m, where s is the Distance from the Centre.*

We take the centre of the disc as origin and its plane as the plane of xy. Then the potential is given by the integral

$$V = \int_0^{2\pi} d\phi \int_0^a s\,ds\, \frac{ks^m}{\sqrt{(r^2+s^2-2rs\cos\gamma)}},$$

where $\cos\gamma = \sin\theta\cos\phi$, and r, θ are the three dimensional polar coordinates of the point at which V is to be found.

We can expand the square root in a series proceeding according to powers of s/r (or r/s, as the case may be), so that

$$\frac{1}{\sqrt{(r^2+s^2-2rs\cos\gamma)}} = \frac{1}{r} + \sum_1^\infty \frac{s^n}{r^{n+1}} P_n(\cos\gamma), \text{ if } s<r,$$

$$\text{or} \quad \frac{1}{s} + \sum_1^\infty \frac{r^n}{s^{n+1}} P_n(\cos\gamma), \text{ if } s>r.$$

Also it is known that

$$\int_0^{2\pi} P_n(\cos\gamma)\,d\phi = 2\pi P_n(0)\, P_n(\cos\theta),$$

and that $P_n(0) = 0$, if n is odd,

or $$P_n(0) = (-1)^m \frac{1\cdot 3\ldots(n-1)}{2\cdot 4\ldots n}, \text{ if } n \text{ is even} = 2m.$$

For brevity in the subsequent work we write

$$C_n = \frac{1\cdot 3\ldots(2n-1)}{2\cdot 4\ldots 2n},$$

and then $$P_{2n}(0) = (-1)^n C_n.$$

On using these facts we get two formulæ for V, according as r is greater or less than a, which we shall denote by V_0 and V_1 respectively; thus, if $r>a$, we have

$$(1)\quad V_0 = 2\pi k \int_0^a s^{m+1} ds \left\{ \frac{1}{r} + \sum_1^\infty (-1)^n C_n \frac{s^{2n}}{r^{2n+1}} P_{2n}(\cos\theta) \right\}$$

$$= 2\pi k a^{m+1} \left\{ \frac{1}{m+2}\frac{a}{r} + \sum_1^\infty (-1)^n \frac{C_n}{m+2n+2} \left(\frac{a}{r}\right)^{2n+1} P_{2n}(\cos\theta) \right\}.$$

But, if $r<a$, we have to divide the integral into two parts, from 0 to r,

and from r to a, and then we find

$$(2) \qquad V_1 = 2\pi k \int_0^r s^{m+1} ds \left\{ \frac{1}{r} + \sum_1^\infty (-1)^n C_n \frac{s^{2n}}{r^{2n+1}} P_{2n}(\cos\theta) \right\}$$

$$+ 2\pi k \int_r^a s^{m+1} ds \left\{ \frac{1}{s} + \sum_1^\infty (-1)^n C_n \frac{r^{2n}}{s^{2n+1}} P_{2n}(\cos\theta) \right\}.$$

Thus, assuming that m is not an odd integer (we have tacitly assumed already that $m+2 > 0$, so that the values excluded now are -1, $+1$, $+3$, $+5$, ...), we find the result

$$(3) \quad V_1 = 2\pi k r^{m+1} \left\{ \frac{-1}{(m+1)(m+2)} + \sum_1^\infty (-1)^n \frac{C_n (4n+1) P_{2n}(\cos\theta)}{(2n+m+2)(2n-m-1)} \right\}$$

$$+ 2\pi k a^{m+1} \left\{ \frac{1}{m+1} - \sum_1^\infty (-1)^n \frac{C_n}{2n-m-1} \left(\frac{r}{a}\right)^{2n} P_{2n}(\cos\theta) \right\}.$$

Since V_1 satisfies Laplace's equation, we are led to conjecture that the first term in V_1 must be of the form $2\pi k A r^{m+1} P_{m+1}(\cos\theta)$; but (since $m+1$ is not an even integer) this function has not the property of remaining unchanged when we change the sign of $\cos\theta$. Thus we amend our conjecture by trying to express the term in the form

$$2\pi k A r^{m+1} P_{m+1}(\cos\theta), \quad \text{when} \quad \cos\theta > 0,$$

or

$$2\pi k A r^{m+1} P_{m+1}(-\cos\theta), \quad \text{when} \quad \cos\theta < 0;$$

and this function has the further property of remaining finite in both regions, even when m is not an integer.

It is now easy to confirm the conjecture, for if we write

$$P_{m+1}(|\mu|) = B_0 + \sum_1^\infty B_n P_n(\mu),$$

where

$$\mu = \cos\theta,$$

we get

$$2B_0 = \int_{-1}^{+1} P_{m+1}(|\mu|)\, d\mu = 2\int_0^1 P_{m+1}(\mu)\, d\mu,$$

and so

$$B_0 = -\frac{1}{(m+1)(m+2)} \left[(1-\mu^2) \frac{dP_{m+1}}{d\mu} \right]_0^1 = -\frac{p}{(m+1)(m+2)},$$

where

$$p = -\left[\frac{dP_{m+1}}{d\mu} \right]_{\mu=0}.$$

Also

$$\frac{2B_n}{2n+1} = \int_{-1}^{+1} P_{m+1}(|\mu|)\, P_n(\mu)\, d\mu = 0 \quad (\text{if } n \text{ is odd})$$

$$= 2\int_0^1 P_{m+1}(\mu)\, P_n(\mu)\, d\mu \quad (\text{if } n \text{ is even}).$$

Now

$$\{n(n+1) - (m+1)(m+2)\} \int P_{m+1} P_n\, d\mu = (1-\mu^2) \left(P_n \frac{dP_{m+1}}{d\mu} - P_{m+1} \frac{dP_n}{d\mu} \right),$$

by using Legendre's equation for the harmonics. Also, when n is even, $dP_n/d\mu$ is zero at $\mu = 0$: and so we find

$$B_n = \frac{(2n+1)\,pP_n(0)}{(n-m-1)(n+m+2)} \quad \text{(if } n \text{ is even)}.$$

Thus
$$B_{2n} = \frac{(4n+1)\,p\,(-1)^n\,C_n}{(2n-m-1)(2n+m+2)},$$

and so we get the result

$$P_{m+1}(|\mu|) = p\left\{-\frac{1}{(m+1)(m+2)} + \sum_1^\infty \frac{(-1)^n\,C_n(4n+1)\,P_{2n}}{(2n-m-1)(2n+m+2)}\right\}.$$

Thus the potential of the disc, when $r < a$, takes the form

$$(4) \qquad V_1 = 2\pi(k/p)\,r^{m+1}P_{m+1}(|\mu|)$$

$$+2\pi k a^{m+1}\left\{\frac{1}{m+1} - \sum_1^\infty (-1)^n \frac{C_n}{2n-m-1}\left(\frac{r}{a}\right)^{2n} P_{2n}(\cos\theta)\right\},$$

where
$$p = -\left(\frac{dP_{m+1}}{d\mu}\right)_0 = -\frac{2}{\sqrt{\pi}}\cos(\tfrac{1}{2}m\pi)\frac{\Gamma(\frac{1}{2}m+\frac{3}{2})}{\Gamma(\frac{1}{2}m+1)} \quad \text{(see p. 104)},$$

or
$$\frac{1}{p} = -\frac{1}{(m+1)(m+2)} + \sum_1^\infty \frac{(-1)^n\,C_n(4n+1)}{(2n+m+2)(2n-m-1)},$$

the second formula for p being obtained by equating the series (3) and (4) in the special case $\theta = 0$.

The form (3) enables us to see at a glance the continuity of V and $\partial V/\partial r$ at the sphere $r = a$; but (4) is useful also, as indicating at once the fact that V satisfies Laplace's equation.

As an illustration we may refer to the familiar case of a *uniform disc*, for which $m = 0$; here $p = -1$, and so (4) gives

$$V_1 = -2\pi kr\,|\mu| + 2\pi k a\left\{1 - \sum_1^\infty (-1)^n \frac{C_n}{2n-1}\left(\frac{r}{a}\right)^{2n} P_{2n}\right\},$$

while (1) gives $\quad V_0 = 2\pi k a\left\{\frac{1}{2}\frac{a}{r} + \sum_1^\infty (-1)^n \frac{C_n}{2n+2}\left(\frac{a}{r}\right)^{2n+1} P_{2n}\right\}.$

The continuity at $r = a$ of these formulæ was established by a special investigation (here extended) in the *Philosophical Magazine* (August, 1901); it will be seen that our present formulæ are in agreement with those found in the ordinary text books, except for the fact that some books seem to suppose that the first term in V_1 is always $-2\pi kr\mu$.

As a further verification of (4), it may be noted that the discontinuity in the normal attraction at the surface of the disc leads to the equation

$$\left(\frac{\partial V_1}{\partial z}\right)_- - \left(\frac{\partial V_1}{\partial z}\right)_+ = 4\pi\sigma,$$

or, in our case,

$$-\frac{1}{r}\left(\frac{\partial V_1}{\partial \mu}\right)_+ = 2\pi\sigma = 2\pi k r^m,$$

which agrees with (4).

We now proceed to discuss the necessary modification of (4), when m is an odd integer; and, on the whole, the simplest method appears to be the application of a limiting process. To carry this out, we shall need to establish first the formula given in (4) for the value of the constant p.

In general the relation

$$(1-\mu^2)\frac{dP_n}{d\mu} = n(P_{n-1}-\mu P_{n+1})$$

is easily established for all values of n; and so, putting $\mu = 0$ and $n = m+1$, we find

$$-p = (m+1)P_m(0).$$

Also, *when μ is positive*, we have

$$P_m(\mu) = \frac{1}{\pi}\int_0^\pi \left\{\mu + \iota\sqrt{(1-\mu^2)}\cos\phi\right\}^m d\phi,$$

where the principal value of the complex power is to be used. Thus, if

$$\mu \to 0 \quad \text{and} \quad 0 \leqslant \phi \leqslant \tfrac{1}{2}\pi,$$

we have as the limiting value

$$\left\{\mu + \iota\sqrt{(1-\mu^2)}\cos\phi\right\}^m \to e^{\frac{1}{2}\iota m\pi}\cos^m\phi,$$

but when $\frac{1}{2}\pi \leqslant \phi \leqslant \pi$, the formula is

$$\left\{\mu + \iota\sqrt{(1-\mu^2)}\cos\phi\right\}^m \to e^{-\frac{1}{2}\iota m\pi}\cos^m\phi_1, \quad \text{if} \quad \phi_1 = \pi-\phi.$$

Consequently we find

$$P_m(0) = \frac{1}{\pi}\int_0^{\frac{1}{2}\pi} e^{\frac{1}{2}\iota m\pi}\cos^m\phi\,d\phi + \frac{1}{\pi}\int_{\frac{1}{2}\pi}^{\pi} e^{-\frac{1}{2}\iota m\pi}\cos^m(\pi-\phi)\,d\phi = \frac{1}{\pi}\int_0^{\frac{1}{2}\pi}\left(e^{\frac{1}{2}\iota m\pi}+e^{-\frac{1}{2}\iota m\pi}\right)\cos^m\phi\,d\phi,$$

or

$$P_m(0) = \frac{2}{\pi}\cos\left(\tfrac{1}{2}m\pi\right)\int_0^{\frac{1}{2}\pi}\cos^m\phi\,d\phi = \frac{1}{\sqrt{\pi}}\cos\left(\tfrac{1}{2}m\pi\right)\frac{\Gamma\left(\frac{1}{2}m+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}m+1\right)}.$$

Thus we find

$$-p = (m+1)P_m(0) = \frac{2}{\sqrt{\pi}}\cos\left(\tfrac{1}{2}m\pi\right)\frac{\Gamma\left(\frac{1}{2}m+\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}m+1\right)},$$

which is the result already stated in (4) above.

To deal with the limiting case, it is convenient to write $m+1 = 2\nu+t$, and then to make t tend to zero; and a glance at (4) shews that we shall need to evaluate p as far as terms in t^2. Thus we write to this order

$$\cos\left(\tfrac{1}{2}m\pi\right) = \sin\left(\nu\pi+\tfrac{1}{2}t\pi\right) = (-1)^\nu\sin\left(\tfrac{1}{2}t\pi\right) = (-1)^\nu\left(\tfrac{1}{2}t\pi\right),$$

and so the quotient of the two Gamma-functions need not be evaluated beyond terms of the first degree in t. Thus we can use the approximate formulæ

$$\Gamma\left(\tfrac{1}{2}m+\tfrac{3}{2}\right) = \Gamma\left(\nu+1+\tfrac{1}{2}t\right) = \Gamma(\nu+1)\left\{1+\tfrac{1}{2}t\psi(\nu+1)\right\},$$

and

$$\Gamma\left(\tfrac{1}{2}m+1\right) = \Gamma\left(\nu+\tfrac{1}{2}+\tfrac{1}{2}t\right) = \Gamma\left(\nu+\tfrac{1}{2}\right)\left\{1+\tfrac{1}{2}t\psi\left(\nu+\tfrac{1}{2}\right)\right\},$$

where

$$\psi(x) = \Gamma'(x)/\Gamma(x).$$

Thus

$$\frac{\Gamma\left(\frac{1}{2}m+\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}m+1\right)} = \frac{\Gamma(\nu+1)}{\Gamma\left(\nu+\frac{1}{2}\right)}\left[1+\tfrac{1}{2}t\left\{\psi(\nu+1)-\psi\left(\nu+\tfrac{1}{2}\right)\right\}\right],$$

and
$$\frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu+1)} = \sqrt{\pi}\, C_\nu,$$

$$\psi(\nu+1)-\psi(\nu+\tfrac{1}{2}) = 2\left(\log 2-1+\tfrac{1}{2}-\tfrac{1}{3}+\ldots+\frac{1}{2\nu}\right) = 2R_\nu, \text{ say};^*$$

with the special values $\Gamma(\frac{1}{2})/\Gamma(1) = \sqrt{\pi}$, $\psi(1)-\psi(\frac{1}{2}) = 2\log 2$.

Thus we obtain the approximate formula

$$-\frac{1}{p} = (-1)^\nu C_\nu\left(\frac{1}{t}-R_\nu\right),$$

which applies also for the case $\nu = 0$, if we write

$$C_0 = 1, \quad R_0 = \log 2.$$

To calculate the limiting form of equation (4), we can write $m+1 = 2\nu$ in all the terms except the first, and the term in the bracket for which $n = \nu$; thus we get†

$$(5)\quad V_1 = \pi k a^{2\nu}\left\{\frac{1}{\nu}-\sum_1^\infty{}'(-1)^n\frac{C_n}{n-\nu}\left(\frac{r}{a}\right)^{2n}P_{2n}\right\}$$

$$+2\pi k a^{2\nu}(-1)^\nu C_\nu \lim_{t\to 0}\left\{\frac{1}{t}\left(\frac{r}{a}\right)^{2\nu}P_{2\nu}-\left(\frac{1}{t}-R_\nu\right)\left(\frac{r}{a}\right)^{2\nu+t}P_{2\nu+t}\right\},$$

where for brevity we are now considering only the case when μ is positive; to deal with negative values of μ, we need only change the final conclusions by writing $|\mu|$ for μ.

Hence for the law of density $ks^{2\nu-1}$, we obtain the formula for the potential of the disc at points for which $r < a$,

$$(6)\quad V_1 = \pi k a^{2\nu}\left\{\frac{1}{\nu}-\sum_1^\infty{}'(-1)^n\frac{C_n}{n-\nu}\left(\frac{r}{a}\right)^{2n}P_{2n}\right\}$$

$$+2\pi k a^{2\nu}(-1)^\nu C_\nu\left\{R_\nu\left(\frac{r}{a}\right)^{2\nu}P_{2\nu}-Y_{2\nu}\right\},$$

where we write

$$(7)\quad Y_n = \frac{\partial}{\partial n}\left\{\left(\frac{r}{a}\right)^n P_n\right\}.$$

An alternative form, equivalent to (6), will be found in (11) below.

---

* See, for instance, my book on *Infinite Series*, p. 475, Ex. 42.

† The symbol Σ' is used in the sense often adopted by Weierstrass and others; the accent implies that certain values of n are exoluded (here $n = \nu$) which would involve infinities in the summation.

In the special case $\nu = 0$, the formula (6) becomes

$$(6a) \qquad V_1 = \pi k \sum_1^\infty (-1)^{n-1} \frac{C_n}{n} \left(\frac{r}{a}\right)^{2n} P_{2n} + 2\pi k (\log 2 - Y_0),$$

which is the potential (when $r < a$, $\mu > 0$) of a disc whose density is k/s.

We shall consider below more fully the form of the function Y_n, which does not appear to have presented itself previously. But before discussing this, it will be useful to remark that the limiting process which we have used can be avoided by writing $m = 2\nu - 1$ in the integral (2) for the potential. It will then be seen that the potential takes the form

$$(8) \qquad \pi k a^{2\nu} \left\{ \frac{1}{\nu} - \sum_1^\infty{}' \frac{(-1)^n C_n}{n-\nu} \left(\frac{r}{a}\right)^{2n} P_{2n} \right\}$$

$$+ 2\pi k r^{2\nu} \left[- \frac{1}{2\nu(2\nu+1)} + \sum_1^\infty{}' \frac{(-1)^n C_n (4n+1)}{(2n+2\nu+1)(2n-2\nu)} P_{2n} \right.$$

$$\left. + (-1)^\nu C_\nu P_{2\nu} \left\{ \frac{1}{4\nu+1} + \log \left(\frac{a}{r}\right) \right\} \right].$$

The bracket in the last line takes the form $(-1)^\nu C_\nu P_{2\nu} \log (a/r) + f(\theta)$; and so we are led to conjecture the existence of a solution of Laplace's equation which takes the form

$$r^n \{ P_n \log (a/r) + f_1(\theta) \},$$

n being an integer. We proceed next (in § II) to investigate the form of this solution, and we shall afterwards connect the solution with the function Y_n defined in (7).

As a matter of fact the form of the potential was originally determined by the second process; but the limiting process has the advantage of indicating more clearly the relation of this case to the general formula (4).

II. *Investigation of certain Harmonics.*

The simplest case of these harmonics corresponds to (8), when we suppose $\nu = 0$ or $m = 1$; the integral (2) then leads to a harmonic of the form

$$1 + \log \left(\frac{a}{r}\right) + \sum_1^\infty (-1)^n C_n \frac{4n+1}{2n(2n+1)} P_{2n}.$$

This series of zonals, as we shall prove in a moment, is equal to

$$2 \log 2 - 1 - \log (1 + |\mu|),$$

so that the harmonic is simply

$$\log\left(\frac{4a}{r+|z|}\right),$$

which is at once seen to be a solution of Laplace's equation—it is, of course, a well-known solution, although not often wanted in mathematical physics.

To verify the identity quoted above, it is easiest to expand the function

$$\log(1+|\mu|)$$

in a series of zonal harmonics. It is readily seen that no odd harmonics can occur, and if we write

$$\log(1+|\mu|) = A_0 + \sum_1^\infty A_n P_{2n}(\mu),$$

we find at once

$$A_0 = \tfrac{1}{2}\int_{-1}^{+1} \log(1+|\mu|)\, d\mu = \int_0^1 \log(1+\mu)\, d\mu$$

$$= \left[(1+\mu)\log(1+\mu) - \mu\right]_0^1 = 2\log 2 - 1.$$

Also

$$A_n = \frac{4n+1}{2}\int_{-1}^{+1} \log(1+|\mu|)\, P_{2n}(\mu)\, d\mu$$

$$= (4n+1)\int_0^1 \log(1+\mu)\, P_{2n}(\mu)\, d\mu.$$

Thus using the differential equation for P_{2n}, we find

$$A_n = -\frac{4n+1}{2n(2n+1)}\left[\log(1+\mu)(1-\mu^2)\frac{dP_{2n}}{d\mu}\right]_0^1 + \frac{4n+1}{2n(2n+1)}\int_0^1 (1-\mu)\frac{dP_{2n}}{d\mu}\, d\mu.$$

The integrated part is zero at both limits, and on integrating by parts again, we find

$$A_n = \frac{4n+1}{2n(2n+1)}\left[(1-\mu)P_{2n}\right]_0^1 + \frac{4n+1}{2n(2n+1)}\int_0^1 P_{2n}\, d\mu.$$

Here the last term is zero, and so

$$A_n = -\frac{4n+1}{2n(2n+1)}P_{2n}(0) = -\frac{4n+1}{2n(2n+1)}(-1)^n C_n,$$

which confirms the result stated.

Returning to the formula (8), we now get the formula for the potential of a surface-density k/s, at points for which $r < a$,

$$(8a) \qquad \pi k \sum_1^\infty (-1)^{n-1}\frac{C_n}{n}\left(\frac{r}{a}\right)^{2n} P_{2n} + 2\pi k \log\left(\frac{4a}{r+|z|}\right),$$

a result which is really the same as (6a).

To obtain the general case of the harmonics suggested by (8) we try to find a harmonic function of the form

$$V_n = Z_n \log\left(\frac{r+z}{2a}\right) + r^n f(\mu),$$

the abbreviation Z_n being used for $r^n P_n(\mu)$; it will be supposed here that z and μ are positive.

Now $\Delta Z_n = 0$, where Δ denotes Laplace's operator; and as we have just seen

$$\Delta \log \{(r+z)/2a\} = 0,$$

so that $$\Delta \left\{ Z_n \log \left(\frac{r+z}{2a}\right)\right\} = 2\left(\frac{\partial Z_n}{\partial x}\frac{x/r}{r+z} + \frac{\partial Z_n}{\partial y}\frac{y/r}{r+z} + \frac{\partial Z_n}{\partial z}\frac{1+z/r}{r+z}\right)$$

$$= \frac{2}{r+z}\left(\frac{nZ_n}{r} + \frac{\partial Z_n}{\partial z}\right).$$

Now it is easy to verify that

$$\frac{\partial Z_n}{\partial z} = nZ_{n-1},$$

and so we have the formula

$$\Delta \left\{ Z_n \log \left(\frac{r+z}{2a}\right)\right\} = \frac{2n}{1+\mu} r^{n-2}(P_n + P_{n-1}).$$

Also $$\Delta \{r^n f(\mu)\} = r^{n-2}\left[n(n+1)f + \frac{d}{d\mu}\left\{(1-\mu^2)\frac{df}{d\mu}\right\}\right],$$

so that the condition $\Delta X_n = 0$ leads to

$$\frac{2n}{1+\mu}(P_{n-1}+P_n) + n(n+1)f + \frac{d}{d\mu}\left\{(1-\mu^2)\frac{df}{d\mu}\right\} = 0.$$

It is easy to see that the first term in this formula is a polynomial of degree $(n-1)$, and that it is equal to twice*

$$(2n-1)P_{n-1} - (2n-3)P_{n-2} + (2n-5)P_{n-3} - \ldots + (-1)^{n-1}.$$

We assume accordingly for f,

$$f(\mu) = A_n P_n + A_{n-1}P_{n-1} + A_{n-2}P_{n-2} + \ldots + A_0,$$

---

* In fact, if the given expression is multiplied by $(1+\mu)$, we get for the product

$$\begin{aligned} & nP_n + (2n-1)P_{n-1} + (n-1)P_{n-2} \\ & - (n-1)P_{n-1} - (2n-3)P_{n-2} - (n-2)P_{n-2} \\ & + (n-2)P_{n-2} + (2n-5)P_{n-3} + (n-3)P_{n-4} \\ & + \ldots + \ldots \\ & = nP_n + nP_{n-1}. \end{aligned}$$

and then our condition yields

$$2\{(2n-1)P_{n-1}-(2n-3)P_{n-2}+\ldots+(-1)^{n-1}P_0\}$$
$$+\sum_{s=0}^{n-1}\{n(n+1)-s(s+1)\}A_sP_s=0.$$

Hence $A_s=(-1)^{n-s}\dfrac{(2s+1)}{(n-s)(n+s+1)}$, for $s=0, 1, \ldots, n-1$;

and so, taking $A_n=0$ for simplicity, we find

$$f(\mu)=-2\left\{\frac{2n-1}{1.2n}P_{n-1}-\frac{2n-3}{2(2n-1)}P_{n-2}+\frac{2n-5}{3(2n-2)}P_{n-3}-\ldots\right.$$
$$\left.+(-1)^{n-1}\frac{1}{n(n+1)}\right\}.$$

Thus our new solution of Laplace's equation takes the form

$$(9)\quad X_n=r^nP_n(\mu)\log\left(\frac{r+z}{2a}\right)$$
$$-2r^n\left\{\frac{2n-1}{1.2n}P_{n-1}-\frac{2n-3}{2(2n-1)}P_{n-2}+\ldots+\frac{(-1)^{n-1}}{n(n+1)}\right\},$$

and, of course, any arbitrary multiple of $r^nP_n(\mu)$ may be added to this without affecting the conclusion.

To obtain the solution commonly denoted by $r^nQ_n(\mu)$, we change the sign of μ in (9), and deduce, after division by $(-1)^n$, a second solution

$$(9a)\ r^nP_n(\mu)\log\left(\frac{r-z}{2a}\right)+2r^n\left\{\frac{2n-1}{1.2n}P_{n-1}+\frac{2n-3}{2(2n-1)}P_{n-2}+\ldots+\frac{1}{n(n+1)}\right\}.$$

Thus on subtracting $(9a)$ from (9) and dividing by 2, we get the solution

$$(9b)\quad \tfrac{1}{2}r^nP_n(\mu)\log\left(\frac{1+\mu}{1-\mu}\right)-2r^n\left\{\frac{2n-1}{1.2n}P_{n-1}+\frac{2n-5}{3(2n-2)}P_{n-3}+\ldots\right\},$$

which is one of the known formulæ for $r^nQ_n(\mu)$.

It will be seen that $(9b)$ contains r only through the power r^n; but, in contrast to this apparent advantage, there is the drawback that $(9b)$ has an infinity when μ is equal to either -1 or $+1$. On the contrary, (9) remains finite at $\mu=+1$, while $(9a)$ is finite at $\mu=-1$; thus we should expect to find (9) of use on the positive side of the plane of xy, while $(9a)$ will serve for points on the negative side of this plane.

In view of these calculations it is evident that when $\mu>0$, we can

write the solution Y_n of (7) in the form

$$Y_n = \{X_n + A_n r^n P_n(\mu)\}/a^n,$$

where the coefficient A_n is not easily found directly, but is obtained indirectly in (10) below.

III. *Specification of Harmonics by their Values on the Axis of Symmetry.*

It is useful to notice that the harmonics which we have been considering are completely specified by their values on the positive axis of symmetry (for which $\mu = 1$). Thus the harmonic $r^{m+1}P_{m+1}(\mu)$ is represented by r^{m+1} on the axis, whatever may be the value of m (whether an integer or not).

Similarly the harmonic defined in (7) by the equation

$$Y_n = \frac{\partial}{\partial n}\left\{\left(\frac{r}{a}\right)^n P_n(\mu)\right\}$$

is represented on the axis by $(r/a)^n \log(r/a)$; while the function X_n, defined in (9) above, is represented by

$$r^n \log\left(\frac{r}{a}\right) - 2r^n\left\{\frac{2n-1}{1.2n} - \frac{2n-3}{2(2n-1)} + \ldots + \frac{(-1)^{n-1}}{n(n+1)}\right\}$$

$$= r^n \log\left(\frac{r}{a}\right) - 2r^n\left(1 - \tfrac{1}{2} + \tfrac{1}{3} - \ldots - \frac{1}{2n}\right).$$

Thus, in particular, we can obtain the relation between X_n and Y_n: on the axis we have the relation

$$a^n Y_n - X_n = 2r^n\left(1 - \tfrac{1}{2} + \tfrac{1}{3} - \ldots - \frac{1}{2n}\right),$$

and so, in general, we have the identity

$$a^n Y_n = X_n + 2r^n P_n(\mu)\left(1 - \tfrac{1}{2} + \tfrac{1}{3} - \ldots - \frac{1}{2n}\right). \tag{10}$$

We shall obtain later (in § V) a direct proof of the relation (10); but it should be noted here that (10) leads to a formula equivalent to (6), for the potential of the disc with density $ks^{2\nu-1}$, namely,

$$V_1 = \pi k a^{2\nu}\left\{\frac{1}{\nu} - \sum_1^{\infty}{}'(-1)^n \frac{C_n}{n-\nu}\left(\frac{r}{a}\right)^{2n} P_{2n}\right\} \tag{11}$$

$$+ 2\pi k(-1)^\nu C_\nu\{r^{2\nu}P_{2\nu}(\log 2 - S_\nu - 2S_{2\nu}) - X_{2\nu}\},$$

where $$S_\nu = 1-\tfrac{1}{2}+\tfrac{1}{3}-\ldots-\frac{1}{2\nu},$$

and $$S_{2\nu} = 1-\tfrac{1}{2}+\tfrac{1}{3}-\ldots-\frac{1}{4\nu}.$$

For instance, with $\nu = 1$, we have

$$S_1 = \tfrac{1}{2}, \quad S_2 = \tfrac{7}{12}, \quad S_1+2S_2 = \tfrac{5}{3};$$

and so the potential corresponding to a density ks is

$$(11a) \qquad V_1 = \pi k a^2 \left\{ 1-\sum_1^\infty{}' (-1)^n \frac{C_n}{n-1} \left(\frac{r}{a}\right)^{2n} P_{2n} \right\}$$

$$-\pi k \left\{ r^2 P_2 (\log 2-\tfrac{5}{3}) - X_2 \right\},$$

a result proposed in the *Mathematical Tripos*, Part II, 1909, and given in the abstract of this paper.

To illustrate the method of utilising the value of the potential on the axis to determine the potential generally,* we may apply the process to a disc whose surface density is ks^m. At a point on the axis, we have

$$V = 2\pi k \int_0^a \frac{s^{m+1}\,ds}{\sqrt{(s^2+r^2)}}.$$

Thus, taking $r > a$, we get

$$V_0 = 2\pi k \int_0^a s^{m+1}\,ds \left\{ \frac{1}{r} + \sum_1^\infty (-1)_n C_n \frac{s^{2n}}{r^{2n+1}} \right\}$$

$$= 2\pi k a^{m+1} \left\{ \frac{1}{m+2} \frac{a}{r} + \sum_1^\infty (-1)_n \frac{C_n}{2n+m+2} \left(\frac{a}{r}\right)^{2n+1} \right\},$$

which leads at once to formula (1) above. But, if $r < a$, we have

$$V_1 = 2\pi k \int_0^r s^{m+1}\,ds \left\{ \frac{1}{r} + \sum_1^\infty (-1)^n C_n \frac{s^{2n}}{r^{2n+1}} \right\}$$

$$+ 2\pi k \int_r^a s^{m+1}\,ds \left\{ \frac{1}{s} + \sum_1^\infty (-1)^n C_n \frac{r^{2n}}{s^{2n+1}} \right\}$$

$$= 2\pi k r^{m+1} \left\{ -\frac{1}{(m+1)(m+2)} + \sum_1^\infty (-1)^n C_n \frac{4n+1}{(2n+m+2)(2n-m-1)} \right\}$$

$$+ 2\pi k a^{m+1} \left\{ \frac{1}{m+1} - \sum_1^\infty (-1)^n \frac{C_n}{2n-m-1} \left(\frac{r}{a}\right)^{2n} \right\},$$

a result which leads at once to (3) above.

It is not very easy, however, to apply a similar method, in general, to obtain (6); the formula obtained is substantially (8), but the coefficient of $r^{2\nu} P_{2\nu}$ is troublesome to express in a simple form. For the special cases $\nu = 0$, $\nu = 1$, we can proceed as follows.

* Of course the method is familiar (and goes back to Legendre) in the case when harmonics of *integral* order suffice to express the potential V_1.

When $\nu = 0$, we have

$$V = 2\pi \int_0^a \frac{k\,ds}{\sqrt{(s^2+r^2)}} = 2\pi k \sinh^{-1}\left(\frac{a}{r}\right) = 2\pi k \log\left\{\frac{a+\sqrt{(a^2+r^2)}}{r}\right\},$$

and thus we find,* if $r < a$,

$$V_1 = 2\pi k\left\{\log\frac{2a}{r} + \frac{1}{2}\frac{r^2}{2a^2} - \frac{1.3}{2.4}\frac{r^4}{4a^4} + \frac{1.3.5}{2.4.6}\frac{r^6}{6a^6} - \ldots\right\},$$

which leads to (6a) at once.

Similarly, when $\nu = 1$, we find

$$V = 2\pi k \int_0^a \frac{s^2 ds}{\sqrt{(s^2+r^2)}} = \pi k\left\{a\sqrt{(a^2+r^2)} - r^2 \sinh^{-1}\left(\frac{a}{r}\right)\right\},$$

and expanding similarly we obtain the special case of (6) which corresponds to $\nu = 1$; the terms of special interest here are those in r^2, which are seen to be

$$\pi k\left\{\tfrac{1}{2}r^2 - r^2 \log(2a/r)\right\},$$

and these lead to the general values

$$\pi k\left\{a^2 Y_2 - (\log 2 - \tfrac{1}{2})\, r^2 P_2\right\}.$$

IV. *The Solutions Inverse to X_n.*

Professor Hobson pointed out to me, in 1909, that the solutions X_n can be derived from known solutions as follows.

It is a familiar fact that if $f(x, y, z)$ is a solution of Laplace's equation, then another solution can be derived by inversion, which leads to the form

$$\frac{1}{r} f\left(\frac{x}{r^2}, \frac{y}{r^2}, \frac{z}{r^2}\right).$$

Now $\log\frac{1}{2}(r+z)$ is a known solution† and consequently we have another

$$\frac{1}{r}\log\left(\frac{r+z}{2r^2}\right).$$

From this we can derive others by differentiation in the form

$$(12)\quad \frac{(-1)^n}{n!}\frac{\partial^n}{\partial z^n}\left\{\frac{1}{r}\log\left(\frac{r+z}{2r^2}\right)\right\}$$
$$= \frac{P_n(\mu)}{r^{n+1}}\log\left(\frac{r+z}{2r^2}\right)$$
$$+ \left\{-\frac{P_{n-1}(\mu)}{r^n}\left(\frac{1}{r} - \frac{2z}{r^2}\right) + \frac{1}{2!}\frac{P_{n-2}(\mu)}{r^{n-1}}\left(-\frac{z}{r^3} - \frac{2}{r^2} + \frac{4z^2}{r^4}\right) - \ldots\right\},$$

* See, for example, my book, *Infinite Series*, p. 163, Ex. 14.

† Actually $\log(r+z) = \int \frac{dz}{r}$, so that this is an obvious solution.

by applying Leibnitz's theorem on repeated differentiation. Now this expression is of the form

$$\frac{P_n(\mu)}{r^{n+1}} \log\left(\frac{r+z}{2r^2}\right) + \frac{f_n(\mu)}{r^{n+1}},$$

where f_n is a polynomial of degree n in μ. Inverting again, we derive a solution of the form

$$r^n P_n(\mu) \log \tfrac{1}{2}(r+z) + r^n f_n(\mu). \tag{13}$$

To actually obtain a formula for $f_n(\mu)$ it is, however, easier to apply the differential equation directly to the function (13); and so we return substantially to the calculations given in § II above. The actual relation of this harmonic to X_n and Y_n can be determined by examining its form at points on the axis, for which (12) becomes

$$\frac{(-1)^n}{n!}\frac{d^n}{dr^n}\left\{\frac{1}{r}\log\left(\frac{1}{r}\right)\right\} = \frac{1}{r^{n+1}}\left(\log\frac{1}{r} + 1 + \tfrac{1}{2} + \ldots + \frac{1}{n}\right).$$

Consequently (13) becomes on the axis

$$r^n\left(\log r + 1 + \tfrac{1}{2} + \ldots + \frac{1}{n}\right),$$

and (13) is therefore (in general) equal to

$$Y_n + r^n P_n(\mu)\left(1 + \tfrac{1}{2} + \ldots + \frac{1}{n}\right),$$

where we have taken $a = 1$ in the definition of Y_n.

V. *A direct Determination of a Formula for* Y_n.

To abbreviate the formulæ, put $a = 1$ in the definition (7), and then

$$Y_n = \frac{\partial}{\partial n}(r^n P_n) = r^n\left(P_n \log r + \frac{\partial P_n}{\partial n}\right). \tag{14}$$

To proceed further we shall use the complex integral

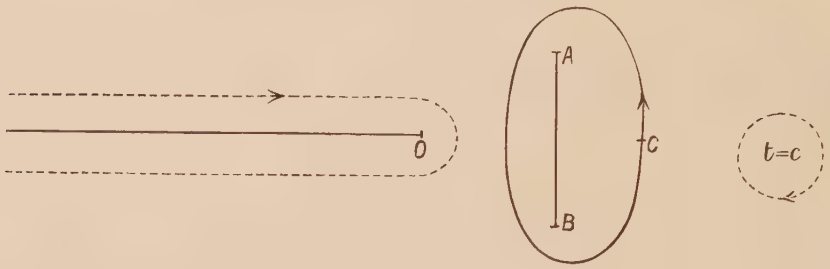
$$P_n(\mu) = \frac{1}{2\pi\iota}\int \frac{t^n dt}{\sqrt{(1-2\mu t+t^2)}}. \tag{15}$$

In order to make the integrand single-valued we make cuts in the t-plane, along the negative half of the real axis, and along the line which joins the branch-points A, B given by

$$1-2\mu t+t^2 = 0,$$

or by $$t = \mu+\iota\sqrt{(1-\mu^2)} \quad \text{and} \quad \mu-\iota\sqrt{(1-\mu^2)}.$$

We assume that μ is positive, and this is in agreement with the applications which have been made of the function Y_n.



Further, we suppose that the integrand is real and positive at the point C in which the path of integration cuts the real axis, on the side of AB which is further from the origin; with this choice of integrand it will be seen that the integral (15) gives the value of $P_n(\mu)$.

A simple proof may be given as follows. Let the integral be called I_n, and then write $t = v/r$. We get

$$r^n I_n = \frac{1}{2\pi\iota}\int \frac{v^n dv}{\sqrt{(r^2-2vz+v^2)}},$$

where the integral extends round a similar path in the v-plane. But

$$r^2-2vz+v^2 = x^2+y^2+(z-v)^2,$$

so that $(r^2-2vz+v^2)^{-\frac{1}{2}}$ satisfies Laplace's equation. Thus the same is true of $r^n I_n$, and so I_n satisfies Legendre's equation in μ. To identify I_n with P_n, we consider the value of I_n when $\mu = 1$. We have then

$$I_n = \frac{1}{2\pi\iota}\int \frac{t^n dt}{t-1},$$

where the path surrounds $t = 1$, and t^n is to be real and positive at the point C (since $t-1$ is there positive); and thus I_n is now seen to be equal to 1, when $\mu = 1$. Hence

$$I_n = P_n(\mu),$$

because $P_n(\mu)$ is the only solution of Legendre's equation which reduces to unity at $\mu = 1$.

It now follows that

$$\frac{\partial P_n}{\partial n} = \frac{1}{2\pi\iota}\int \frac{t^n \log t\, dt}{\sqrt{(1-2\mu t+t^2)}},$$

taken round the same path as (15); and so if $|c| > 1$, we can write

$$\sum_1^\infty \frac{\partial P_n}{\partial n}\frac{1}{c^{n+1}} = \frac{1}{2\pi\iota}\int \frac{\log t\, dt}{(c-t)\sqrt{(1-2\mu t+t^2)}}. \tag{16}$$

The last integrand is, when $|t|$ is large, of the order $(\log t)/t^2$, so that the value of the integral taken round any portion of a large circle tends to

zero as the radius tends to infinity; and thus the original path of integration can be transformed into the dotted paths indicated in the diagram.

From the circle round $t = c$, we get*

$$(\log c)/\sqrt{(1-2\mu c+c^2)} = (\log c)/q, \text{ say};$$

where the square root q will be real and positive, if c is real and greater than 1, as we shall suppose for simplicity of statement.

On the part of the dotted path which lies *above* the negative axis, we have (in the limiting form of the path)

$$t = -x, \quad \log t = \log x + \pi\iota, \quad \sqrt{(1-2\mu t+t^2)} = -\sqrt{(1+2\mu x+x^2)},$$

and so this part of the integral yields

$$+\frac{1}{2\pi\iota}\int_{\infty}^{0} \frac{(\log x+\pi\iota)\,dx}{(c+x)\sqrt{(1+2\mu x+x^2)}} = -\frac{1}{2\pi\iota}\int_{0}^{\infty} \frac{(\log x+\pi\iota)\,dx}{(c+x)\sqrt{(1+2\mu x+x^2)}}.$$

For the path *below* the negative axis

$$t = -x, \quad \log t = \log x - \pi\iota, \quad \sqrt{(1-2\mu t+t^2)} = -\sqrt{(1+2\mu x+x^2)},$$

and so this part of the integral gives

$$+\frac{1}{2\pi\iota}\int_{0}^{\infty} \frac{(\log x-\pi\iota)\,dx}{(c+x)\sqrt{(1+2\mu x+x^2)}}.$$

Thus on combination, we get, from (16), the equation

$$(16a) \qquad \sum_{0}^{\infty} \frac{1}{c^{n+1}} \frac{\partial P_n}{\partial n} = \frac{\log c}{q} - \int_{0}^{\infty} \frac{dx}{(c+x)\sqrt{(1+2\mu x+x^2)}}.$$

The last integral is easily evaluated by elementary methods and is equal to†

$$\frac{2}{q}\tanh^{-1}\left(\frac{q}{1+c}\right),$$

so that

$$(17) \qquad \sum_{0}^{\infty} \frac{1}{c^{n+1}} \frac{\partial P_n}{\partial n} = \frac{1}{q}\left\{\log c - 2\tanh^{-1}\left(\frac{q}{1+c}\right)\right\} = \frac{U}{q}, \text{ say.}$$

---

* The fact that the circle is described in the *clockwise* sense just neutralizes the change of sign introduced by our having $(c-t)$ in place of $(t-c)$ in the denominator of (16).

† See, for instance, *Messenger of Mathematics*, Vol. 35, p. 131.

Then, if we differentiate out the bracket, we find

$$\frac{\partial U}{\partial c} = \frac{1}{c} - \frac{2}{q}\,\frac{(1+c)(c-\mu)-(1-2\mu c+c^2)}{(1+c)^2-(1-2\mu c+c^2)}$$

$$= \frac{1}{c} - \frac{c-1}{c}\,\frac{1}{q}$$

$$= \sum_1^\infty \frac{1}{c^{n+1}}\,(P_{n-1}-P_n).$$

Hence
$$U = U_0 + \sum_1^\infty \frac{1}{nc^n}\,(P_n - P_{n-1}),$$

where U_0 is the limit of U when c tends to infinity. But

$$U = \log \frac{c\,(1+c-q)}{1+c+q} = \log \frac{c\,.\,2c\,(1+\mu)}{(1+c+q)^2},$$

so that
$$U_0 = \log \tfrac{1}{2}\,(1+\mu),$$

and
$$U = \log \tfrac{1}{2}(1+\mu) + \sum_1^\infty \frac{1}{nc^n}\,(P_n - P_{n-1}).$$

If we multiply the last series by

$$\frac{1}{q} = \frac{1}{c} + \sum_1^\infty \frac{1}{c^{n+1}}\,P_n,$$

we obtain, from (17), the formula*

$$\frac{\partial P_n}{\partial n} = P_n \log \tfrac{1}{2}(1+\mu) + P_{n-1}(P_1-P_0) + \tfrac{1}{2}P_{n-2}(P_2-P_1) + \ldots + \frac{1}{n}\,(P_n - P_{n-1}). \tag{18}$$

It is clear that the last part of (18) is a polynomial of degree n in μ, and so can be arranged in the form

$$A_n P_n + A_{n-1} P_{n-1} + \ldots + A_0 P_0,$$

but it is not easy to obtain the form of these coefficients A_s directly; although one relation is evident by writing $\mu = 1$, namely,

$$A_n + A_{n-1} + \ldots + A_0 = 0, \tag{19}$$

a result which is also confirmed by the fact that $P_n(1) = 1$ for all values of n, so that, when μ is 1,

$$\frac{\partial P_n}{\partial n} = 0.$$

---

* In the special case $n = 0$, the value of the differential coefficient reduces to $\log \frac{1}{2}(1+\mu)$; this special case was obtained by a different method in 1906 (*Proc. London Math. Soc.*, Ser. 2, Vol. 4, p. 222, footnote).

On substituting from (18) in (14), we see that

$$Y_n = r^n \{P_n \log \tfrac{1}{2}(r+z) + A_n P_n + A_{n-1} P_{n-1} + \ldots + A_0 P_0\},$$

and since Y_n satisfies Laplace's equation, it follows from § II, that

$$A_{n-1} = -2\left(\frac{2n-1}{1.2n}\right) = -2\left(1-\frac{1}{2n}\right),$$

$$A_{n-2} = +2\left(\frac{2n-3}{2(2n-1)}\right) = +2\left(\frac{1}{2}-\frac{1}{2n-1}\right),$$

$$A_{n-3} = -2\left(\frac{2n-5}{3(2n-2)}\right) = -2\left(\frac{1}{3}-\frac{1}{2n-2}\right), \quad \&c.,$$

$$A_0 = (-1)^n \frac{2}{n(n+1)} = (-1)^n 2\left(\frac{1}{n}-\frac{1}{n+1}\right).$$

Thus, from (19), we have the result

$$A_n = -(A_{n-1}+A_{n-2}+\ldots+A_0) = 2\left(1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\ldots-\frac{1}{2n}\right),$$

a result which we found before in § III by comparison of the values of Y_n and X_n on the axis of symmetry.

VI. *Potential of a Hemisphere, with a Law of Density ks^{m-1}, where s is the Distance from the Centre.*

We take the base of the hemisphere as the plane of xy, the centre as the origin, and the hemisphere as lying on the positive side of the plane. Then the potential of the hemisphere is

$$V = \int_0^a k s^{m+1} ds \int_0^{\frac{1}{2}\pi} \sin\theta' \, d\theta' \int_0^{2\pi} \frac{d\phi}{\sqrt{(r^2+s^2-2rs\cos\gamma)}}, \tag{20}$$

where $$\cos\gamma = \cos\theta'\cos\theta + \sin\theta'\sin\theta\cos\phi.$$

Now $$\int_0^{2\pi} P_n(\cos\gamma) = 2\pi P_n(\cos\theta')\, P_n(\cos\theta),$$

and $$\int_0^{\frac{1}{2}\pi} \sin\theta' \, d\theta' \, P_n(\cos\theta') = \int_0^1 P_n(\mu')\, d\mu',$$

so that the value of this integral will be zero if n is even. Also we have

$$\int_0^1 P_0(\mu')\, d\mu' = 1, \qquad \int_0^1 P_{2n+1}(\mu')\, d\mu' = (-1)^n K_n,$$

if $$K_n = \frac{1.3\ldots(2n-1)}{2.4\ldots(2n+2)} \quad \text{and} \quad K_0 = \tfrac{1}{2}.$$

These integrals are readily obtained by integrating the series

$$\frac{1}{\sqrt{(1-2\mu' t+t^2)}} = 1+\sum_1^\infty P_n(\mu')\,t^n \quad (\text{where } |t|<1),$$

from $\mu'=0$ to $\mu'=1$; and this gives

$$1+\sum_1^\infty t^n\int_0^1 P_n(\mu')\,d\mu' = \frac{1}{t}\{\sqrt{(1+t^2)}-(1-t)\} = 1+\tfrac{1}{2}t+\sum_1^\infty(-t)^n K_n t^{2n+1},$$

from which the results stated are obvious.

If now $r>a$, we can expand the square root in (20) in the form

$$\frac{1}{r}+\sum_1^\infty \frac{s^n}{r^{n+1}}P_n(\cos\gamma),$$

and so, when $r>a$, we find the potential of the hemisphere,

$$(21)\quad V_0 = 2\pi k a^{m+1}\left\{\frac{1}{m+2}\,\frac{a}{r}+\sum_0^\infty(-1)^n\frac{K_n}{2n+m+3}\left(\frac{a}{r}\right)^{2n+2}P_{2n+1}(\mu)\right\}.$$

It is perhaps worth while noting that if we write

$$V_0 = F(r,\mu),$$

then

$$F(r,\mu)+F(r,-\mu) = \frac{4\pi k a^{m+2}}{(m+2)r},$$

which is the potential of a complete sphere having the same law of density.

We now return to the case $r<a$; for which (as in the case of the discs) we must divide the integral with respect to s into two parts, from 0 to r and from r to a, and then use the two different series

$$\frac{1}{r}+\sum_1^\infty\frac{s^n}{r^{n+1}}P_n(\cos\gamma) \quad\text{and}\quad \frac{1}{s}+\sum_1^\infty\frac{r^n}{s^{n+1}}P_n(\cos\gamma),$$

in these two parts.

After carrying out the integrations and rearranging the series in the same way as (3), we get*

$$(22)\quad V_1 \text{ or } V_2 = 2\pi k r^{m+1}\left\{-\frac{1}{(m+1)(m+2)}+\sum_0^\infty(-1)^n\frac{(4n+3)K_n}{(2n+m+3)(2n-m)}P_{2n+1}(\mu)\right\}$$
$$+2\pi k a^{m+1}\left\{\frac{1}{m+1}-\sum_0^\infty(-1)^n\frac{K_n}{2n-m}\left(\frac{r}{a}\right)^{2n+1}P_{2n+1}(\mu)\right\}.$$

* It will be necessary to distinguish between the potentials on the positive and negative sides of the base; and we denote by V_1 the potential when $\mu>0$, by V_2 the potential when $\mu<0$, so that V_1 refers to points inside the hemisphere and V_2 to external points.

Now since V_2 satisfies Laplace's equation it is suggested that the coefficient of r^{m+1} will be proportional to $P_{m+1}(-\mu)$ when μ is negative.

To investigate this conjecture we shall find the coefficients L_n, so that

$$\sum_0^\infty L_n P_{2n+1}(\mu) = P_{m+1}(\mu) - P_{m+1}(0) \qquad (\mu > 0),$$

and

$$= P_{m+1}(0) - P_{m+1}(-\mu) \quad (\mu < 0).$$

The function so defined is clearly an odd function of μ, which is zero at $\mu = 0$, and hence is expressible as a series of *odd* harmonics only. Thus we have

$$\frac{2L_n}{4n+3} = 2\int_0^1 \{P_{m+1}(\mu) - P_{m+1}(0)\} P_{2n+1}(\mu)\, d\mu.$$

Now Legendre's equation gives

$$\{(2n+1)(2n+2) - (m+1)(m+2)\} \int_0^1 P_{m+1}(\mu) P_{2n+1}(\mu)\, d\mu$$

$$= \left[(1-\mu^2)\left\{P_{2n+1}(\mu)\frac{dP_{m+1}}{d\mu} - P_{m+1}(\mu)\frac{dP_{2n+1}}{d\mu}\right\}\right]_0^1$$

$$= P_{m+1}(0)\left(\frac{dP_{2n+1}}{d\mu}\right)_0.$$

But $$(1-\mu^2)\frac{dP_{2n+1}}{d\mu} = (2n+1)(P_{2n} - \mu P_{2n+1}),$$

so that $$\left(\frac{dP_{2n+1}}{d\mu}\right)_0 = (2n+1)P_{2n}(0) = (-1)^n \frac{1.3\ldots(2n+1)}{2.4\ldots 2n}$$

$$= (-1)^n (2n+1)(2n+2) K_n.$$

Hence $$\int_0^1 P_{m+1}(\mu) P_{2n+1}(\mu)\, d\mu = (-1)^n \frac{(2n+1)(2n+2)}{(2n+1)(2n+2) - (m+1)(m+2)} K_n P_{m+1}(0),$$

and similarly $$\int_0^1 P_{m+1}(0) P_{2n+1}(\mu)\, d\mu = (-1)^n K_n P_{m+1}(0).$$

Thus $$\int_0^1 \{P_{m+1}(\mu) - P_{m+1}(0)\} P_{2n+1}(\mu)\, d\mu = (-1)^n \frac{(m+1)(m+2) K_n P_{m+1}(0)}{(2n-m)(2n+m+3)},$$

and so we find $$L_n = (-1)^n \frac{(4n+3) K_n}{(2n-m)(2n+m+3)} (m+1)(m+2) P_{m+1}(0).$$

Hence we have the formulæ

$$\sum_0^\infty \frac{(-1)^n (4n+3) K_n}{(2n-m)(2n+m+3)} P_{2n+1}(\mu) = \frac{P_{m+1}(\mu) - P_{m+1}(0)}{(m+1)(m+2) P_{m+1}(0)}, \quad \text{if} \quad \mu > 0,$$

or

$$\frac{P_{m+1}(0) - P_{m+1}(-\mu)}{(m+1)(m+2) P_{m+1}(0)}, \quad \text{if} \quad \mu < 0.$$

Thus, on substituting in (22), we get the potentials, when $r < a$,

$$V_1 = \frac{2\pi k r^{m+1}}{(m+1)(m+2)} \left\{-2 + \frac{P_{m+1}(\mu)}{P_{m+1}(0)}\right\} + F(r, \mu), \tag{23}$$

$$V_2 = \frac{2\pi k r^{m+1}}{(m+1)(m+2)} \left\{-\frac{P_{m+1}(-\mu)}{P_{m+1}(0)}\right\} + F(r, \mu), \tag{24}$$

where

$$(25)\quad F(r,\mu) = 2\pi k a^{m+1}\left\{\frac{1}{m+1} - \sum_0^\infty (-1)^n \frac{K_n}{2n-m}\left(\frac{r}{a}\right)^{2n+1} P_{2n+1}(\mu)\right\}.$$

These formulæ carry with them certain obvious means of verification; thus we get

$$\Delta V_1 = -4\pi k r^{m-1}, \quad \Delta V_2 = 0,$$

in agreement with Poisson's and Laplace's equations. Again, at the base of the hemisphere ($\mu = 0$), we have

$$V_1 = V_2, \qquad \frac{1}{r}\frac{\partial V_1}{\partial \mu} = \frac{1}{r}\frac{\partial V_2}{\partial \mu},$$

which should be the case in view of the general properties of the potential of a solid.

As an illustration, we may consider the case of a *uniform hemisphere*, with $m = 1$; here

$$P_{m+1}(\mu) = \tfrac{1}{2}(3\mu^2-1), \qquad P_{m+1}(0) = -\tfrac{1}{2},$$

and so

$$V_1 = -\tfrac{1}{3}\pi k r^2 (1+3\mu^2) + F(r,\mu),$$

$$V_2 = -\tfrac{1}{3}\pi k r^2 (1-3\mu^2) + F(r,\mu),$$

where

$$F(r,\mu) = 2\pi k a^2\left\{\tfrac{1}{2} - \sum_0^\infty (-1)^n \frac{K_n}{2n-1}\left(\frac{r}{a}\right)^{2n+1} P_{2n+1}(\mu)\right\}.$$

Expressed in terms of x, y, z, these functions take the forms (as far as terms of the second degree)

$$V_1 = \pi k\left\{a^2 + az - \tfrac{1}{8}(x^2+y^2+4z^2)\right\},$$

$$V_2 = \pi k\left\{a^2 + az - \tfrac{1}{8}(x^2+y^2-2z^2)\right\},$$

results originally given, I believe, by Mr. R. R. Webb.

When m is an even integer (2ν), the formulæ (23), (24) require modification; and the most direct method is to apply again the same limiting process as in § 1 above by writing $m = 2\nu + t$, and making t tend to zero. Now we proved on p. 104 that

$$(m+2)P_{m+1}(0) = \frac{2}{\sqrt{\pi}}\cos\left\{\tfrac{1}{2}(m+1)\pi\right\}\frac{\Gamma(\frac{1}{2}m+2)}{\Gamma(\frac{1}{2}m+\frac{3}{2})},$$

and so we find

$$(m+1)(m+2)P_{m+1}(0) = -\frac{4}{\sqrt{\pi}}\sin(\tfrac{1}{2}m\pi)\frac{\Gamma(\frac{1}{2}m+2)}{\Gamma(\frac{1}{2}m+\frac{1}{2})}.$$

Thus

$$\frac{1}{(m+1)(m+2)P_{m+1}(0)} = (-1)^{\nu-1}\frac{\sqrt{\pi}}{4\sin(\frac{1}{2}t\pi)}\frac{\Gamma(\nu+\frac{1}{2}+\frac{1}{2}t)}{\Gamma(\nu+2+\frac{1}{2}t)}.$$

and, approximating as on pp. 104, 105, the last formula becomes

$$(-1)^{\nu-1}\frac{1}{t}K_\nu\left[1+\tfrac{1}{2}t\left\{\psi(\nu+\tfrac{1}{2})-\psi(\nu+2)\right\}\right],$$

because $$\frac{1}{2\sqrt{\pi}}\frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu+2)}=\frac{1.3\ldots(2\nu-1)}{2.4\ldots(2\nu+2)}=K_\nu.$$

Also $$\psi(\nu+\tfrac{1}{2})-\psi(\nu+2)=-2R_\nu-1/(\nu+1),$$

where, as before, $$R_\nu=\log 2-\left(1-\tfrac{1}{2}+\tfrac{1}{3}-\ldots-\frac{1}{2\nu}\right).$$

Thus, to the necessary degree of accuracy, we can write in (23) and (24),

$$\frac{1}{(m+1)(m+2)\,P_{m+1}(0)}=(-1)^{\nu-1}K_\nu\left(\frac{1}{t}-R_\nu-\frac{1}{2\nu+2}\right).$$

And now, on taking the limit as t tends to zero, we obtain *the potential of a hemisphere whose density is* $ks^{2\nu-1}$, in the form

$$(26)\quad V_1=-\frac{2\pi k r^{2\nu+1}}{(2\nu+1)(\nu+1)}+G(r,\mu)$$
$$+2\pi k a^{2\nu+1}(-1)^{\nu-1}K_\nu\left\{Y_{2\nu+1}-\left(R_\nu+\frac{1}{2\nu+2}\right)\left(\frac{r}{a}\right)^{2\nu+1}P_{2\nu+1}(\mu)\right\},$$

$$(27)\quad V_2=G(r,\mu)$$
$$+2\pi k a^{2\nu+1}(-1)^{\nu-1}K_\nu\left\{-Y_{2\nu+1}-\left(R_\nu+\frac{1}{2\nu+2}\right)\left(\frac{r}{a}\right)^{2\nu+1}P_{2\nu+1}(\mu)\right\},$$

where

$$(28)\quad G(r,\mu)=\pi k a^{2\nu+1}\left\{\frac{2}{2\nu+1}-\sum_0^{\infty\prime}(-1)^n\frac{K_n}{n-\nu}\left(\frac{r}{a}\right)^{2n+1}P_{2n+1}(\mu)\right\}.$$

In (27) the value of $Y_{2\nu+1}(|\mu|)$ is to be used; in fact, the function Y_n has been defined only for positive values of the argument.

As an illustration, let us take the case $\nu=0$, which corresponds to the law of density k/s. Then we get

$$R_0=\log 2,$$

and $$aY_1=z\log\left(\frac{r+z}{2a}\right)-r+z,\quad\text{if}\quad z>0,$$

or $$-z\log\left(\frac{r-z}{2a}\right)-r-z,\quad\text{if}\quad z<0,$$

so that our formulæ reduce to

$$V_1=2\pi k(a-r)+\pi k\left\{z\log\left(\frac{4a}{r+z}\right)+r-\tfrac{1}{2}z\right\}+G_1,$$

$$V_2=2\pi k a\qquad+\pi k\left\{z\log\left(\frac{4a}{r-z}\right)-r-\tfrac{1}{2}z\right\}+G_1,$$

where $$G_1 = \pi ka \sum_0^\infty{}' (-1)^{n-1} \frac{K_n}{n} \left(\frac{r}{a}\right)^{2n+1} P_{2n+1}$$

$$= \pi ka \left\{ \frac{1}{2.4} \left(\frac{r}{a}\right)^3 P_3 - \frac{1.3}{2.4.6} \frac{1}{2} \left(\frac{r}{a}\right)^5 P_5 + \ldots \right\} .$$

As a verification it is not difficult to calculate directly the potential on the axis, which is

$$V_1 = 2\pi \int_0^a k\, s\, ds \int_0^{\frac{1}{2}\pi} \frac{\sin\theta'\, d\theta'}{\sqrt{(r^2 - 2rs\cos\theta' + s^2)}},$$

or

$$V_2 = 2\pi \int_0^a k\, s\, ds \int_0^{\frac{1}{2}\pi} \frac{\sin\theta'\, d\theta'}{\sqrt{(r^2 + 2rs\cos\theta' + s^2)}}.$$

Hence we find $$V_1 = \frac{2\pi k}{r} \int_0^a \left\{ \sqrt{(r^2+s^2)} - |\, r - s\, | \right\} ds$$

$$= \frac{\pi k}{r} \left\{ a\sqrt{(r^2+a^2)} + r^2 \sinh^{-1}\left(\frac{a}{r}\right) - r^2 - (a-r)^2 \right\},$$

and $$V_2 = \frac{2\pi k}{r} \int_0^a \left\{ (r+s) - \sqrt{(r^2+s^2)} \right\} ds$$

$$= \frac{\pi k}{r} \left\{ (a+r)^2 - r^2 - a\sqrt{(r^2+a^2)} - r^2 \sinh^{-1}\left(\frac{a}{r}\right) \right\}.$$

We can then expand in powers of r/a, using the series quoted on p. 112 for $\sinh^{-1}(a/r)$, which leads to the results (on the axis)

$$V_1 = 2\pi k\,(a-r) + \pi kr \left\{ \log(2a/r) + \tfrac{1}{2} \right\} + H(r),$$

$$V_2 = 2\pi ka \quad - \pi kr \left\{ \log(2a/r) + \tfrac{1}{2} \right\} - H(r),$$

where $$H(r) = \pi ka \left\{ \frac{1}{2.4} \left(\frac{r}{a}\right)^3 - \frac{1.3}{2.4.6} \frac{1}{2} \left(\frac{r}{a}\right)^5 + \frac{1.3.5}{2.4.6.8} \frac{1}{3} \left(\frac{r}{a}\right)^7 - \ldots \right\}.$$

These harmonize with the results obtained from our general formulæ.

Appendix (*January*, 1913).

Since the foregoing pages were written out I have had the opportunity of seeing Mr. G. N. Watson's formulæ, which were published in the Abstracts of the November meeting of the Society.

It is there remarked that (in the case of discs) the formulæ for the potentials V_1, V_0 along the axis of symmetry can be converted into one another by the use of complex integrals containing Gamma functions.

By a slight modification it appears to be easy to utilise this method, not only on the axis, but for any point in space; and the process can also be applied to the case of the hemisphere examined in § VI, so as to connect the potentials V_2 and V_0.

I have briefly indicated the necessary calculations below; but it does not appear obvious that the transformation involved in the passage from (29) to (30) would necessarily convert the potential V_0 into the potential V_1, although it is evident that the integral (30) gives a solution of

Laplace's equation, and it is also evident that with a more highly convergent integrand the two integrals (29) and (30) would be equal.

It is easy to modify the integrals given below so as to represent potentials corresponding to densities of the types mentioned by Mr. Watson; thus, if $\sigma = ks^m \{\log(a/s)\}^{l-1}$ in the disc, the integrals (29) and (30) will be multiplied by $\Gamma(l)$, and the denominators will be changed to $(m+2t+2)^l$ and $(m+1-2v)^l$ respectively. A corresponding modification can be made in (32) and (33), when the density of the hemisphere is $\rho = ks^{m-1} \{\log(a/s)\}^{l-1}$.

We begin by considering the potential of the disc defined in Section I. Taking the series (1) we can write

$$\frac{(-1)^n C_n}{m+2n+2} = \frac{(-1)^n \Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})\,\Gamma(n+1)(m+2n+2)},$$

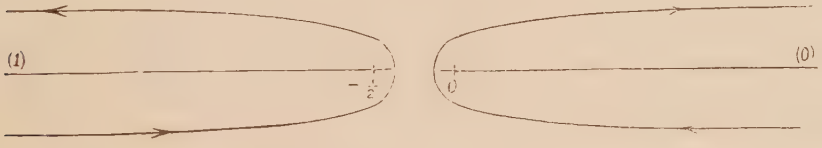
and this is the residue at $t = n$ of the function

$$-\frac{\Gamma(-t)\,\Gamma(t+\frac{1}{2})}{\Gamma(\frac{1}{2})(m+2t+2)}.$$

Thus the series (1) can be expressed in the form*

$$V_0 = \frac{2\pi k a^{m+1}}{\Gamma(\frac{1}{2})} \left\{ \frac{1}{2\pi\iota} \int \frac{\Gamma(-t)\,\Gamma(t+\frac{1}{2})}{m+2t+2} Z_{-(2t+1)}\, dt \right\}, \tag{29}$$

where the integral is taken round the path marked (0) in the diagram.



The integral (29) is convergent so long as $r > a$; but when $r < a$ the convergence fails. To convert the integral into a convergent form the path can be changed into (1); the value of the integral then gives

---

* We use here the notation for zonal harmonics

$$Z_n = (r/a)^n P_n(|\cos\theta|),$$

so that

$$Z_{-(n+1)} = (a/r)^{n+1} P_n(|\cos\theta|);$$

because in general $P_n(\cos\theta)$ is defined by the hypergeometric series $F[-n, n+1, 1, \frac{1}{2}(1-\cos\theta)]$, so that

$$P_n(\cos\theta) = P_{-(n+1)}(\cos\theta).$$

the series (4) for V_1. To verify the truth of this statement write $t = -(\frac{1}{2}+v)$; then the path (1) of the t-plane is changed into the path (0) of the v-plane (allowing for the change of sign implied by the equation $dt = -dv$). Thus we find the integral

$$(30) \qquad \frac{2\pi k a^{m+1}}{\Gamma(\frac{1}{2})} \left\{ \frac{1}{2\pi\iota} \int \frac{\Gamma(v+\frac{1}{2})\,\Gamma(-v)}{m+1-2v} Z_{2v}\, dv \right\},$$

and when m is not an odd integer the integral (30) at once yields the series

$$(31) \qquad \frac{2\pi k a^{m+1}}{\Gamma(\frac{1}{2})} \left\{ \sum_0^\infty \frac{(-1)^n\,\Gamma(n+\frac{1}{2})}{(m+1-2n)\,\Gamma(n+1)} Z_{2n} + \tfrac{1}{2} Z_{m+1}\,\Gamma\left(\frac{m}{2}+1\right)\Gamma\left(-\frac{m+1}{2}\right) \right\}.$$

The series (31) can be identified with (4) by noting that

$$\Gamma\left(\frac{m}{2}+1\right)\Gamma\left(-\frac{m+1}{2}\right) = -\frac{\pi\,\Gamma(\frac{1}{2}m+1)}{\cos(\frac{1}{2}m\pi)\,\Gamma(\frac{1}{2}m+\frac{3}{2})} = +\frac{2\sqrt{\pi}}{p},$$

where p is the constant defined on p. 104 above.

When m is an odd integer the integral (30) leads to the series (6); but the work involved hardly differs from that already given on pp. 104, 105.

In regard to the hemisphere of § VI, the present method will not serve to obtain V_1 from V_0, because V_1 does not satisfy Laplace's equation, whereas any integral such as (29) or (30) does satisfy it. On the other hand, it is not difficult to obtain V_2 from V_0 by using a complex integral.

For this purpose it will be simplest to change the sign of μ in (21) and in (24), so that the region now considered is that for which μ is *positive*; but the hemisphere is on the *negative* side of the plane $z = 0$. We can then write (21) in the form

$$(21a) \qquad V_0 = 2\pi k a^{m+1} \left\{ \frac{1}{m+2} Z_{-1} + \sum_0^\infty (-1)^{n+1} \frac{K_n}{2n+m+3} Z_{-(2n+2)} \right\},$$

where $$K_n = \frac{\Gamma(n+\frac{1}{2})}{2\Gamma(\frac{1}{2})\,\Gamma(n+2)}.$$

The last series leads at once to the integral

$$(32) \qquad V_0 = \frac{2\pi k a^{m+1}}{2\Gamma(\frac{1}{2})} \left\{ \frac{1}{2\pi\iota} \int \frac{\Gamma(t)\,\Gamma(-t-\frac{1}{2})}{2t+m+2} Z_{-2t-1}\, dt \right\},$$

taken round the path (0) in the t-plane.

Applying the same transformations as before, we find, when $r < a$, the integral

$$(33) \qquad \frac{2\pi k a^{m+1}}{2\Gamma(\frac{1}{2})} \left\{ \frac{1}{2\pi\iota} \int \frac{\Gamma(v)\,\Gamma(-v-\frac{1}{2})}{m+1-2v} Z_{2v} dv \right\},$$

taken round the path (0) in the v-plane; and this integral leads to the formula

$$(34) \quad 2\pi k a^{m+1} \left\{ \frac{Z_0}{m+1} + \sum_0^\infty \frac{(-1)^n K_n Z_{2n+1}}{2n-m} + \frac{\Gamma(\frac{1}{2}m+\frac{1}{2})\,\Gamma(-\frac{1}{2}m-1)}{4\Gamma(\frac{1}{2})} Z_{m+1} \right\}.$$

By using the formula for $(m+1)(m+2)P_{m+1}(0)$ given on p. 120, the series (34) can be identified with the series (24) for V_2, when the sign of μ is changed in the latter.

Similarly we can derive the series corresponding to (27) from (33); but the work is essentially the same as that already given on pp. 120, 121.

To obtain the series for V_1 (in either case) it seems to be only possible to use the fact that the potential of the complete sphere at internal points is equal to

$$\int_0^r k s^{m-1} \frac{4\pi s^2 ds}{r} + \int_r k s^{m-1} . 4\pi s\, ds = \frac{4\pi k}{m+1} \left(a^{m+1} - \frac{r^{m+1}}{m+2} \right),$$

and so by subtracting from this expression the potential (34), we can infer a formula for V_1 which reduces to (23) or (26), according to the form of m.

ON THE IRREDUCIBILITY OF LEGENDRE'S POLYNOMIALS

(SECOND PAPER.)

By J. B. HOLT.

[Received October 31st, 1912.—Read November 14th, 1912.]

1. In a previous paper* on this subject it was shown that the n-th Legendrian Polynomial $P_n(x)$, divided by x when n is odd, is irreducible when n has any of the forms

$$p+1, \quad p, \quad p-1, \quad p-2, \quad 2p-1, \quad 2p-2,$$

p being an odd prime. In extension of these results it will now be proved that, if $P_n(x)$ have a rational factor ax^2+b, this must be $3x^2-1$; and that, for certain forms of n, P_n, if reducible, must have such a factor, but does not vanish when $3x^2 = 1$. In such cases P_n is irreducible.

In this way it appears that $p+3$, $p-4$, $2p-4$ are degrees of irreducible cases, and that $p+2$, $p-3$, $2p-3$ are so on condition that $P_n(x)$ and $3x^2-1$ do not vanish together. A method of testing the last point is given, which always succeeds so far as it has been tried, up to $n = 1000$ It is also shewn that if p, q be successive primes for which $q-p < 10$, and n be one of the numbers p, $p+1$, ..., q, then P_n is irreducible. This at once shews that P_n is irreducible for $n \leqslant 113$, and further examination shews that for $n < 200$ there are only three doubtful cases: $n = 122, 185, 186$.

2. $$P_n(x) = \frac{1}{\pi}\int_0^\pi (x+y\cos\phi)^n\, d\phi \quad (y^2 = x^2-1)$$

$$= x^n + A_1 \tfrac{1}{2} x^{n-2}y^2 + A_2 \frac{1.3}{2.4} x^{n-4}y^4 + \ldots,$$

each A being a positive integer.

Put $$z = \frac{4x^2}{x^2-1},$$

so that $$x^2 = \frac{z}{z-4}, \qquad y^2 = \frac{4}{z-4}.$$

* *Proc. London Math. Soc.*, Ser. 2, Vol. 11, pp. 351–356.

Then, if $n = 2m$ or $2m+1$, the equation $P_n(x) = 0$ is equivalent to

$$z^m + 2^2 A_1 \tfrac{1}{2} z^{m-1} + 2^4 A_2 \frac{1.3}{2.4} z^{m-2} + \ldots = 0.$$

The expression $\dfrac{1.3 \ldots 2r-1}{2.4 \ldots 2r}$ on reduction is a fraction whose denominator is a power of 2, namely, the greatest which divides $2r!$. Its index is then $< 2r$, so that in the last written equation every coefficient after the first is a positive even integer. A rational root of the equation must then be negative, integral, and even, so that $z = -2s$, and $x^2 = s/s+2$.

Hence if $P_n(x) = 0$, and x^2 be rational, it must $= s/s+2$, where s is an odd integer; for the first and last coefficients of $P_n(x)$, when this is integralised, are odd.

3. It will now be shewn that s cannot exceed 1. When n is even the equation $P_n(x) = 0$ is

$$1 - \frac{n.n+1}{2!} x^2 + \frac{n-2.n.n+1.n+3}{4!} x^4 - \ldots = 0.$$

If p be an odd prime, $\dfrac{p^m}{2m!}$ is, on reduction, a fraction whose numerator is divisible by p. For the greatest power of p which divides $2m!$ is of index $< \dfrac{2m}{p-1}$; and for $p > 2$ this is $\leqslant m$. If then x^2 be rational, and p a prime divisor of its numerator s, each of $x^2/2!$, $x^4/4!$, ..., on reduction, has a denominator prime to p, and a numerator divisible by p. The left-hand side of the equation is, on reduction, a fraction whose numerator is prime to p, and therefore it cannot vanish. Hence for n even s is not > 1.

For n odd, the equation $P_n(x) = 0$ is

$$1 - \frac{n-1.n+2}{3!} x^2 + \frac{n-3.n-1.n+2.n+4}{5!} x^4 - \ldots = 0.$$

Here again $p^m/2m+1!$ on reduction has a denominator prime to p, and a numerator divisible by p, if $p > 3$. For the greatest power of p which divides $2m+1!$ is of index $< \dfrac{2m+1}{p-1}$, which is $< m$ if $p > 3$. As before, if p divide s, each of $x^2/3!$, $x^4/5!$, ..., has a numerator divisible by p, and the left-hand side cannot vanish; s is not divisible by p. When $p = 3$, the index of the highest power of 3 in $2m+1!$ is m at most, so

that the denominator of $3^m/2m+1!$ is prime to 3. If, then, s be divisible by 3^2, the left-hand side cannot vanish.

Hence s cannot exceed 3 when n is odd. If $s=3$, $x^2=3/5$, and the equation is

$$1-\frac{n-1.n+2}{3!}\frac{3}{5}+\frac{n-3.n-1.n+2.n+4}{5!}\frac{3^2}{5^2}-\ldots=0.$$

The sum of the first two terms is $-(n-3)(n+4)/10$. If $n>3$, we may divide out $(n-3)(n+4)$; and the left-hand side becomes, with a little rearrangement,

$$-\frac{1}{10}+\frac{n-1.n+2}{5^2}\frac{3^2}{5!}-\frac{n-1.n+2.n-5.n+6}{5^3}\frac{3^3}{7!}+\ldots.$$

It has been seen that $3^2/5!$, $3^3/7!$, ..., have on reduction denominators prime to 3. The second term has a numerator divisible by 3, and the later ones are all divisible by 3, since one of $n+2$, $n-5$, $n+6$ is so, and these occur in all. Hence the whole has a numerator prime to 3 and cannot vanish, so that s is not $=3$.

It follows that if $P_n(x)=0$ and x^2 is rational, P_n is divisible by $3x^2-1$.

4. In what follows use will be made of congruences such as

$$ax^n+bx^{n-1}+\ldots\equiv 0 \pmod p,$$

in which the x is undetermined. The meaning of this is that each of a, b, ..., is $\equiv 0 \pmod p$. The coefficients a, b need not be integers, provided that their denominators are prime to p. By the expression $c/d \pmod p$ is meant, as usual, the solution of the congruence

$$dz\equiv c \pmod p.$$

Since in $P_m(x)$ the denominators of the coefficients are all powers of 2, the identity

$$mP_m=(2m-1)xP_{m-1}-(m-1)P_{m-2}$$

may be replaced by a congruence to any odd prime modulus p. Suppose, then, that $m=n+r$, n being divisible by p, and r less than p. Then for mod p

$$rP_{n+r}\equiv(2r-1)xP_{n+r-1}-(r-1)P_{n+r-2}.$$

Put $r=1$: then $P_{n+1}\equiv xP_n\equiv P_1P_n$,

$r=2$: $2P_{n+2}\equiv 3xP_{n+1}-P_n$

$\equiv(3x^2-1)P_n\equiv 2P_2P_n$,

and $P_{n+2}\equiv P_2P_n$.

In general, if

$$P_{n+r-1} \equiv P_{r-1}P_n,$$

$$P_{n+r-2} \equiv P_{r-2}P_n,$$

then

$$rP_{n+r} \equiv [(2r-1)\,xP_{r-1} - (r-1)\,P_{r-2}]\,P_n$$

$$\equiv rP_rP_n,$$

and as $r < p$,

$$P_{n+r} \equiv P_rP_n \pmod{p}.$$

Again, with n, r as before,

$$(-r+1)\,P_{n-r+1} \equiv (-2r+1)\,xP_{n-r} + rP_{n-r-1} \pmod{p},$$

or

$$rP_{n-r-1} \equiv (2r-1)\,xP_{n-r} - (r-1)\,P_{n-r+1}.$$

Putting $r = 1, 2, \ldots$ successively

$$P_{n-2} \equiv P_1P_{n-1},$$

$$P_{n-3} \equiv P_2P_{n-1},$$

and

$$P_{n-r} \equiv P_{r-1}P_{n-1}.$$

5. Since in P_p every coefficient but the first is divisible by p,

$$P_p(x) \equiv Ax^p \pmod{p},$$

and by putting

$$x = 1, \quad A \equiv 1.$$

So, too, in P_{p-1} every coefficient but the last is divisible by p, so that

$$P_{p-1} \equiv B \equiv 1.$$

Hence

$$\left.\begin{aligned} P_{p+r} &\equiv x^pP_r \\ P_{p-r} &\equiv P_{r-1} \end{aligned}\right\} \pmod{p}.$$

If in these indeterminate congruences we give x^2 a definite rational value, they still hold as congruences in the ordinary sense. Put, then, $x = \omega$, $3\omega^2 = 1$. If $P_r(\omega)$ be not $\equiv 0 \pmod{p}$, neither $P_{p+r}(\omega) = 0$, nor $P_{p-r-1}(\omega) = 0$. Now the values of $P_r(\omega)$ may be easily calculated in succession from the difference equation

$$rP_r = (2r-1)\,\omega P_{r-1} - (r-1)\,P_{r-2},$$

or still more conveniently from the relation

$$(r+1)(r+2)(2r-1)\,P_{r+2} + \tfrac{2}{3}r\,(r+1)(2r+1)\,P_r + r\,(r-1)(2r+3)\,P_{r-2} = 0,$$

which is easily deduced from the difference equation. It is found that for

$$r = 3, \quad 4, \quad 5, \quad 6, \quad 7, \quad 8, \quad 9, \quad 10, \quad 11,$$

$$P_r(\omega) = \frac{-2}{3}\omega, \quad \frac{-7}{18}, \quad \frac{-\omega}{6}, \quad \frac{2}{9}, \quad \frac{5\omega}{9}, \quad \frac{11}{72}, \quad \frac{-7.19}{2^3.3^4}\omega, \quad \frac{-5.13}{3^5}, \quad \frac{-5.7}{2^2.3^3}\omega,$$

so that for $r = 3, 4, \ldots, 11$, $P_r(\omega)$ is not $\equiv 0$ for any prime > 20.

This then gives a test by which it is usually possible to see that ω is not a root of $P_n(x) = 0$.

6. When n is of any of the forms $p+3$, $p+2$, $p-3$, $p-4$, it follows from the rule that in each case, if P_n be reducible, it has an irreducible factor of degree $n-2$ or $n-3$ as n is even or odd; and consequently has a factor of the form ax^2+b, which must be $3x^2-1$. But as

$$\left.\begin{aligned} P_{p+3} &\equiv x^p P_3 \\ P_{p-4} &\equiv P_3 \end{aligned}\right\} \pmod{p},$$

it follows from the last section that ω is not a root; and that P_{p+3}, P_{p-4} are irreducible. For $p+2$, $p-3$ the question is not settled in general, but in practice offers little difficulty. For instance, $115 = 113+2$; it is also $109+6$, and so $P_{115} \equiv x^{109}P_6$, and this case is irreducible.

7. From the result that P_n is irreducible when n is of any of the forms

$$p-4, \quad p-2, \quad p-1, \quad p, \quad p+1, \quad p+3,$$

a convenient rule may be deduced. Let p, q be successive primes, whose difference is < 10. If $q-p$ be 2, 4, or 6, every number from p to q inclusive is the degree of an irreducible case. If $q-p = 8$, this is still evident except for $p+2$, $p+5$. But $p+2$ is irreducible, unless

$$P_{p+2}(\omega) = 0;$$

and as $p+2 = q-6$,

$$P_{p+2} \equiv P_5 \equiv \frac{-\omega}{6} \pmod{q};$$

and ω is not a root. Again, $p+5 = q-3$, and P_{p+5} is irreducible, unless $P_{p+5}(\omega) = 0$. But

$$P_{p+5}(\omega) \equiv \omega^5 P_5 \equiv \frac{-\omega^6}{6} \pmod{p},$$

and ω is not a root.

Hence, if n lie between two primes whose difference is < 10, P_n is

irreducible. As the first interval not of this kind is $113 \ldots 127$, P_n is proved irreducible for $n \leqslant 113$.

8. It remains to prove that P_n is irreducible when $n = 2p-1$, $2p-2$, or $2p-4$; and also when $n = 2p-3$, if only $P_n(\omega)$ do not vanish.

Consider the case $n = 2p-r$, r being $< p/5$. Since $E(n/p) = 1$, $E(2x/p) = 3$, the first $p-2r+1$ coefficients of P_n are divisible by p, and also the last $p+1$ or $p-1$, as r is even or odd. If there be an irreducible factor of this type, it is of degree at least $2p-2r+2$ or $2p-2r$, and its co-factor in P_n is of degree $r-2$ or r at most. Since in the second case r is odd, we may say that if P_n be irreducible there is a factor of even degree $< r$.

If this be not so there must be two distinct factors of degrees $> n/3$; for $3(p-2r+1) \geqslant 2p-r$ if $p \geqslant 5r-3$. But,* as P_n must in any case have an irreducible factor of degree $> 2n/3$, it cannot have two of degrees $> n/3$. Hence this alternative must be rejected.

For small values of r, then, P_n is irreducible unless it have a divisor of even degree $< r$. For $r = 1$ or 2 it must be irreducible, and for $r = 3$ or 4 it is so unless $P_n(\omega) = 0$. For $r = 4$ this is impossible, since $P_{2p-4} \equiv x^p P_{p-4} \equiv x^p P_3 \bmod p$.

9. In making an enumeration of the cases where P_n is proved irreducible by these methods it is only necessary to consider intervals between primes differing by 10 at least. Thus, for $n < 200$ the only intervals are 113 to 127, 139 to 149, 181 to 191, and it is found that all cases are irreducible except possibly $x = 122, 185, 186$.

For $n < 1000$ it appears that over 90 per cent. of the cases are certainly irreducible.

10. The lemma which formed the starting point of the former paper may be generalised as follows :—

If $f(x)$ be a polynomial whose first r coefficients are divisible by p, but whose first coefficient is not divisible by p^{a+1}, then it has an irreducible factor of degree $\geqslant r/a$.

For let $g(x)$, $h(x)$ be polynomials of degrees m, n respectively, the first s coefficients of $g(x)$ and the first t of $h(x)$ being divisible by p.

* Cf. § 9 of former paper.

To modulus p they are of degrees $m-s$, $n-t$ respectively, and their product of degree $m+n-s-t$. Hence the first $s+t$ terms of the product $g(x)\,h(x)$ are divisible by p. This plainly holds for any number of factors; so that if the first r coefficients of $f(x)$ and the first s of a typical irreducible factor be divisible by p, then $r = \Sigma s$.

As the leading coefficient of $f(x)$ is not divisible by p^{a+1}, $f(x)$ has at most a factors for which $s > 0$; that is, for which the first coefficient is divisible by p. Hence one at least of the numbers s must be $\geqslant r/a$.

COR. 1.—If $a = 1$, this is the lemma referred to. If in addition $r = n$, the degree of $f(x)$, the latter is irreducible. This is Eisenstein's theorem.

COR. 2.—If $r > a$, the roots of $f(x)$ are not all rational.

COR. 3.—If every coefficient but one of $f(x)$ be divisible by p, the same is true of every irreducible factor of $f(x)$. For, if a typical irreducible factor reduce for mod p to the product of a power of x and a polynomial of degree s, $f(x)$ reduces to the product of a power of x and a polynomial of degree Σs. If this $= 0$, each $s = 0$.

ERRATA, VOL. 11, SER. 2, PP. 351–356.

P. 355, line 10, *for* a_2 *read* α_2,
,, 11, *for* a_r *read* α_r,
,, 16, *for* z^2 *read* z.

THE THEORY OF FUNCTIONS OF A REAL VECTOR

By E. CUNNINGHAM.

[Received August 20th, 1912.—Read November 14th, 1912.]

CONTENTS.

§§ 1–3. Introductory and Definitions.
4–7. Continuous Vector Functions.
8–12. Vector Integration.
13–19. Vector Differentiation.
20–22. Differentiation of a Product. Green's Theorem.
23–29. Generalized Mean-value Theorems.

1. In recent years several treatises on so-called Vector Analysis have been produced as the result of a need for the expression in a condensed and convenient symbolism of many results which are required in Mathematical Physics, especially in electromagnetic theory. Many of these results, though usually arrived at by means of Cartesian analysis, are independent of the particular system of coordinates used. The fundamental theorems of Gauss, Green, and Stokes have clearly a significance beneath their expression in any special set of space co-ordinates, and one which should not depend for its elucidation on a complex demonstration of an invariant of a transformation from one set to another. Such an elementary significance can only be surely arrived at and clearly demonstrated by a systematic building up from the beginning of a calculus of vectors as functions of a position vector. An attempt to do this will naturally be guided by those simple results which have emerged from mathematical physics; but it will naturally require to come into line with the more developed theory of functions of a single real scalar variable which, ideally, should become but a particular case. The present paper seeks to give a first outline of such an attempt. Much of it will appear as an obvious generalization of the well established theory; other parts, while not giving any new results, attempt to give a more systematic place to old ones; and, finally, theorems are given which are generalizations of the classical mean-value theorems of the differential and integral calculus.

The point which is most insisted upon is the classification of functions not by their properties treated as functions of three separate quantities varying independently, but as functions of a single varying vector. Thus, the usual partial differential coefficients are not mentioned, and it is found that a classification, according to the existence or non-existence of the quantities commonly known as the *gradient* of a scalar, and the *divergence* and *curl* of a vector is vital, and, in very many respects, analogous to the usual classification of functions of a real variable as possessing or not possessing a differential coefficient for various values of the variable.

The Independent Variable.

2. The quantities to be considered are such as vary from point to point in space. It is presupposed that there is an absolute standard of direction and an absolute zero of position—a point O called the *origin*. The position of any point P is given by the magnitude and direction of the line OP. This vector OP is the independent variable, and will be denoted by $\mathbf{x}$.

Vector and Scalar Functions.

3. Any quantity which has a defined value at each point P of a given region of space is called a function of $\mathbf{x}$. Such functions may be *scalar functions*, possessing the property of magnitude only, or *vector functions* possessing also the property of direction. To distinguish between these a scalar function will be represented thus—$f(\mathbf{x})$—and a vector function by $\mathbf{f}(\mathbf{x})$.

It will be convenient often to think of the value of a vector function as represented in a subsidiary *vector diagram*, taking a fixed point o and a line op to represent the value of the function in magnitude and direction.

The *magnitude* or *absolute value* of a vector will be denoted by $|\mathbf{f}|$.

The *scalar product* of two vectors $\mathbf{f}, \mathbf{g}$ is the quantity $|\mathbf{f}||\mathbf{g}|\cos\theta$, where θ is the angle between their directions, and is denoted by $(\mathbf{fg})$.

The *vector product* of the same two vectors is a vector $\mathbf{F}$ such that $|\mathbf{F}| = |\mathbf{f}||\mathbf{g}|\sin\theta$, and in a direction perpendicular to both $\mathbf{f}$ and $\mathbf{g}$ in such sense that a positive rotation about $\mathbf{F}$ is a rotation from $\mathbf{f}$ to $\mathbf{g}$. $\mathbf{F}$ is denoted by the symbol $[\mathbf{fg}]$.

Continuous Functions.

4. A function (vector or scalar) is said to be continuous at the point $\mathbf{x}$

if, corresponding to any arbitrarily assigned small quantity ϵ, a number δ, not zero (depending on ϵ), can be found such that

$$|\mathbf{f}(\mathbf{x}')-\mathbf{f}(\mathbf{x})| < \epsilon$$

for all values of $\mathbf{x}'$ which are within the given region and are such that

$$|\mathbf{x}'-\mathbf{x}| < \delta.$$

A Curve.

If the vector $\mathbf{x}$ is a function of a scalar variable x, then, as x varies from a to b, $\mathbf{x}$ varies, say, from $\mathbf{a}$ to $\mathbf{b}$, and the point whose position vector is $\mathbf{x}$ is said to describe *a curve* from the former point to the latter. If $\mathbf{x}$ is a continuous function of x, within a continuous region of x, then the curve is said to be *continuous* between its extreme points.

5. THEOREM.—*If a scalar function $f(\mathbf{x})$ is continuous in a continuous region of space S, and if f_1, f_2 are the values of $f(\mathbf{x})$ at points P_1, P_2, within or upon the boundary of S, then on any continuous curve drawn within S from P_1 to P_2 there is at least one point at which $f(\mathbf{x})$ takes any arbitrary value intermediate between f_1 and f_2.*

For, along the continuous curve, $\mathbf{x}$ is a continuous function of x and $f(\mathbf{x})$ is a continuous function of $\mathbf{x}$; therefore, any quantity ϵ being assigned, a quantity δ can be found such that

$$|f(\mathbf{x}')-f(\mathbf{x})| < \epsilon$$

for all values of $\mathbf{x}'$ such that

$$|\mathbf{x}'-\mathbf{x}| < \delta;$$

and again a quantity γ can be found such that

$$|\mathbf{x}'-\mathbf{x}| < \delta$$

for all values of x' such that

$$|x'-x| < \gamma.$$

Hence, for any ϵ, γ can be found such that

$$|f(\mathbf{x}')-f(\mathbf{x})| < \epsilon$$

for all values of x' such that

$$|x'-x| < \gamma,$$

and therefore along the continuous curve $f(\mathbf{x})$ is a continuous function of x.

The given theorem is thus reduced to the ordinary theorem of the function of one variable.

6. *Corresponding Theorem for a Vector Function.*

If a vector function $\mathbf{f}(\mathbf{x})$ *is continuous within a given region and* $\mathbf{f}_1$, $\mathbf{f}_2$ *are the values of* $\mathbf{f}$ *at two points* P_1, P_2 *within the region or upon the boundary, and if* op_1, op_2 *represent* $\mathbf{f}_1$ *and* $\mathbf{f}_2$, *then, if a continuous curve* C *be drawn from* P_1 *to* P_2, *the aggregate of values taken by* $\mathbf{f}$ *at all the points of* C *are represented in the vector diagram by all the points of a continuous curve from* p_1 *to* p_2.

For if any continuous curve be drawn from P_1 to P_2 by making $\mathbf{x}$ a continuous function of a scalar variable x within a continuous range a to b, then $\mathbf{f}(\mathbf{x})$ is, just as in the preceding theorem, a continuous function of x, and therefore the representative point p in the vector diagram given by the position vector $\mathbf{f}$ describes a continuous curve from p_1 to p_2.

7. The following theorem is the generalization of the well known property of a continuous function that its limiting values are actually reached.

The values of a continuous vector function within a continuous region are represented in a vector diagram by all the points of a continuous region, including all points on the boundary.

Let P be a point $\mathbf{x}_1$ in the field, and let the value of a continuous function $\mathbf{f}$ tend to the limit $\mathbf{F}$ represented in the vector diagram by the point p as the point P is approached along any continuous curve. If possible, let the value of $\mathbf{f}$ at P be $\mathbf{G}$, a vector not equal to $\mathbf{F}$.

Since $\mathbf{f}$ is continuous, if ϵ is any arbitrary small quantity a finite quantity δ can be found such that for all values of $\mathbf{x}$ such that

$$|\mathbf{x}-\mathbf{x}_1| < \delta$$

$$|\mathbf{f}-\mathbf{G}| < \epsilon.$$

But, since $\mathbf{F}$ is the limit of $\mathbf{f}$ as P is approached, a finite quantity δ_1 can be found such that for all points on the path by which P is approached which are such that

$$|\mathbf{x}-\mathbf{x}_1| < \delta_1$$

$$|\mathbf{f}-\mathbf{F}| < \epsilon;$$

that is within the smaller of the regions

$$|\mathbf{x}-\mathbf{x}_1| < \delta \quad \text{and} \quad |\mathbf{x}-\mathbf{x}_1| < \delta_1$$

there would be points for which

$$|\mathbf{f}-\mathbf{F}| < \epsilon \quad \text{and} \quad |\mathbf{f}-\mathbf{G}| < \epsilon,$$

no matter how small ϵ may be.

Hence $\mathbf{F}$ must be equal to $\mathbf{G}$.

It follows that all limiting values of the function are actually taken, that the function tends to the same limit in approaching a given point by all possible paths, and that the points on the boundary of the region occupied by the representative points represent values actually taken. By the preceding theorem the region must be a continuous one.

Integration of Vector Functions.

8. In the theory of functions of a single variable it has been common practice to deal with differentiation before integration. The reason is to be found in the history of the theory. There is now little reason to be alleged against an inversion of the order, inasmuch as integration is applicable to a much larger class of functions than that which possesses the property of differentiability. In the ideas which will now be developed it is essential to introduce first the conception of integration.

Three kinds of integrals arise according as the region of integration is a curve, a surface, or a volume; a further subdivision occurs according as the subject of integration is scalar or vector.

Volume Integrals.

9. Let $\mathbf{f}$ be any vector function defined for a continuous volume V, and let V be divided into any number n of smaller volumes v_r such that the greatest diameter of each of them is less than an arbitrarily chosen quantity d. Let $\mathbf{F}_r$ be any value taken by $\mathbf{f}$ within v_r.* Let $\mathbf{I}_n = \Sigma \mathbf{F}_r v_r$, where the summation extends to all the n subvolumes.

By a process of repeated subdivision, n being constantly increased and d constantly diminished, let the volumes v be made to tend to zero; then, if $\mathbf{I}_n$ tends to a definite limit $\mathbf{I}$, which is independent of the particular manner in which the process of subdivision is carried out, the

* A more general statement would be: Let $\mathbf{F}_r$ be any value within the smallest closed convex continuous region in the vector diagram which includes all the values taken by $\mathbf{f}$ within v_r.

function $\mathbf{f}(\mathbf{x})$ is said to be integrable in the volume V, and $\mathbf{I}$ is said to be the value of the volume integral of $\mathbf{f}$ through V and is denoted by $\int_V \mathbf{f}\,dv$.

The necessary and sufficient condition of integrability can be readily shewn to be that *if* $\mathbf{F}_r$ *and* $\mathbf{G}_r$ *are any two values whatever of* $\mathbf{f}$ *within* v_r, *the sum* $\Sigma(\mathbf{G}_r - \mathbf{F}_r)v_r$ *shall tend to zero as the subdivision advances.*

The simplest method of proof of the sufficiency of this condition is, perhaps, to consider the components of the vectors in an arbitrary direction and to apply Jordan's method of the upper and lower integrals to the scalar integral so obtained.

Surface Integrals.

10. In the same way a *surface integral* is defined. Let S be any finite portion of a continuous surface, and let $\mathbf{f}$ be a vector function defined at every point of the surface.

Let the surface be divided into any number of small areas of greatest diameter not greater than d.

Let $\mathbf{S}_r$ be a vector whose magnitude is the area of one of these smaller areas and whose direction is that of the normal at any point of the same area taken always in the same sense with respect to S; let $\mathbf{F}_r$ be the value of $\mathbf{f}$ at any point of the area.

Then, if $\Sigma(\mathbf{F}_r \mathbf{S}_r)$ tends to a definite limit as the process of subdivision is carried further in any way so that d tends to zero, independently of the manner of subdivision, this limit is said to be the surface integral of $\mathbf{f}$ over the surface S and is denoted by $\int_S \mathbf{f}d\mathbf{S}$, and $\mathbf{f}$ is said to be integrable over S. A condition of integrability analogous to that for the volume integral can at once be laid down.

Line Integrals.

11. In exactly a similar way we define the *line integral* of a vector $\mathbf{f}$ taken along a curve S. Let $\boldsymbol{\sigma}_r$ be a vector whose magnitude is equal to the length of a small arc of S of length less than d, and whose direction is that of the curve at some point of this arc, and let $\mathbf{f}_r$ be be the value of $\mathbf{f}$ at any point whatever of the same arc.

Then, if $\Sigma(\mathbf{f}_r \boldsymbol{\sigma}_r)$ has a definite limit as d tends to zero independently of the particular mode of carrying out the subdivision, this limit is called the line integral of $\mathbf{f}$ along $\mathbf{s}$, and is denoted by

$$\int_s \mathbf{f}d\mathbf{s}.$$

Again a condition of integrability may be laid down.

It is clear that the conditions of integrability for the three types of integrals are not equivalent.

If a function is integrable through a volume it is not necessarily integrable over any surface included in that volume, and, if it is in-

tegrable over a surface, it is not necessarily integrable along a curve drawn on that surface.

But it is easy to conceive of functions which satisfy all these conditions : in particular, a function which is continuous in a given volume is integrable for any volume, surface of finite area, or curve of finite length contained within it.

Scalar Integrals.

12. There is hardly need to speak here of the corresponding integrals when the subject of integration is a scalar function ; in such a case the elements of area and arc dS, and ds are also treated as scalar quantities. The theory of such integrals is well established, but we may note as above that there are distinct conditions of integrability for each of the three types of integral.

Space Differentiation.

13. *The gradient of a scalar.*

Let $f(\mathbf{x})$ be any scalar function of $\mathbf{x}$.

In the case of many simple functions the surfaces $f =$ constant which pass near to any point P are such that a single normal can be drawn from P to each of them, and that this normal tends to a definite direction—viz., that of the normal to the surface through P as the surface approaches P.

In such a case let a vector $\mathbf{g}$ be defined whose magnitude is

$$\mathrm{Lt}_{P' \to P} \frac{f_{P'} - f_P}{PP'}$$

as P' approaches P along this normal (assuming that the limit exists), and whose direction is that of the normal. $\mathbf{g}$ is known as the *gradient* of f.

If the conditions above mentioned are satisfied and Q is a point near to P on a line making an angle θ with the same normal and on the surface $f = f_{P'}$, then

$$\begin{aligned} \mathrm{Lt}_{Q \to P} \frac{f_Q - f_P}{PQ} &= \mathrm{Lt}_{P' \to P} \frac{f_{P'} - f_P}{PP'} \frac{PP'}{PQ} \\ &= |\mathbf{g}| \cos\theta \\ &= (\mathbf{g}, \mathbf{r}_1), \end{aligned}$$

where $\mathbf{r}_1$ is a unit vector in the direction of PQ.

In other words, the rate of increase of f in any direction is the component of $\mathbf{g}$ in that direction.

The conditions required for the existence of $\mathbf{g}$ can, of course, be expressed in terms of the behaviour of f treated as a function of Cartesian

coordinates; but, for the present purposes, it is sufficient to simply classify scalar functions as those which do or do not possess a gradient at certain specified or at all points in a given region, just as functions of a single variable are classified according to the existence of a differential coefficient.

A function which possesses a gradient at a point is obviously continuous at that point.

A notation that is often used for the gradient of a function f is ∇f.

The Divergence of a Vector.

14. A quantity which is of great importance in connexion with vector space functions, and which, it will be seen, is in many ways analogous to the differential coefficient of a function of one variable, is the so-called *divergence* of a function. This will be defined as follows:—

If $\mathbf{f}$ *is any function which is integrable over any surface in its domain,* P *is any point in the domain and* V *is any small continuous volume including* P*, and bounded by a closed surface* S *whose greatest diameter is* d*, and if the quantity* $\frac{1}{V}\int_S \mathbf{f}\,d\mathbf{S}$ *tends to a definite limit as* d *tends to zero, whatever be the sequence of forms taken by* S *during the approach to zero, this limit is called the* DIVERGENCE *of* $\mathbf{f}$ *at the point* P *and is denoted by div* $\mathbf{f}$.

Total Divergence.

It will be convenient at times to speak of $\int \mathbf{f}\,d\mathbf{S}$ taken over the boundary of any closed surface as *the total divergence* in the enclosed volume, and we may speak of div $\mathbf{f}$ at P as the limit of the *average divergence* in a small volume including P. If P is on the boundary of the region for which $\mathbf{f}$ is defined, V must be taken to be a volume partly bounded by the boundary of the domain of $\mathbf{f}$, so that P in this case lies on the surface S.

15. It may be noted that it would be equally possible to define div $\mathbf{f}$ at a point within the region by taking S to be always a surface passing through P. For, if $\frac{1}{V}\int_S \mathbf{f}\,d\mathbf{S}$ has the same limit A for surfaces S_1 and S_2, including two volumes V_1 and V_2, which together make up a volume V of which P is an interior point, we have

$$\int_{S_1} \mathbf{f}\,d\mathbf{S} = V_1(A+\epsilon_1),$$

$$\int_{S_2} \mathbf{f}\,d\mathbf{S} = V_2(A+\epsilon_2),$$

and $$\int_S \mathbf{f}\,d\mathbf{S} = \int_{S_1} \mathbf{f}\,d\mathbf{S} + \int_{S_2} \mathbf{f}\,d\mathbf{S} = (V_1 + V_2)A + V_1\epsilon_1 + V_2\epsilon_2,$$

where ϵ_1 and ϵ_2 both tend to zero with d. Hence

$$\frac{1}{V}\int_S \mathbf{f}\,d\mathbf{S} = A + \frac{V_1\epsilon_1 + V_2\epsilon_2}{V_1 + V_2};$$

and therefore $$\text{Lt}\,\frac{1}{V}\int_S \mathbf{f}\,d\mathbf{S} = A,$$

whatever the ratio in which V_1 and V_2 vanish.

Conversely, if $\frac{1}{V}\int_S \mathbf{f}\,d\mathbf{S}$ tends to a definite limit independently of the shape of S, $\frac{1}{V_1}\int_{S_1} \mathbf{f}\,d\mathbf{S}$ and $\frac{1}{V_2}\int_{S_2} \mathbf{f}\,d\mathbf{S}$ must each have a definite limit, and these limits must be the same. For, if not, let

$$\int_{S_1} \mathbf{f}\,d\mathbf{S} = V_1F_1,$$

and $$\int_{S_2} \mathbf{f}\,d\mathbf{S} = V_2F_2.$$

Then $$\frac{1}{V}\int_S \mathbf{f}\,d\mathbf{S} = \frac{V_1F_1 + V_2F_2}{V_1 + V_2}.$$

Then, if F_1 has a definite limit which is not equal to lim F_2, since the ratio of V_1 to V_2 is arbitrary, $(V_1F_1 + V_2F_2)/(V_1 + V_2)$ can be made to approach any arbitrary limit between F_1 and F_2; and, if F_1 or F_2 or both have no definite limit, it is impossible that

$$(V_1F_1 + V_2F_2)/(V_1 + V_2)$$

should have a definite limit.

A similar remark may be made in respect of the ordinary differential coefficient of a function of one variable. In order that the differential coefficient of a function $f(x)$ shall exist at $x = a$, a point within the region of definition of f, it is necessary and sufficient that h and k being two positive quantities

$$\frac{f(a+h) - f(a-k)}{h+k}$$

shall have a definite limit as h and k tend to zero in any arbitrary ratio. This condition includes the identity of the left and right differential coefficients.

At an end point of the domain of f either h or k is to be taken permanently to be zero.

The Curl of a Vector.

16. Let $\mathbf{f}$ be a vector function which is assumed to be integrable in the third sense, *i.e.* along any line drawn in the domain of $\mathbf{f}$. If P is any point, let S be any small continuous plane area including P and bounded by a closed curve s, the greatest diameter of S being d. Let it be assumed that

$$\frac{1}{S}\int_s \mathbf{f}\,d\mathbf{s}$$

tends to a definite limit as d tends to zero, and that independently of the sequence of forms through which s shrinks up to P_1.

Further, let it be assumed that this is so for all directions of the normal to the area S, and that for each direction the limit so obtained is continuous for displacements of P in that direction and uniformly so for all directions round P.*

With these conditions it may be shown that at each point P there is one direction for which the limit has a maximum value, and if a vector $\mathbf{c}$ be defined with this direction and of magnitude equal to this maximum value, then the limit for other direction at P is the component of $\mathbf{c}$ along that direction.†

Proof.—Describe a small tetrahedron $ABCD$ to include P, the edges DA, DB, DC being mutually at right angles, and the greatest edge of the tetrahedron being less than an assigned quantity d. Let the faces be called α, β, γ, δ.

Then, if the integrals are taken in the same sense round the four faces relatively to the normals drawn outward from the tetrahedron,

$$\int_\alpha \mathbf{f}\,\mathbf{ds}+\int_\beta \mathbf{f}\,\mathbf{ds}+\int_\gamma \mathbf{f}\,\mathbf{ds}+\int_\delta \mathbf{f}\,\mathbf{ds}$$

is identically zero.

If c_1, c_2, c_3, c_4 are the limits of the quantities $\int \mathbf{f}\,\mathbf{ds}/S$ at P for the directions of the outward drawn normals, by choosing d sufficiently small we may make

$$\int_\alpha \mathbf{f}\,\mathbf{ds} = \alpha(c_1+\epsilon_1),$$

$$\int_\beta \mathbf{f}\,\mathbf{ds} = \beta(c_2+\epsilon_2),$$

$$\int_\gamma \mathbf{f}\,\mathbf{ds} = \gamma(c_3+\epsilon_3),$$

$$\int_\delta \mathbf{f}\,\mathbf{ds} = \delta(c_4+\epsilon_4),$$

where ϵ_1, ϵ_2, ϵ_3, ϵ_4 are all less than an assigned quantity ϵ.

Thus assuming that α, β, γ are not all zero—in which case δ would be zero whatever the direction of the face ABC,

$$-(c_4+\epsilon_4) = \frac{\alpha}{\delta}(c_1+\epsilon_1)+\frac{\beta}{\delta}(c_2+\epsilon_2)+\frac{\gamma}{\delta}(c_3+\epsilon_3).$$

---

* Possibly these requirements may be to a certain extent tautologous, but care has been taken not to understate the limitations imposed on the function.

† The proof here given involves a certain return to Cartesian methods, but it is not easy to see how this could be avoided entirely at this step.

Calling l, m, n the direction cosines of the outward normal to ABC referred to the directions DA, DB, DC, and, proceeding to the limit, we find that

$$c_4 = lc_1 + mc_2 + nc_3,$$

with the condition that $l^2+m^2+n^2 = 1$.

To make c_4 a maximum for varying values of l, m, n

$$0 = c_1\delta l + c_2\delta m + c_3\delta n,$$

subject to

$$0 = l\delta l + m\delta m + n\delta n\,;$$

so that $\frac{l}{c_1} = \frac{m}{c_2} = \frac{n}{c_2} = \frac{1}{|\mathbf{c}|}$ gives the direction of the maximum value of c_4, and c_4 is zero for any direction perpendicular to this. Taking the direction so found for one of the edges of the tetrahedron, so that

$$c_1 = |\mathbf{c}|, \quad c_2 = 0, \quad c_3 = 0, \quad \text{then} \quad c_4 = |\mathbf{c}|\cos\theta,$$

where θ is the angle between the normal to the face δ and $\mathbf{c}$.

The required conditions for the existence of $\mathbf{c}$ being satisfied, we call $\mathbf{c}$ the *curl* of the vector $\mathbf{f}$, and denote it by curl $\mathbf{f}$.

Thus we have a further classification of functions by the existence or non-existence of the *curl* at specified points or at all points of a given region.

Circulation.

The value of the integral $\int \mathbf{f}\,\mathbf{ds}$ round a closed curve has been called the *circulation* round the curve. It will sometimes be useful to retain the term. The vector *curl* may be defined as *the mean circulation per unit area* of a small circuit at a point.

17. It is clear that the *divergence* and *curl* of a vector function are, in certain senses, extensions of the conception of the differential coefficient of a function of one variable.

If the vector function has the same direction at all points of the region, and the same value at all points of a plane perpendicular to this direction, it becomes effectively equivalent to a scalar function of one variable—viz., a coordinate in the direction of the vector—and the divergence reduces to the differential coefficient of this function with respect to this variable.

In the same way, if the vector has the same direction at all points, and the same value at all points of a set of parallel planes to which this direction is parallel, the *curl* reduces to the differential coefficient of the magnitude of the vector with respect to a coordinate perpendicular to the planes.

The analogy and generalization will become clearer in the light of theorems now to be proved.

Mean-value Theorems.

18. An extension will now be given of the theorem that if $f(x)$ is continuous in the interval $(x, x+h)$ and at every point within this interval $f'(x)$ exists and is finite,* then

$$f(x+h)-f(x) = hf'(x+\theta h),$$

where θ is a number such that $0 < \theta < 1$.

The theorem is as follows :—

If $\mathbf{f}$ *is continuous in a continuous volume* V *bounded by the closed surface* S, *and if div* $\mathbf{f}$ *exists at all points within* V, *then there is a point* P *within* S *such that*

$$\int_S \mathbf{f}\,\mathbf{dS} = V \operatorname{div}_P \mathbf{f},$$

where $\operatorname{div}_P \mathbf{f}$ *denotes the value of div* $\mathbf{f}$ *at* P.

The theorem may be reduced to the following particular case :

If $\int_S \mathbf{f}\,\mathbf{dS} = 0$, *then there is a point within* S *at which div* $\mathbf{f}$ *vanishes*,

the conditions as to continuity and the existence of div $\mathbf{f}$ being unchanged.

This reduction is effected thus :

If $\mathbf{r}$ denotes the position vector of a variable point from any fixed point,

$$\int_S \tfrac{1}{3}\mathbf{r}\,\mathbf{dS} = V.$$

Hence, if

$$\int_S \mathbf{f}\,\mathbf{dS} = KV,$$

$$\int_S (\mathbf{f}-\tfrac{1}{3}K\mathbf{r})\,\mathbf{dS} = 0.$$

Thus, if

$$\mathbf{f}' = \mathbf{f}-\tfrac{1}{3}K\mathbf{r},$$

$$\int_S \mathbf{f}'\,\mathbf{dS} = 0.$$

To save writing, the sign D_S will be used in the following proof to denote $\int_S \mathbf{f}'\,\mathbf{dS}$, which has been called above the *total divergence* of $\mathbf{f}'$ in V.

Suppose now V is divided into any two portions V_1 and V_2 by a plane,

* The case of $f'(x)$ becoming infinite with fixed sign is not here considered.

and that the complete boundaries of these two portions are called S_1 and S_2.

Then $$D_{S_1} + D_{S_2} = D_S = 0.$$

Hence either D_{S_1} or D_{S_2} is positive and the other negative or both are zero.

In the first case suppose D_{S_1} is positive.

Now divide V again into two parts V_3 and V_4 by any plane which cuts the former plane within the region; call the complete boundaries S_3 and S_4. As before, either D_{S_3} and D_{S_4} are both zero or they are of opposite signs.

Let D_{S_3} be negative.

Since the planes cut within the region, V_1 and V_3 have a common part. Call this part V_5 and its boundary S_5. Let D_{S_5} be positive.

Then V_5 is a portion of V_3 and D_{S_5} and D_{S_3} have opposite signs. But, if we consider the plane which cuts off V_5 from V_3 to move continuously so that V_5 expands towards V_3, D_{S_5} will vary continuously (since $\mathbf{f}'$ is continuous), and therefore it must for some position of the plane take the value zero.

A similar argument holds if D_{S_5} is negative, and therefore of opposite sign from D_{S_1}.

Hence in all cases it is possible to find a region which is a part of V, say V', for which the total divergence $D_{S'}$ is still zero.

In the same way a part V'' of V' can be found for which the total divergence $D_{S''}$ is zero, and so on. Also, owing to the arbitrariness of the initial dividing plane, it could, for instance be always secured that

$$V' \leqslant \tfrac{1}{2}V, \quad V'' \leqslant \tfrac{1}{2}V',$$

and so on.

Thus the sequence of volumes will have a limit zero, and, since the directions of the dividing planes are arbitrary, it can also be secured not only that the volumes have a limit zero, but that their greatest diameters have a limit zero. Therefore, since each volume in the sequence is interior to the preceding one, the sequence must converge to a limiting point in a definite position. If this point is P, it has therefore been proved that a surface can be found surrounding P whose greatest diameter is less than any assigned small quantity d for which the total divergence is zero, and therefore it follows that, since div $\mathbf{f}'$ exists at P, it is zero at that point.

Hence $$\operatorname{div}_P (\mathbf{f} - \tfrac{1}{3}K\mathbf{r}) = 0,$$

or $$\operatorname{div}_P \mathbf{f} = \tfrac{1}{3}K \operatorname{div} \mathbf{r} = K$$
$$= \frac{1}{V}\int_S \mathbf{f}\, d\mathbf{S}.$$

It may be noted that, with the modification of the definition of an ordinary differential coefficient given in § 15, a proof of the ordinary mean-value theorem can at once be constructed on the lines of this proof.

19. A similar theorem may be proved in a similar manner in respect of curl $\mathbf{f}$ in case it exists at all points of a surface S.

The statement of the theorem is :

If $\mathbf{f}$ *is continuous over a continuous unclosed surface* S *within the region of existence of* $\mathbf{f}$, *and if curl* $\mathbf{f}$ *exists at all points of* S, *then if* $\int \mathbf{f}\,d\mathbf{s}$ *taken round the rim of* S *is equal to zero, a point* P *can be found on* S *such that*

$$\int_S \mathbf{f}\,d\mathbf{s} = S(\mathbf{n} \operatorname{curl} \mathbf{f})_P,$$

where $\mathbf{n}$ *is a unit vector in the direction of the normal at* P.

Differentiation of a Product and Integration by Parts.

20. The analogues of these most important processes are practically contained in the well known Green's Theorem.

This may be expressed in vector notation in the form :

If f *is any scalar, and* $\mathbf{g}$ *a vector, then, subject to certain restrictions which will be obtained below,*

$$\operatorname{div}(f\mathbf{g}) = f \operatorname{div} \mathbf{g} + (\nabla f . \mathbf{g}).$$

This is strictly analogous to the formula for differentiating a product, with the remark that, in the case of a mixture of scalars and vectors, the appropriate kind of differentiation must be applied to each.

To obtain this formula we revert to the integral expression whence the divergence is obtained, $\int_S f\mathbf{g}\,d\mathbf{S}$, where S is a closed surface enclosing a point P. It is, of course, assumed that f and $\mathbf{g}$ are separately integrable, and that therefore their product is so.*

---

* This is readily shewn.

Let f_P, $\mathbf{g}_P$ be the values of f and $\mathbf{g}$ at P, and let

$$f = f_P + f', \quad \mathbf{g} = \mathbf{g}_P + \mathbf{g}'.$$

Then
$$\int_S f(\mathbf{g}\,\mathbf{dS}) = \int_S f_P(\mathbf{g}\,\mathbf{dS}) + \int f'(\mathbf{g}_P\mathbf{dS}) + \int f'(\mathbf{g}'\mathbf{dS}). \qquad (A)$$

In the first integral on the right-hand side, f_P, being a constant scalar, may be put before the integral sign as a multiplier.

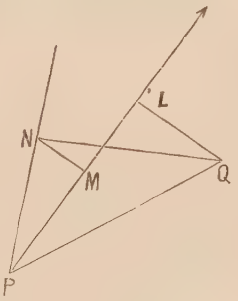
In treating the second integral the restriction will be made that f', and therefore also f, possesses a gradient $\nabla_P f$ at P. If $\mathbf{r}$ is the position vector of δS from P, a quantity d can be chosen so that for all values of $|\mathbf{r}|$ which are less than d

$$f' = \mathbf{r}\nabla_P f + \epsilon\,|\mathbf{r}|,$$

where ϵ is an arbitrarily chosen small quantity.

Consider now $\int (\mathbf{r}\nabla_P f)(\mathbf{g}_P\mathbf{dS})$.

In the figure let PQ be $\mathbf{r}$, QN the perpendicular from Q on PN the direction of $\mathbf{g}_P$, QL, NM the perpendiculars from Q and N on PM the direction of $\nabla_P f$.



Then
$$\begin{aligned}\mathbf{r}\nabla_P f &= |\nabla_P f|\,PL \\ &= |\nabla_P f|\,(PM + ML) \\ &= |\nabla_P f|\,(PN\cos\psi + ML),\end{aligned}$$

where ψ is the angle between $\mathbf{g}$ and $\nabla_P f$.

Also
$$(\mathbf{g}_P\mathbf{dS}) = |\mathbf{g}_P|\,d\sigma,$$

where $d\sigma$ is the projection of dS on the plane perpendicular to $\mathbf{g}_P$.

Hence
$$\begin{aligned}\int PN\,(\mathbf{g}_P\mathbf{dS}) &= |\mathbf{g}_P| \int PN\,d\sigma \\ &= |\mathbf{g}_P|\,V.\end{aligned}$$

Also
$$\int ML\,d\sigma = 0,$$

because, since the surface is closed, to every element $d\sigma$ must correspond another with opposite sign (as in the ordinary proof of Gauss's theorem) with the same QN in direction and magnitude.

Hence
$$\begin{aligned}\int (\mathbf{r}\nabla_P f)(\mathbf{g}_P\mathbf{dS}) &= |\mathbf{g}_P|\,|\nabla_P f|\,V\cos\psi \\ &= (\mathbf{g}_P\nabla_P f)\,V.\end{aligned}$$

In treating the third integral on the right-hand side of A, the simplest assumption that can be made is that $\mathbf{g}$ is continuous. If this is so, then for any arbitrarily chosen small quantity η, the greatest diameter d of V can be chosen so that $|\mathbf{g}'| < \eta$ at all points of S.

Also $$|f'| < |\mathbf{r}||\nabla_P f + \epsilon|.$$

Hence $$\left|\int f'(\mathbf{g}\,\mathbf{dS})\right| < \eta\,|\nabla_P f + \epsilon| \int |\mathbf{r}||\,\mathbf{dS}|,$$

and provided S is not a surface of infinite corrugation,

$$\int |\mathbf{r}||\,\mathbf{dS}| = kV,$$

where k is a finite quantity, so that

$$\left|\int f'(\mathbf{g}\,\mathbf{dS})\right| < \eta k V\,|\,\nabla_P f + \epsilon\,|.$$

In the same way the remaining part of the second integral in (A), viz.,

$$\left|\int \epsilon\,|\mathbf{r}|\,(\mathbf{g}_P\,\mathbf{dS})\right| < \epsilon k V |g_P|.$$

Hence, from (A),

$$\left|\frac{1}{V}\int_S f(\mathbf{g}\,\mathbf{dS}) - \left\{f_P\,\frac{1}{V}\int_S (\mathbf{g}\,\mathbf{dS}) + (\mathbf{g}_P \nabla_P f)\right\}\right| < \eta k\,|\nabla_P f + \epsilon\,| + \epsilon k V |\mathbf{g}_P|,$$

so that, passing to the limit, if div $\mathbf{g}$ exists,

$$\text{Lt}\ \frac{1}{V}\int_S f(\mathbf{g}\,\mathbf{dS})$$

also exists, and is equal to

$$f_P \text{ div } \mathbf{g} + \mathbf{g}_P \nabla_P f;$$

that is, dropping the subscript P, div $(f\mathbf{g})$ exists, and is equal to

$$f \text{ div } \mathbf{g} + (\mathbf{g}\,\nabla f).$$

It may be well to repeat the assumptions that have been made in obtaining this equation.

(i) f is finite and integrable over any surface;

(ii) ∇f exists at P;

(iii) div $\mathbf{g}$ exists and is finite at all points in a small neighbourhood surrounding P;

(iv) $\mathbf{g}$ is continuous.*

The theorem as thus proved is of greater generality than Green's Theorem as ordinarily proved by means of Cartesian Coordinates, as it is possible to conceive of functions for which the divergence exists, while the partial differential coefficients do not.

21. An exactly similar proof can be given of the following proposition :—

If (i) *f is finite and integrable round any curve,*

(ii) *∇f exist at a point P,*

(iii) *curl $\mathbf{g}$ exists and is finite at all points in a small neighbourhood surrounding P,*

then curl $f\mathbf{g}$ exists at P and is equal to

$$f \operatorname{curl} \mathbf{g} + [\nabla f, \mathbf{g}].$$

22. At this point it may be appropriate to refer to a well known result which follows immediately from that of Green, and which now appears as a generalization of the simple theorem that *a function can be uniquely determined whose values $f(a)$, $f(b)$ are given for the end points (a, b) of an interval and whose differential coefficient exists and is a given integrable function $f'(x)$, provided that*

$$f(b) - f(a) = \int_a^b f'(x)\, dx.$$

The generalization is

A vector function $\mathbf{f}$ *can be uniquely determined by the following conditions :—*

(i) $\mathbf{f}$ *is equal to a given function* $\mathbf{f}_0$ *on a closed surface S;*

---

* It is possible that conditions (iii) and (iv) are to a certain extent tautologous, but this point is left for future investigation. This section has been slightly altered from the form in which it was read.—[*March 4th*, 1913.]

(ii) *div* $\mathbf{f}$ *and curl* $\mathbf{f}$ *exist and are given at all points within* S *and on the boundary;*

(iii) *provided that* $$\int_S \mathbf{f}_0 \mathbf{dS} = \int_V \operatorname{div} \mathbf{f}\, dv.$$

Some Mean-value Theorems in Vector Integration.

23. *If* $\mathbf{f}$ *is a vector which possesses a volume integral for a given region, and if a vector diagram be drawn to represent all the values of* $\mathbf{f}$ *in the region, and if* S *is the smallest, everywhere convex, closed surface which can be drawn in the vector diagram to contain all the representative points, then* $$\int_V \mathbf{f}\, dv = V\mathbf{f}_1,$$

where $\mathbf{f}_1$ *is a vector represented by a point within or upon* S.

This theorem follows at once from a consideration of the component of the integral in a given direction. If $\mathbf{t}$ is an arbitrary fixed unit vector

$$\mathbf{t}\int \mathbf{f}\, dv = \int (\mathbf{tf})\, dv$$
$$= VA,$$

where A is a quantity not greater than the greatest and not less than the least value of $\mathbf{tf}$. The greatest and least values of $\mathbf{tf}$ can be found by drawing planes perpendicular to $\mathbf{t}$ through the points representing the values of $\mathbf{f}$ and taking the two extreme planes so obtained.

The volume contained between the pairs of planes so obtained for all directions of $\mathbf{t}$ is the smallest, everywhere convex, closed surface containing all the representative points.

Hence the theorem.

If $\mathbf{f}$ is constant in direction, and therefore reduces to a scalar function, then $\mathbf{f}_1$ becomes a value between the greatest and least values of $\mathbf{f}$. In particular, if $\mathbf{f}$ is also continuous, $\mathbf{f}_1$ is a value actually taken.

But it is to be noted that in general $\mathbf{f}_1$ need not be a vector to which $\mathbf{f}$ is equal at some point within V.

24. *If* f, g *are two scalar functions of which* g *is everywhere positive, limited, and integrable in a given region* V, *and* f *is continuous, then* $$\int_V fg\, dv = f_1 \int_V g\, dv,$$

where f_1 *is a value taken by* f *at some point in the region* V.

25. *If* $\mathbf{f}$ *is any vector function and* g *is a scalar which is positive, limited, and integrable in a given region* V, *then*

$$\int_V \mathbf{f} g\, dv = \mathbf{f}_1 \int_V g\, dv,$$

where $\mathbf{f}_1$ *is a vector represented by a point within or upon the convex surface* S *drawn as in* § 23.

The proof of this is a simple extension of the theorem of § 23.

26. *If* $\mathbf{f}$, $\mathbf{g}$ *are two vectors which are continuous in a given region* V, *then*

$$\int_V (\mathbf{fg})\, dv = V\mathbf{f}_1\mathbf{g}_1,$$

where $\mathbf{f}_1$, $\mathbf{g}_1$ *are values taken by* $\mathbf{f}$ *and* $\mathbf{g}$ *at some point in* V.

For $(\mathbf{fg})$ is a continuous scalar function.

27. The next theorem to be proved is the extension of the theorem of Weierstrass and Du Bois Reymond. If $f(x)$ be monotone in the interval (a, b) and if $\phi(x)$ be limited and integrable in the same interval, then

$$\int_a^b f(x)\,\phi(x)\,dx = f(a)\int_a^\xi \phi(x)\,dx + f(b)\int_\xi^b \phi(x)\,dx,$$

where ξ is some point in the interval (a, b).

If $\mathbf{f}(\mathbf{x})$ *is a function which is integrable through a closed volume* V *and whose total divergence in any volume which is part of* V *has a constant sign, and if* $\mathbf{g}$ *is a function which is integrable along any curve in the region, and whose circulation round any closed curve is zero, then a point* P *can be found in* V *such that*

$$\int_V (\mathbf{fg})\, dv = \int_S \left(\int_P^S \mathbf{g}\,\mathbf{ds}\right) \mathbf{f}\,\mathbf{dS},$$

where $\int_P^S \mathbf{g}\,\mathbf{ds}$ *denotes the line integral of* $\mathbf{g}$ *along any curve from* P *to the element* dS.

It will, perhaps, make the theorem a little more clear if a proof is first given based on Green's Theorem, just as Weierstrass first proved his theorem by the analogous process of integration by parts.

Let G be a scalar function whose gradient is $\mathbf{g}$. This introduces no

fresh restriction on $\mathbf{g}$, and implies that G is continuous; and let it be first assumed that div $\mathbf{f}$ exists and is of constant sign.

Then, by Green's theorem,

$$\int \mathbf{f} \nabla G \, dv = \int G \mathbf{f} \mathbf{dS} - \int \operatorname{div} \mathbf{f} \, G \, dv.$$

Now, since div $\mathbf{f}$ is everywhere one-signed, and G is continuous in V,

$$\int_V \operatorname{div} \mathbf{f} \, G \, dv = G_P \int_V \operatorname{div} \mathbf{f} \, dv$$
$$= G_P \int_S \mathbf{f} \mathbf{dS},$$

where G_P is a value actually taken by G at some point in V.

Hence
$$\int \mathbf{fg} \, dv = \int_V \mathbf{f} \nabla G \, dv$$
$$= \int_S G \mathbf{f} \mathbf{dS} - G_P \int_S \mathbf{f} \mathbf{dS}$$
$$= \int_S (G - G_P) \mathbf{f} \mathbf{dS}$$
$$= \int_S \left(\int_P^S \mathbf{g} \, \mathbf{ds} \right) \mathbf{f} \mathbf{dS}.$$

Thus the theorem is proved in case div $\mathbf{f}$ exists.

28. This restriction will now be removed by a process which goes back to the method by which Green's Theorem was proved (§ 20), with the removal of certain limitations introduced for that special purpose.

Let the volume V be divided into a large number of smaller volumes $V_1, V_2, \ldots$, each of which has a greatest diameter not greater than an assigned small quantity d, and let the bounding surfaces of these volumes be called $S_1, S_2, \ldots$. Let it be assumed further that if $\mathbf{r}$ is the position vector of a point on the surface of S_n referred to an interior point

$$\int_{S_n} |\mathbf{r}| \, |\mathbf{dS}_n| < k V_n,$$

where k is a finite number, the same for all values of n. This can be easily secured by choosing the method of subdivision suitably, *e.g.*, it is always true if the division is by planes or families of surfaces having finite curvature at all points; and, to avoid complication, it will be

assumed that the surface S is such that the same relation is satisfied for those of the volumes V_1, V_2, ..., which meet the surface S.

As above, let G be a scalar function whose gradient is $\mathbf{g}$, the only restriction placed on G being the existence of $\mathbf{g}$. This implies that G is continuous, and therefore is integrable in each sense.

Let Q_n be any point within S_n.

Then as in the proof of Green's Theorem,

$$\int_{S_n} G(\mathbf{f}\,d\mathbf{S}) = \int_{S_n} G_{Q_n}(\mathbf{f}\,d\mathbf{S}) + \int_{S_n} G'_n(\mathbf{f}_{Q_n}\,d\mathbf{S}) + \int_{S_n} G'_n(\mathbf{f}'_n\,d\mathbf{S}),$$

where $$\mathbf{f} = \mathbf{f}_{Q_n} + \mathbf{f}'_n, \qquad G = G_{Q_n} + G'_n.$$

Since G'_n possesses a gradient, d may be chosen so small that

$$G'_n = \mathbf{r}(\nabla_{Q_n} G + \boldsymbol{\gamma}_n),$$

where $|\boldsymbol{\gamma}_n| < \epsilon$, an arbitrary small quantity. Therefore

$$\begin{aligned}\int_{S_n} G'_n(\mathbf{f}_{Q_n}\,d\mathbf{S}) &= V_n(\mathbf{f}\nabla G)_{Q_n} + \int_{S_n} (\mathbf{r}\boldsymbol{\gamma}_n)\,\mathbf{f}_{Q_n}\,d\mathbf{S} \\ &= V_n(\mathbf{f}\nabla G)_{Q_n} + |\mathbf{f}_{Q_n}|\,\eta_n V_n,\end{aligned}$$

where $$\eta_n < \epsilon k.$$

Again, $$\begin{aligned}\int_{S_n} G'_n(\mathbf{f}'_n\,d\mathbf{S}) &= \int_{S_n} \mathbf{r}\,(\nabla_{Q_n} G + \boldsymbol{\gamma})\}\,(\mathbf{f}'_n\,d\mathbf{S}) \\ &< \theta_n k V_n(|\nabla_{Q_n} G| + |\epsilon|),\end{aligned}$$

where θ_n is a quantity equal to the greatest value of $|\mathbf{f}'_n|$ at all points of S_n.

Now, let the corresponding expressions for all the surfaces S_n be added together.

Then

$$\int G(\mathbf{f}\,d\mathbf{S}) = \Sigma G_{Q_n} \int_{S_n} \mathbf{f}\,d\mathbf{S} + \Sigma V_n(\mathbf{f}\nabla G)_{Q_n} + \Sigma \eta_n\,\mathbf{f}_{Q_n}|\,V_n + R,$$

where $$R < \Sigma k\,\theta_n V_n(|\nabla_{Q_n} G| + |\epsilon|).$$

The number of subdivisions is now to be increased and d is to diminish to zero.

If $\mathbf{f}$ is integrable through V, since $\mathbf{f}'_n$ is the difference between its

values at two points of V_n,

$$\text{Lt } \Sigma\theta_n V_n = 0.$$

Therefore, if ∇G is less than any fixed finite quantity at all points of V,

$$\text{Lt}_{d\to 0} R = 0.$$

Further, if $\mathbf{f}$ is always less than a fixed finite quantity, or more generally if $|\mathbf{f}|$ is integrable,

$$\text{Lt } \Sigma\eta_n |\mathbf{f}_{Q_n}| V_n = 0,$$

and

$$\text{Lt } V_n (\mathbf{f}_n \nabla G)_{Q_n} = \int (\mathbf{f}\nabla G)\, dv.$$

Lastly, if $\int_{S_n} \mathbf{f}d\mathbf{S}$ is always of one sign, $\Sigma G_{Q_n} \int_{S_n} \mathbf{f}d\mathbf{S}$ is intermediate in value between the greatest and least values of G_Q, each multiplied by $\Sigma \int_{S_n} \mathbf{f}d\mathbf{S}$, that is, by $\int_S \mathbf{f}d\mathbf{S}$; therefore

$$\Sigma G_{Q_n} \int_{S_n} \mathbf{f}d\mathbf{S} = G_P \int \mathbf{f}d\mathbf{S},$$

where G_P is the value of G at some point P within S, since all the values G_Q are taken within S, and G is continuous within S.

Hence altogether

$$\int_S G(\mathbf{f}\,d\mathbf{S}) = G_P \int_S \mathbf{f}d\mathbf{S} + \int_V (\mathbf{f}\nabla G)\, dv,$$

or

$$\int_S (G - G_P)\, \mathbf{f}d\mathbf{S} = \int_V (\mathbf{f}\nabla G)\, dv,$$

or

$$\int_S \left(\int_P^S \mathbf{g}\,d\mathbf{s}\right) \mathbf{f}d\mathbf{S} = \int_V \mathbf{f}\mathbf{g}\, dv.$$

Thus the theorem is proved with the restrictions that

(i) $\mathbf{f}$ *is finite and integrable through the volume* V *and over any surface enclosed within it;*

(ii) *the total divergence of* $\mathbf{f}$ *is positive for any part of* V;

(iii) $\mathbf{g}$ *is a vector everywhere finite which is the gradient of a scalar.*

29. A theorem which bears exactly the same relation to the last that the equation

$$\text{curl } \mathbf{f}\mathbf{g} = f \text{ curl } \mathbf{g} + [\nabla f, \mathbf{g}]$$

bears to $\quad \operatorname{div} f\mathbf{g} = f \operatorname{div} \mathbf{g} + (\nabla f \mathbf{g})$

may be proved in the same manner.

If (i) $\mathbf{f}$ *is finite and integrable over a surface* S *and round any closed curve on* S,

(ii) *the circulation of* $\mathbf{f}$ *round any closed curve on* S *has always the same sign,*

(iii) $\mathbf{g}$ *is a vector everywhere finite which on* S *is the gradient of a scalar function,*

then a point P *can be found on* S *such that*

$$\int_S [\mathbf{fg}]\,\mathbf{dS} = \int_S \left(\int_P^S \mathbf{g}\,\mathbf{ds}' \right) \mathbf{f}\,\mathbf{ds},$$

where $\int_P^S \mathbf{g}\,\mathbf{ds}'$ *means the line integral of* g *along any curve from* P *to the element* $\mathbf{ds}$.

ON THE REPRESENTATION OF A SUMMABLE FUNCTION BY A SERIES OF FINITE POLYNOMIALS

By E. W. HOBSON.

[Received October 31st, 1912.—Read November 14th, 1912.]

THE principal object of the present communication is to give an extension of the well known theorem of Weierstrass, that a function of a real variable that is continuous in a given interval may be represented by a series of which the terms are finite polynomials, in such wise that the series converges to the value of the function uniformly in the interval. It is here shewn that any summable function $f(x)$, whether limited or unlimited in the finite interval for which it is defined, may be represented in that interval by a series of which the terms are finite polynomials, in the sense that the series converges to the value of the function at every point of the interval with the possible exception of those points which belong to an exceptional set of which the measure is zero. It is further shewn that the polynomials can be so determined that, for each value of x which does not belong to the exceptional set, the terms of the series are all positive from and after some fixed term dependent in general on the value of x. Moreover, it is shewn that the convergence is such as to be uniform in a set of points of which the measure is arbitrarily nearly equal to that of the interval of representation of the function. It is also shewn that the series may be subjected to term-by-term integration. The results are applicable to the case of a summable function of any number of real variables.

The theorems here established provide a method by which, in suitable cases, properties of integrals involving a summable function may be deduced from properties of the corresponding integrals which involve a finite polynomial.* As an example of this process I have given a simple proof, by means of the main results of this paper, of a general theorem of convergence which I established in an earlier paper and which in the same

* Another method of making extensions of this kind, depending upon the use of monotone series of semi-continuous functions, has been developed by Prof. W. H. Young; see his paper 'On a New Method in Integration,'' *Proc. London Math. Soc.*, Ser. 2, Vol. 9, 1911.

and subsequent papers I applied to the consideration of questions relating to the convergence of Sturm-Liouville series, series of Bessel's and Legendre's functions, and to other cases. I have given in the present paper an application of the methods here developed to obtain a proof of the result, due to Haar and Mercer, that the convergence of a Sturm-Liouville series at a point or in an interval is dependent on conditions identical with those which govern the case of an ordinary Fourier's series.

1. (α) Let us consider the function $f(x)$ defined for the interval (a, b) of the real variable x by the law $f(x) = 1$, in an interval (c, d) contained in (a, b), and $f(x) = 0$ at each point of (a, b) external to (c, d). Let the continuous function $\psi(x)$ be defined by $\psi(x) = 0$ in the interval $(a, c-\epsilon)$, $\psi(x) = (x-c+\epsilon)/\epsilon$ in the interval $(c-\epsilon, c)$, $\psi(x) = 1$ in the interval (c, d), $\psi(x) = (\epsilon-x+d)/\epsilon$ in the interval $(d, d+\epsilon)$, $\psi(x) = 0$ in the interval $(d+\epsilon, b)$, where ϵ is an arbitrarily chosen positive number less than $c-a$ and less than $b-d$. In case $c = a$ the interval $(c-\epsilon, c)$ is to be suppressed, and in case $d = b$ the interval $(d+\epsilon, b)$ is to be suppressed.

We have then

$$\int_a^b |f(x)-\psi(x)|\,dx = \epsilon,$$

or $\frac{1}{2}\epsilon$, in case c coincides with a, or in case d coincides with b.

(β) Next let $f(x)$ be defined to have the value 1 at all points of the intervals $\delta_1, \delta_2, \ldots, \delta_r$ of a finite non-overlapping set of intervals contained in (a, b) and let it $= 0$ everywhere else in (a, b). If $f_s(x)$ is the function which is $= 1$ in the interval δ_s for $1 \leqslant s \leqslant r$, and is elsewhere zero, we can, as above, determine a continuous function $\psi_s(x)$ such that

$$\int_a^b |f_s(x)-\psi_s(x)|\,dx < \frac{1}{r}\eta,$$

where η is a prescribed positive number arbitrarily small. Since

$$f(x) = f_1(x)+f_2(x)+\ldots+f_s(x),$$

we have, denoting the continuous function $\psi_1(x)+\psi_2(x)+\ldots+\psi_s(x)$ by $\psi(x)$,

$$\int_a^b |f(x)-\psi(x)|\,dx \leqslant \sum_{s=1}^{s=r} \int_a^b |f_s(x)-\psi_s(x)|\,dx < \eta.$$

(γ) Let $f(x) = 1$ at all points of a measurable set E of points contained in (a, b), and let $f(x) = 0$ at every point of the interval (a, b) which does not belong to E. A set Δ of non-overlapping intervals $\delta_1, \delta_2, \ldots, \delta_r, \ldots,$

exists such that E is contained in the intervals of the set and such that $m(\Delta)-m(E) < \frac{1}{2}\eta$, where η is an arbitrarily fixed positive number. Also r can be so determined that $m(\Delta)-m(\delta_1+\delta_2+\ldots+\delta_r) < \frac{1}{2}\eta$. Hence

$$|\, m(E)-m(\delta_1)-m(\delta_2)-\ldots-m(\delta_r)\,| < \eta.$$

Let $\chi(x)$ be that function which $=1$ in the intervals $\delta_1, \delta_2, \ldots, \delta_r$ and $=0$ everywhere else, and let $\psi(x)$ be that continuous function determined as in (β), such that $\int_a^b |\chi(x)-\psi(x)|\,dx < \eta$. We have then

$$\int_a^b |f(x)-\psi(x)|\,dx \leqslant \int_a^b |f(x)-\chi(x)|\,dx + \int_a^b |\chi(x)-\psi(x)|\,dx.$$

Now
$$\int_a^b |f(x)-\chi(x)|\,dx < \eta,$$

since $f(x)$ and $\chi(x)$ differ from one another at points forming a set of which the measure is less than η; therefore the continuous function $\psi(x)$ is such that

$$\int_a^b |f(x)-\psi(x)|\,dx < 2\eta.$$

(δ) Let $f(x)$ have the value c_1 at all points of a measurable set E_1; let it have the value c_2 at the points of the measurable set E_2, and so on, there being r such sets $E_1, E_2, \ldots, E_r$, in which $f(x)$ has the values $c_1, c_2, \ldots, c_r$ respectively. At all points x which do not belong to any of the sets $E_1, E_2, \ldots, E_r$, let $f(x) = 0$. We may write

$$f(x) = c_1 f_1(x)+c_2 f_2(x)+\ldots+c_r f_r(x),$$

where $f_s(x)$, for $1 \leqslant s \leqslant r$, is defined by the rule $f_s(x) = 1$ in E_s, and $f_s(x) = 0$ everywhere else. Let $\psi_s(x)$ be the continuous function corresponding to $f_s(x)$ determined as in (γ), so that

$$\int_a^b |f_s(x)-\psi_s(x)|\,dx < \epsilon/(|\,c_1\,|+|\,c_2\,|+\ldots+|\,c_r\,|).$$

If, now,
$$\psi(x) = c_1\psi_1(x)+c_2\psi_2(x)+\ldots+c_r\psi_r(x);$$

we have
$$\int_a^b |f(x)-\psi(x)|\,dx \leqslant \sum_{s=1}^{s=r} |\,c_r\,| \int_a^b |f_s(x)-\psi_s(x)|\,dx$$
$$< \epsilon.$$

Therefore a continuous function $\psi(x)$ can be so determined that

$$\int_a^b |f(x)-\psi(x)|\,dx$$

is less than the arbitrarily chosen positive number ϵ.

(ϵ) Next let $f(x)$ be any summable function that is limited in (a, b). Another summable function $f_\epsilon(x)$ can be found such that

$$|f(x)-f_\epsilon(x)| < \epsilon/2(b-a)$$

for every value of x, and which is such that it takes only a finite number of values $c_1, c_2, \ldots, c_r$ in the interval (a, b); this is in accordance with the mode in which a Lebesgue integral is defined. Let $\psi(x)$ be that continuous function determined as in (δ), so that

$$\int_a^b |f_\epsilon(x)-\psi(x)|\,dx < \tfrac{1}{2}\epsilon.$$

We have then

$$\int_a^b |f(x)-\psi(x)|\,dx \leqslant \int_a^b |f(x)-f_\epsilon(x)|\,dx + \int_a^b |f_\epsilon(x)-\psi(x)|\,dx < \epsilon.$$

(ζ) Lastly, let $f(x)$ be an unlimited summable function. Let E be that set of points (x) at which $|f(x)|$ exceeds a positive number N. In accordance with a known property of Lebesgue integrals, N may be chosen so great that $\int_{(E)} |f(x)|\,dx$ is less than the arbitrarily chosen positive number $\frac{1}{2}\epsilon$. Let $f(x) = \phi_1(x)+\phi_2(x)$, where $\phi_2(x) = f(x)$ at all points x which do not belong to E, and $\phi_2(x) = 0$ in E. Let $\psi(x)$ be the continuous function corresponding to $\phi_2(x)$ determined as in (ϵ), so that

$$\int_a^b |\phi_2(x)-\psi(x)|\,dx < \tfrac{1}{2}\epsilon.$$

We have now

$$\begin{aligned}\int_a^b |f(x)-\psi(x)|\,dx &\leqslant \int_a^b |\phi_1(x)|\,dx + \int_a^b |\phi_2(x)-\psi(x)|\,dx \\ &\leqslant \int_{(E)} |f(x)|\,dx + \int_a^b |\phi_2(x)-\psi(x)|\,dx \\ &< \epsilon.\end{aligned}$$

The following theorem has now been established :—

If $f(x)$ be any function, whether limited or unlimited, which is summable in (a, b), a function $\psi(x)$ continuous in the interval can be so determined that $\int_a^b |f(x)-\psi(x)|\,dx$ is less than an arbitrarily prescribed positive number ϵ.

It may be observed that

$$\left| \int_{(G)} \{f(x)-\psi(x)\}\, dx \right| \leqslant \int_{(G)} |f(x)-\psi(x)|\, dx < \epsilon\,;$$

where G is any measurable set of points contained in (a, b), and which may be the whole of the interval (a, b).

In accordance with the well known theorem of Weierstrass, a finite polynomial $P(x)$ can be so determined that

$$|\psi(x)-P(x)| < \epsilon/(b-a),$$

and thus that $$\int_a^b |\psi(x)-P(x)|\, dx < \epsilon\,;$$

it then follows that $$\int_a^b |f(x)-P(x)|\, dx < 2\epsilon.$$

Hence it has been shown that

If $f(x)$ be summable in (a, b), whether it be limited or not, a finite polynomial $P(x)$ can be so determined that $\int_a^b |f(x)-P(x)|\, dx$ is less than an arbitrarily prescribed positive number.

It is easily seen that the whole of the reasoning by which these theorems have been established applies, without essential modification, to the case of functions of any number of real variables, the cell, which in the case of two dimensions consists of a rectangle with its sides parallel to to the axes of x and y, taking the place of an interval.

2. Let $f(x)$ denote a summable function defined for a finite interval which we may take, without loss of generality, to be the interval (0, 1) of the variable x. Let us choose a fixed positive number $\epsilon(< 4/5)$. Corresponding to the summable function $f(x)-\epsilon$ a finite polynomial $P_1(x)$ can be so determined that

$$\int_0^1 |f(x)-\epsilon-P_1(x)|\, dx < \epsilon^4. \tag{1}$$

Let E_1 denote that set of points x in (0, 1), for which the condition

$$|f(x)-\epsilon-P_1(x)| < \epsilon^2$$

is satisfied; for the points of the set $C(E_1)$ complementary to E_1, we have

$$|f(x)-\epsilon-P_1(x)| \geqslant \epsilon^2.$$

The measure $m[C(E_1)]$ of $C(E_1)$ must, in accordance with (1), be such that

$$\epsilon^2 m[C(E_1)] < \epsilon^4, \quad \text{or} \quad m[C(E_1)] < \epsilon^2,$$

whence we have $m(E_1) > 1-\epsilon^2$. At each point of E_1, $f(x)-\epsilon-P_1(x)$ lies between ϵ^2 and $-\epsilon^2$, or $f(x)-P_1(x)$ lies between $\epsilon+\epsilon^2$ and $\epsilon-\epsilon^2$, and has necessarily a positive value.

Proceed in a similar manner with the numbers $\frac{1}{2}\epsilon$, $\frac{1}{2^2}\epsilon$, ..., $\frac{1}{2^r}\epsilon$, ... in place of ϵ. Denoting by $E_r(x)$ that set of points at which

$$|f(x)-\epsilon/2^{r-1}-P_r(x)| < (\epsilon/2^{r-1})^2,$$

where the finite polynomial $P_r(x)$ is so chosen that

$$\int_a^b |f(x)-\epsilon/2^{r-1}-P_r(x)|\,dx < (\epsilon/2^{r-1})^4,$$

we see that $$m(E_r) > 1-(\epsilon/2^{r-1})^2.$$

Consider the infinite sequence of such sets $E_1, E_2, E_3, \ldots$; those points x which do not belong to all the sets form a set of which the measure is less than the limiting sum of

$$\epsilon^2+(\epsilon/2)^2+(\epsilon/2^2)^2+\ldots+(\epsilon/2^{r-1})^2+\ldots,$$

or than $\frac{4}{3}\epsilon^2$. It follows that a set H_1 of measure $> 1-\frac{4}{3}\epsilon^2$ exists, such that each point of H_1 belongs to all the sets $E_1, E_2, E_3, \ldots$. Similarly, it is seen that a set H_r of measure $> 1-\frac{4}{3}\left(\frac{\epsilon}{2^{r-1}}\right)^2$ exists, such that each point of H_r belongs to all the sets $E_r, E_{r+1}, E_{r+2}, \ldots$.

Let H be the set consisting of all those points x, each of which belongs to all the sets $E_1, E_2, E_3, \ldots$, from and after some fixed one of those sets. We see then that $m(H) = 1$, for $m(H) > 1-\frac{4}{3}\left(\frac{\epsilon}{2^{r-1}}\right)^2$, for every integral value of r. The set of points complementary to H, in case it exists, has the measure zero.

If x be any point belonging to H, for some integer s, in general dependent on x, $f(x)-P_s(x)$ lies between

$$(\epsilon/2^{s-1})-(\epsilon/2^{s-1})^2 \quad \text{and} \quad (\epsilon/2^{s-1})+(\epsilon/2^{s-1})^2;$$

$f(x)-P_{s+1}(x)$ lies between $(\epsilon/2^s)-(\epsilon/2^s)^2$ and $(\epsilon/2^s)+(\epsilon/2^s)^2$, and so on. Hence $f(x)-P_s(x)$, $f(x)-P_{s+1}(x)$, $f(x)-P_{s+2}(x)$, ... are all positive. Moreover

$$P_s(x) < P_{s+1}(x) < P_{s+2}(x) < \ldots,$$

since $$(\epsilon/2^n)-(\epsilon/2^n)^2 > (\epsilon/2^{n+1})+(\epsilon/2^{n+1})^2$$

for all values of n, provided $\epsilon-\epsilon^2 > \frac{1}{2}\epsilon^2+\frac{1}{4}\epsilon^2$, which is the case when $\epsilon < 4/5$.

It follows that the sequence $P_s(x)$, $P_{s+1}(x)$, ..., is monotone and that it converges to $f(x)$ as limit.

It has now been shewn that, at each point x belonging to the set H of measure equal to that of the whole interval, the sequence of numbers $P_1(x)$, $P_2(x)$, $P_3(x)$, converges to the value of $f(x)$; moreover, the sequence is monotone from and after some element $P_r(x)$, where r in general depends on x.

Moreover, a set of points of which the measure is less than that of the whole interval by an arbitrarily small amount (> 0) can be determined such that the convergence of the sequence to $f(x)$ is uniform in the set. This kind of convergence may be described as *quasi-uniform* convergence.*

The following theorem has now been established :—

If $f(x)$ be a limited or unlimited function, defined for a finite interval in which it is summable, a sequence $P_1(x)$, $P_2(x)$, ..., $P_n(x)$, ..., of finite polynomials can be determined such that the sequence converges to $f(x)$ at every point of the interval with the possible exception of the points of a set of which the measure is zero. Moreover, the convergence is quasi-uniform, and at each point not belonging to the exceptional set the sequence is monotone from and after some definite value of n dependent in general on x. Sets in which the convergence is uniform exist in which the

* The term "wesentlich gleichmässige Convergenz" has been employed by Weyl to denote this mode of convergence. See *Math. Annalen*, Vol. 67, p. 225. The definition of quasi-convergence given by Weyl is that the convergence of the sequence of functions shall be such that a sequence of sets E_1, E_2, ..., E_n, ... of points can be so determined that each set is contained in the next, that the measure of E_n converges to that of the whole interval as n is indefinitely increased, and that the sequence of functions converges uniformly in each of the sets of points. It can be shewn that the definition in the text is equivalent to this one. Taking the definition given in the text, let E_1 be a set of measure $l_1 (< 1)$ in which the convergence is uniform; a set F_1 of measure $> l_1 + \alpha_1$ can be determined in which the convergence is also uniform, where α_1 is any assigned positive number such that $l_1 + \alpha_1 < 1$. Take E_2 to consist of all the points of E_1, together with those points of F_1 which do not belong to E_1; it is then clear that the measure of E_2 is $> l_1 + \alpha_1$, say l_2. Moreover, E_2 contains E_1, and the convergence of the sequence of functions is uniform in E_2. If α_2 be a positive number such that $l_2 + \alpha_2 < 1$, we can in a similar manner determine a set E_3 which contains E_2, and such that $m(E_2) > l_2 + \alpha_2 > l_1 + \alpha_1 + \alpha_2$, and in which the convergence is uniform. Proceeding in this manner, and choosing α_1, α_2, ... so that $l_1 + \alpha_1 + \alpha_2 + \ldots$ converges to 1, we determine a sequence of sets E_1, E_2, ..., each of which is contained in the next, such that their measures converge to 1, and such that the sequence of functions converges uniformly in each set.

[*Added February* 28, 1913.]—My attention has been directed to a note by Egoroff (*Comptes Rendus*, January 1911), where it is shown that every sequence of functions which converges everywhere except at points of a set of measure zero has the property of quasi-uniform convergence.

number n can be determined so as to be the same for all points x in one such set.

In accordance with a well known result due to Baire, the exceptional set can be absent only when the function is point-wise discontinuous with respect to every perfect set contained in the interval. In that case $f(x)$ is a function belonging to Baire's first class.

If $f(x)$ is continuous in the interval for which it is defined, a monotone sequence of finite polynomials can be determined which converges uniformly to $f(x)$ in the interval.

This theorem, which has been proved incidentally above, may be regarded as an extension of that of Weierstrass, which in its original form does not include the condition that the sequence shall be monotone.

The theorems which have been here proved hold also in the case of functions of two or more variables.

3. The function $f(x)$ denoting any summable function, and $P_1(x)$, $P_2(x)$, ..., denoting a sequence of polynomials corresponding to $f(x)$ in accordance with the general theorem of § 2, it will be shewn that $\int_0^x P_n(x)\,dx$ converges to $\int_0^x f(x)\,dx$, or, more generally, that $\int_{(G)} P_n(x)\,dx$ converges to $\int_{(G)} f(x)\,dx$, where G is any measurable set of points contained in the interval (0, 1) for which $f(x)$ is defined.

Since $$\int_0^1 |f(x)-\epsilon/2^{n-1}-P_n(x)|\,dx < (\epsilon/2^{n-1})^4,$$

we have $$\int_0^1 |f(x)-P_n(x)|\,dx \leqslant \int_0^1 |f(x)-\epsilon/2^{n-1}-P_n(x)|\,dx+\int_0^1 \epsilon/2^{n-1}\,dx$$
$$< (\epsilon/2^{n-1})+(\epsilon/2^{n-1})^4,$$

and therefore $\int_0^1 |f(x)-P_n(x)|\,dx$ converges to zero as n is indefinitely increased.

Now

$$\left|\int_{(G)} \{f(x)-P_n(x)\}\,dx\right| \leqslant \int_{(G)} |f(x)-P_n(x)|\,dx \leqslant \int_0^1 |f(x)-P_n(x)|\,dx;$$

therefore $\int_{(G)} \{f(x)-P_n(x)\}\,dx$ converges to zero as n is indefinitely increased, which proves the result.

If $\phi(x)$ is any limited summable function, it can be shewn that $\int_{(G)} P_n(x)\,\phi(x)\,dx$ converges to $\int_{(G)} f(x)\,\phi(x)\,dx$.

For $$\left|\int_{(G)} \phi(x)\,\{f(x)-P_n(x)\}\,dx\right| \leqslant \bar{\phi}\int_{(G)} |f(x)-P_n(x)|\,dx,$$

where $\bar{\phi}$ is the upper limit of $|\phi(x)|$ in the interval (0, 1), and therefore $\int_{(G)} \phi(x)\,\{f(x)-P_n(x)\}\,dx$ converges to zero.

If $\phi(x)$ is summable, but unlimited, we have, provided $\phi(x)f(x)$ is summable,

$$\int_{(G)} \phi(x)\,\{f(x)-P_n(x)\}\,dx$$
$$= \int_{(E'_n)} \phi(x)\,\{f(x)-P_n(x)\}\,dx + \int_{(G-E'_n)} \phi(x)\,\{f(x)-P_n(x)\}\,dx,$$

where E'_n is that part of G which belongs to E_n, and is therefore such that for every point x belonging to it,

$$|f(x)-P_n(x)| < (\epsilon/2^{n-1})+(\epsilon/2^{n-1})^2.$$

We have now

$$\left|\int_{(G)} \phi(x)\,\{f(x)-P_n(x)\}\,dx \leqslant \{\epsilon/2^{n-1}+(\epsilon/2^{n-1})^2\}\int_{(E'_n)} |\phi(x)|\,dx\right.$$
$$+\int_{(G-E'_n)} |\phi(x)f(x)|\,dx + \int_{(G-E'_n)} |\phi(x)\,P_n(x)|\,dx.$$

As n is indefinitely increased, $m(G-E'_n)$ converges to zero, and therefore, in accordance with a known theorem, $\int_{(G-E'_n)} |\phi(x)f(x)|\,dx$ converges to zero. It does not, however, follow that $\int_{(G-E'_n)} |\phi(x)\,P_n(x)|\,dx$ converges to zero, for the upper limit of $|P_n(x)|$ may increase indefinitely as n is indefinitely increased. We cannot therefore in this case infer that $\int_{(G)} \phi(x)\,\{f(x)-P_n(x)\}\,dx$ converges to zero. We have thus obtained the following theorem :—

If the polynomials $\{P_n(x)\}$ correspond to the summable function $f(x)$,

and $\phi(x)$ is any limited summable function, the integral $\int_{(G)} \phi(x) P_n(x)\, dx$ converges to the value $\int_{(G)} \phi(x) f(x)\, dx$, as n is indefinitely increased. In particular, $\int_0^x \phi(x) P_n(x)\, dx$ converges to the value $\int_0^x \phi(x) f(x)\, dx$.

4. The results obtained as to the representation of a summable function as the limit of a sequence of finite polynomials may be expressed in the language of the representation of the function by a series. If $Q_n(x) = P_n(x) - P_{n-1}(x)$ we see that, whenever $P_n(x)$ converges to a definite limit as n is indefinitely increased, the series $Q_1(x) + Q_2(x) + \ldots$ converges to the same limit. The results obtained in §§ 2, 3 may thus be expressed as follows :—

Any given summable function defined in a finite interval of the variable x is representable in that interval by a series of which the terms are finite polynomials ; and this series converges to the value of the function at every point with the exception at most of those points which belong to some set of which the measure is zero. For each value of x the terms of the series all have positive values from and after some fixed term dependent in general on the value of x. The series is quasi-uniformly convergent in the interval of representation in the sense that a sequence of sets of points can be determined such that the measures of the sets converge to that of the interval and such that the series converges uniformly in each set of the sequence. Term-by-term integration is applicable throughout the whole interval or through any measurable set of points contained therein. Moreover, term-by-term integration is applicable when the terms of the series are multiplied by any limited summable function. The corresponding results hold for a summable function of any number of dimensions defined in a rectangular cell.

5. Since a series of continuous functions which converges in a set of points uniformly has a sum that is continuous relative to the set of points, we see that any summable function has the property that a set of points, of which the measure is as nearly equal to that of the whole interval of representation as we please, can be determined so that the function is continuous relatively to the set of points. This does not, of course, imply that the points of the set are points of continuity of the function considered as defined in the whole interval, but only when the functional values at points not belonging to the set in question are dis-

regarded. Any measurable set contains a closed set of measure arbitrarily near to that of the measurable set ; hence there exist closed sets of measures arbitrarily near to that of the whole interval in which the convergence of the series is uniform. A closed set contains in general a perfect component of the same measure as itself. It thus appears that there exists a perfect set of which the measure is arbitrarily near that of the whole interval relatively to which a given summable function is continuous. The following theorem has therefore been proved :—

A closed set of points, or in particular a perfect set, of measure as nearly as we please equal to that of the whole interval for which a given summable function $f(x)$ is defined, can be so determined that the summable function is continuous relatively to the closed set or to the perfect set.

6. As an example of the application of the results obtained in § 3 to the investigation of properties of integrals, I propose here to give a simple proof of a general convergence theorem which I established in a paper * "On a General Convergence Theorem and the Theory of the Representation of a Function by Series of Normal Functions." The theorem may be stated as follows :—

Let $f(x')$ be a limited or unlimited function, summable in the interval (α, β) of x'. Let $F(x', x, n)$ be a function defined for all values of x' in (α, β), and for all values of x belonging to a certain set of points G contained in (α, β), and for positive integral values of n. Let $F(x', x, n)$ satisfy the following conditions :—(1) That $| F(x', x, n) |$ does not exceed some fixed positive number $\bar{F}$, for all values of x' and x such that $| x'-x |$ is not less than a fixed positive number $\mu (< \beta-\alpha)$, where x' belongs to (α, β) and x to G, and for all the values of n; (2) that $\int_{\alpha_1}^{\beta_1} F(x', x, n)\,dx'$ exists as a Lebesgue integral for all values of α_1, β_1 such that $\alpha \leqslant \alpha_1 < \beta_1 \leqslant \beta$, for each value of x belonging to G, but not interior to the interval $(\alpha_1-\mu, \beta_1+\mu)$, and that it converges to the limit zero as n is indefinitely increased uniformly for all the values of α_1, β_1, and x. Then

$$\int_{\alpha}^{x-\mu} f(x')\,F(x', x, n)\,dx'$$

converges as n is indefinitely increased, uniformly to zero, for all values

* See *Proc. London Math. Soc.*, Ser. 2, Vol. 6, 1908, p. 349.

of x belonging to G, and in the interval $(\alpha+\mu, \beta)$. Also

$$\int_{x+\mu}^{\beta} f(x')\,F(x', x, n)\,dx'$$

converges uniformly to zero for all values of x in G and in the interval $(\alpha, \beta-\mu)$. The set G may consist of the whole interval (α, β).

If ϵ be a prescribed positive number, a polynomial $P(x')$ can be so determined that $\int_{\alpha}^{\beta} |f(x')-P(x')| < \epsilon$. The interval (α, β) may be divided into a number of parts (α, α_1), (α_1, α_2), ..., (α_{r-1}, β) such that in any one of these parts the fluctuation of the function $P(x')$ is $< \epsilon/(\beta-\alpha)$. Let $\chi(x')$ be the function that is defined by the condition that it has a constant value at all the interior points of any one of the above intervals (α_s, α_{s+1}), this constant value being equal to the value of $P(x')$ at the middle point of the interval. It is indifferent how we define the values of $\chi(x)$ at the end-points $\alpha, \alpha_1, \alpha_2, \ldots, \beta$ of the r intervals. We may, for example, assume that it vanishes at all those points. We have now

$$\int_{\alpha}^{\beta} |P(x')-\chi(x')|\,dx' < \epsilon;$$

and therefore

$$\int_{\alpha}^{\beta} |f(x')-\chi(x')|\,dx' < 2\epsilon.$$

Now

$$\int_{\alpha}^{x-\mu} f(x')\,F(x', x, n)\,dx'$$

$$= \int_{\alpha}^{x-\mu} \chi(x')\,F(x', x, n)\,dx' + \int_{\alpha}^{x-\mu} \{f(x')-\chi(x')\}\,F(x', x, n)\,dx'.$$

The second integral on the right-hand side is numerically less than $2\epsilon\bar{F}$. The first integral may be expressed by

$$c_1 \int_{\alpha}^{\alpha_1} F(x', x, n)\,dx' + c_2 \int_{\alpha_1}^{\alpha_2} F(x', x, n)\,dx' + \ldots + c_s \int_{\alpha_{s-1}}^{x-\mu} F(x', x, n)\,dx',$$

where $c_1, c_2, \ldots, c_s, \ldots, c_r$ are the values of $\chi(x')$ in the r intervals into which (α, β) has been divided, and where $x-\mu$ falls in the interval (α_{s-1}, α_s). On account of the condition (2) in the enunciation of the theorem, a value n_1 of n may be so chosen that the absolute value of the integral of $F(x', x, n)$ taken over any one of the intervals

$$(\alpha, \alpha_1), \quad (\alpha_1, \alpha_2), \quad \ldots, \quad (\alpha_{s-1}, x-\mu),$$

is less than $\epsilon/\{|c_1|+|c_2|+\ldots+|c_s|+\ldots+|c_r|\}$,

provide $n \geqslant n_1$. It follows that, for $n \geqslant n_1$,

$$\left| \int_a^{x-\mu} \chi(x')\, F(x', x, n)\, dx' \right| < \epsilon\,;$$

and therefore

$$\left| \int_a^{x-\mu} f(x')\, F(x', x, n)\, dx' \right| < 2\epsilon\bar{F}+\epsilon,$$

or

$$\overline{\lim_{n=\infty}} \left| \int_a^{x-\mu} f(x')\, F(x', x, n)\, dx' \right| \leqslant 2\epsilon\bar{F}+\epsilon,$$

and since ϵ is arbitrarily small, we therefore conclude that

$$\lim_{n=\infty} \int_a^{x-\mu} f(x')\, F(x', x, n)\, dx' = 0\,;$$

the convergence being uniform for all the values of x concerned. The convergence of $\int_{x+\mu}^{\beta} f(x')\, F(x', x, n)\, dx'$ to zero is established in the same manner.

7. Let us now consider the case in which the condition (1) in the enunciation of the theorem of § 5 is satisfied for all values of $|\, x'-x\,|$, so that $|\, F(x', x, n)\,|$ is $< \bar{F}$, for all values of x' in (α, β), for all values of x in G, and for all values of n. This is equivalent to replacing the positive number μ by zero. In this case $\int_a^{\beta} f(x')\, F(x', x, n)\, dx'$ converges, uniformly for all values of x in G to the limit zero as n is indefinitely increased. But in this case we may, in considering the integral

$$\int_a^{\beta} f(x')\, F(x', x, n)\, dx',$$

replace the condition (2) of the theorem of § 5 by the simpler one that $\int_{\alpha_1}^{\beta_1} F(x', x, n)\, dx'$ shall exist for each value of x in G, and that for each pair of values of α_1, β_1 it shall converge uniformly for all the values of x to the limit zero, as n is indefinitely increased. The uniformity of the convergence with respect to α_1, β_1 is in this case unnecessary. For, in order to estimate the value of the integral, we have to consider

$$\int_a^{\beta} \chi(x')\, F(x', x, n)\, dx'$$

or $$c_1\int_{\alpha}^{\alpha_1} F(x', x, n)\,dx' + \ldots + c_r\int_{\alpha_{r-1}}^{\beta} F(x', x, n)\,dx'.$$

Since each of the r integrals $\int_{\alpha_{s-1}}^{\alpha_s} F(x', x, n)\,dx'$ converges uniformly with respect to x to the limit zero, it is clear that n_1 may be so chosen that for $n \geqslant n_1$, the condition

$$\left| \int_{\alpha}^{\beta} \chi(x')\, F(x', x, n)\,dx' \right| < \epsilon$$

is satisfied.

We thus obtain the following theorem :—

Let $f(x')$ be a limited or unlimited function, summable in the interval (α, β) of x'. Let $F(x', x, n)$ be defined for all values of x' in (α, β), and for all values of x belonging to a set G contained in (α, β), and for positive integral values of n. Let $F(x', x, n)$ satisfy the following conditions :—(1) *That $|F(x', x, n)|$ does not exceed a fixed positive number $\bar{F}$ for all the values of x', x, and n concerned;* (2) *that $\int_{\alpha_1}^{\beta_1} F(x', x, n)\,dx'$ exists as a Lebesgue integral for all intervals (α_1, β_1) in (α, β), and that for each pair of values of α_1, β_1 it converges to the limit zero as n is indefinitely increased uniformly for all values of x in G. Then*

$$\int_{\alpha}^{\beta} f(x')\, F(x', x, n)\,dx'$$

converges to the limit sero, as n is indefinitely increased, uniformly for all the values of x in G.

In the particular case in which the set of points G contains only a single point, x has only a single constant value, and thus the function $F(x', x, n)$ can be replaced by $F(x' n)$. The theorem then asserts that

If $f(x')$ be a function, summable in the interval (α, β), and if $F(x', n)$ satisfies the conditions (1) *that $|F(x', n)|$ is less than a fixed positive number $\bar{F}$ for all the values of x' and n, and* (2) *that $\int_{\alpha_1}^{\beta_1} F(x', n)\,dx'$ exists as a Lebesgue integral for every pair of values of α_1, β_1 such that $\alpha \leqslant \alpha_1 < \beta_1 \leqslant \beta$, and is such that for each pair of values of α_1 and β_1 it converges to zero as n is indefinitely increased, then $\int_{\alpha}^{\beta} f(x') F(x', n)\,dx'$ converges to zero when n is indefinitely increased.*

This last theorem has been given by Lebesgue,* who has further shewn that the conditions as to the function $F(x', n)$ stated in the theorem are not only sufficient, but are also necessary in order that the convergence of $\int_\alpha^\beta f(x') F(x', n)\, dx'$ may hold for all summable functions $f(x')$.

8. In the paper to which reference has already been made, I applied the general convergence theorem in § 5 to the consideration of the question of the convergence of the Sturm-Liouville series corresponding to a given summable function at a point of the interval of representation, and to the question of uniform convergence in an interval. I shewed that if in a neighbourhood of a point the function is of limited total fluctuation, then the series converges at the point to the mean of the functional limits at that point; and, further, that in any interval of continuity of the function that is contained in the interior of an interval in which the function has limited total fluctuation (*à variation bornée*) the convergence of the series to the value of the function is uniform. In later writings by Haar† and Mercer‡ it was shown that the behaviour of the series in the neighbourhood of a point is the same as that of the Fourier's cosine series corresponding to the same function, and thus that all the known sufficient conditions for the convergence of the Fourier's series are applicable without change to the case of the Sturm-Liouville series. I propose here to shew that this very general result may be obtained in a simple manner by employing the methods of the present paper. It is shewn (*loc. cit*) that the differential equation for the Sturm-Liouville functions can be reduced to the normal form

$$\frac{d^2U}{dz^2} + \rho^2 U = l_1 U,$$

where the functions U satisfy the boundary conditions

$$\frac{dU}{dz} - h'U = 0, \quad \text{for} \quad z = 0,$$

and

$$\frac{dU}{dz} + H'U = 0, \quad \text{for} \quad z = \pi;$$

the interval of representation being taken to be $(0, \pi)$. It is assumed that h' and H' are both finite. In case either h' or H' is taken to be

---

* See his memoir, "Sur les intégrales singulières," *Annales de Toulouse*, 1909.

† See *Math. Annalen*, Vol. 69, p. 355.

‡ See *Phil. Trans.*, Vol. 211 A, p. 181.

infinite, the boundary condition becomes $U = 0$, and some modification in the results is required; accordingly it will be here supposed that this case is excluded. It was shewn (*loc. cit.*) that the normal function $\phi_n(z)$ which denotes $U_n(z)\left[\int_0^\pi \{U_n(z)\}^2\right]^{-\frac{1}{2}}$ is given asymptotically by

$$\phi_n(z) = \sqrt{\frac{2}{\pi}} \cos nz \left\{1 + \frac{\alpha(z, n)}{n^2}\right\} + \sin nz \left\{\frac{\gamma(z)}{n} + \frac{\beta(z, n)}{n^2}\right\},$$

where $\alpha(z, n)$, $\beta(z, n)$, $\gamma(z)$ are all numerically less than fixed positive numbers, independent of z and n.

Let $\phi(z', z, n)$ denote

$$\phi_1(z)\,\phi_1(z') + \phi_2(z)\,\phi_2(z') + \ldots + \phi_n(z)\,\phi_n(z') - \frac{2}{\pi}\left(\tfrac{1}{2} + \cos z \cos z' + \cos 2z \cos 2z' + \ldots + \cos nz \cos nz'\right).$$

It can be shewn that $|\phi(z', z, n)|$ is less than a fixed positive number M for all values of z', z, and n. For

$$\phi_n(z)\,\phi_n(z') - \frac{2}{\pi}\cos nz \cos nz' = \frac{2}{\pi}\left\{\cos nz \sin nz' \frac{\gamma(z')}{n} + \cos nz' \sin nz \frac{\gamma(z)}{n}\right\} + \frac{2}{\pi}\frac{\delta(z, z', n)}{n^2},$$

where $|\delta(z, z', n)|$ is less than some fixed positive number independent of z, z', n. The series $\sum_{n=1} \frac{\delta(z, z', n)}{n^2}$ is uniformly convergent, and thus its sum-function is a continuous, and therefore limited, function of (z, z'). The series $\sum_1^n \frac{\sin n(z+z')}{n}$, $\sum_1^n \frac{\sin n(z-z')}{n}$ are known to be numerically less than fixed numbers independent of n, z, and z'; hence the expressions

$$\sum_1^n \cos nz \sin nz' \frac{\gamma(z')}{n}, \qquad \sum_1^n \cos nz' \sin nz \frac{\gamma(z)}{n}$$

have the same property. The result has thus been established.

Let $P(z')$ be a finite polynomial such that $\int_0^\pi |f(z') - P(z')|\,dz$ is less than the arbitrarily chosen positive number ϵ, then

$$\int_0^\pi f(z')\,\phi(z', z, n)\,dz' = \int_0^\pi P(z')\,\phi(z', z, n)\,dz' + \int_0^\pi \{f(z') - P(z')\}\,\phi(z', z, n)\,dz';$$

the second integral on the right-hand side is numerically less than $M\epsilon$ for all values of z and n. Since ϵ is arbitrarily small, so also is $M\epsilon$; there-

fore, when n is indefinitely increased, $\int_0^\pi f(z')\,\phi(z', z, n)\,dz'$ has the same limits of indeterminancy as $\int_0^\pi P(z')\,\phi(z', z, n)$, and if one of these integrals converges to a definite value, then the other converges to the same value.

Now

$$\int_0^\pi P(z')\,\phi(z', z, n)\,dz'$$

$$= \frac{2}{\pi} \sum_{r=1}^{r=n} \frac{1}{r} \cos rz \int_0^\pi P(z')\,\gamma(z') \sin rz'\,dz' + \frac{2}{\pi} \Sigma \frac{1}{r^2} \int_0^\pi P(z')\,\delta(z, z', r)\,dz'$$

$$+ \frac{2}{\pi}\,\gamma(z)\,\Sigma \frac{1}{r} \sin rz \int_0^\pi P(z') \cos rz'\,dz'.$$

The first series on the right-hand side is obtainable by integrating with respect to z the Fourier's sine series

$$\Sigma \frac{1}{r} \sin rz \int_0^\pi P(z')\,\gamma(z') \sin rz'\,dz',$$

and is uniformly convergent; the second series is clearly uniformly convergent, since $|\,\delta(z, z', r)\,|$ is limited. The third series like the first is uniformly convergent. It follows that $\int_0^\pi P(z')\,\phi(z', z, n)\,dz'$ converges uniformly to the value of some function $\chi(z)$, continuous in the interval $(0, \pi)$. We see that

$$\int_0^\pi \chi(z)\,\phi_r(z)\,dz = 0,$$

since

$$\int_0^\pi \phi_r(z)\,\phi_s(z)\,dz = 0, \quad \text{for} \quad r \neq s,$$

and

$$\int_0^\pi \{\phi_r(z)\}^2\,dz = 1,$$

and since the Fourier's cosine series

$$\frac{1}{\pi} \int P(z')\,dz' + \frac{2}{\pi} \sum_{r=1} \cos rz \int_0^\pi P(z') \cos nz'\,dz',$$

when multiplied by $\phi_r(z)$, and integrated between 0 and π, converges to the value $\int_0^\pi P(z)\,\phi_r(z)$. It is a known theorem in the theory of orthogonal functions that if $\chi(z)$ be a continuous function, such that $\int_0^\pi \chi(z)\,\phi_r(z)\,dz$ is

zero for all values 1, 2, 3, ... of r, then $\chi(z)$ vanishes. It has thus been shewn that $\int_0^\pi P(z')\,\phi(z', z, n)\,dz'$ converges uniformly to zero, and therefore that $\int_0^\pi f(z')\,\phi(z', z, n)\,dz'$ converges uniformly to zero.

It has now been shewn that, at any point z, the two series

$$\frac{1}{\pi} + \frac{2}{\pi} \sum_{r=1} \cos rz \int_0^\pi \cos rz'\, f(z')\, dz',$$

$$\sum_{r=1} \phi_n(z) \int_0^\pi \phi_n(z') f(z')\, dz',$$

are both convergent or that both are non-convergent, and that in the latter case they have the same limits of indeterminancy. It is thus clear that all the known sufficient conditions of convergence of the Fourier's cosine series relating to the nature of the function $f(z)$ in an arbitrarily small neighbourhood of a point are applicable without change as sufficient conditions for the convergence of the Sturm-Liouville series.

AN EXTENSION OF A THEOREM ON OSCILLATING SERIES

By G. H. HARDY.

[Received November 11th, 1912.—Read December 12th, 1912.]

1. Suppose that $\lambda_1, \lambda_2, \lambda_3, \ldots$

is an ascending sequence of positive* numbers increasing beyond all limit with n. Let

$$C(\omega) = \sum_{\lambda_n \leqslant \omega} c_n,$$

so that $$C(\omega) = c_1 + c_2 + \ldots + c_p,$$

if $\lambda_p \leqslant \omega < \lambda_{p+1}$; and let

$$C^\kappa(\omega) = \sum_{\lambda_n \leqslant \omega} (\omega - \lambda_n)^\kappa c_n,$$

κ being any positive number. It will easily be verified that

$$C^\kappa(\omega) = \kappa \int_0^\omega C(\tau)(\omega - \tau)^{\kappa-1}\, d\tau.$$

Then, if $$\omega^{-\kappa} C^\kappa(\omega) \to C,$$

as $\omega \to \infty$, we shall say that the series Σc_n is *summable* (R, λ, κ)† *to sum* C.

If $\lambda_n = n$, this definition of the sum of an oscillating series is equivalent to Cesàro's.‡

---

* λ_1 may be zero.

† That is to say, *by Riesz's means of type* λ *and order* κ. These methods of summation were introduced by M. Riesz in a note in the *Comptes Rendus* of 5 July, 1909. A more systematic account of them, and of their applications to the theory of Dirichlet's series, will be given in the *Cambridge Tract* on "The General Theory of Dirichlet's Series" that Dr. Riesz and I are now preparing in collaboration.

‡ Riesz, *Comptes Rendus*, 12 June, 1911. When I speak of Cesàro's methods of summation I include the methods of non-integral order whose theory has been developed by Knopp and Chapman (Knopp, *Inaugural Dissertation*, Berlin, 1907; Chapman, *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 369).

If, in the general expression for $C^\kappa(\omega)$, we take $\kappa = 1$, $\omega = \lambda_p$, we obtain

$$\frac{C^\kappa(\omega)}{\omega} = \frac{(\lambda_2-\lambda_1)C_1+(\lambda_3-\lambda_2)C_2+\ldots+(\lambda_p-\lambda_{p-1})C_{p-1}}{\lambda_p},$$

where $$C_n = c_1+c_2+\ldots+c_n.$$

Thus, when $\kappa = 1$, Riesz's definition is the natural generalisation of Cesàro's which arises when we attach *weights* to the successive partial sums C_n.

2. My principal object in this note is to prove the following theorem.

THEOREM 1.—*If* Σc_n *is summable* (R, λ, κ), *and*

$$c_n = O\left(\frac{\lambda_n-\lambda_{n-1}}{\lambda_n}\right), \tag{1}$$

then Σc_n *is convergent. In other words, no series which satisfies the condition* (1) *can be summable by Riesz's methods without being convergent.*

This theorem is the generalisation for Riesz's methods of summation of what Mr. Littlewood and I have called *the general Cesàro-Tauber theorem*, the theorem* which asserts that a series for which $c_n = O(1/n)$ cannot be summable by Cesàro's means without being convergent.

I have already published a proof† of the special case of this theorem in which $\kappa = 1$, assuming, however, that the λ's are subject to the restriction

$$\lambda_{n+1}/\lambda_n \to 1. \tag{2}$$

When this last condition is satisfied my theorem, and its extension to general values of κ, may be deduced as a corollary from Mr. Littlewood's extension of Tauber's theorem.‡ This method of procedure is, however, open to several objections. In the first place, Mr. Littlewood's theorem is a more difficult theorem than that which we are using it to prove. Secondly, the proof also depends on another theorem of which no proof has yet been published, viz., the theorem that, *if* Σc_n *is summable* (R, λ, κ), *to sum* C, *and* $\Sigma c_n e^{-\lambda_n x}$ *is convergent for* $x > 0$, *then*

$$\Sigma c_n e^{-\lambda_n x} \to C,$$

---

* Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 8, p. 307.

† *L.c.*, p. 313.

‡ Littlewood, *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 434.

as $x \to 0$.* Finally, until a proof is given of Mr. Littlewood's theorem, which shall be free from restrictions on the λ's, we are compelled to adhere to the assumption (2). The presence of this unnecessary restriction is at any rate an æsthetic blemish in the theorem.

The idea of obtaining a direct and general proof of the theorem, which should be free from any restriction on the λ's, was suggested to me by Dr. Riesz. Dr. Riesz himself indicated to me the general lines of a proof in the case $\kappa = 1$. The form of this proof was different from that which I adopted in my previous paper. I find, however, that the line of argument which I then followed can be adapted so as to lead to the desired result, and I have followed it here, as the preliminary transformations on which it is based seem to be of some interest in themselves, and can be applied for other purposes.

3. I proceed now to the proof of the theorem. I write

$$b_n = \lambda_n c_n,$$

and use $B(\omega)$, $B^\kappa(\omega)$ to denote the sums formed from the b's in the way in which $C(\omega)$, $C^\kappa(\omega)$ were formed from the c's. It is evident that there is no loss of generality in supposing κ to be integral. Further, we may suppose our series to be *real*.†

We have

$$\frac{C^\kappa(\omega)}{\omega^\kappa} - \frac{C^{\kappa+1}(\omega)}{\omega^{\kappa+1}} = \sum_{\lambda_n \leqslant \omega} \left(1 - \frac{\lambda_n}{\omega}\right)^\kappa \left\{1 - \left(1 - \frac{\lambda_n}{\omega}\right)\right\} a_n$$

$$= \frac{B^\kappa(\omega)}{\omega^{\kappa+1}}.$$

Hence (α) *the necessary and sufficient condition that the series* Σc_n, *if known to be summable* $(R, \lambda, \kappa+1)$, *should be summable* (R, λ, κ), *is that*

$$B^\kappa(\omega) = o(\omega^{\kappa+1}).$$

---

* For the case $\kappa = 1$, see Hardy, *l.c.*, pp. 311 *et seq.* Dr. Riesz has found a proof of a theorem a good deal more general than that stated in the text, which will be published in the *Cambridge Tract* already referred to.

The steps of the deduction referred to in the text are as follows. From (1) we infer the convergence of $\Sigma c_n e^{-\lambda_n x}$ for $\kappa > 0$. From the summability of Σc_n, and the theorem stated in the text, we infer the existence of the limit as $x \to 0$. Finally, from (1), (2) and Mr. Littlewood's theorem, we infer the convergence of Σc_n.

† Cf. Hardy, *l.c.*, p. 303.

Again,
$$\frac{d}{d\omega}\left\{\frac{C^{\kappa+1}(\omega)}{\omega^{\kappa+1}}\right\} = \frac{\kappa+1}{\omega^2}\sum_{\lambda_n\leqslant\omega}\left(1-\frac{\lambda_n}{\omega}\right)^{\kappa}\lambda_n a_n$$
$$= (\kappa+1)\frac{B^{\kappa}(\omega)}{\omega^{\kappa+2}}.$$

Hence (β) *the necessary and sufficient condition that the series* Σc_n *should be summable* $(R, \lambda, \kappa+1)$ *is that the integral*
$$\int^{\infty}\frac{B^{\kappa}(\omega)}{\omega^{\kappa+2}}\,d\omega$$
should be convergent.

I shall now show that, when the c's satisfy condition (1) of the theorem, (β) implies (α). From this it will obviously follow that the summability of the series implies its convergence.

We have, in the first place,

(1) $$|c_n| < K(\lambda_n-\lambda_{n-1})/\lambda_n.$$

Now, let us suppose that (α) is not true, and that *e.g.*, the upper limit of $\omega^{-\kappa-1}B^{\kappa}(\omega)$ is positive. Then there is a positive constant H, such that

(2) $$B^{\kappa}(\omega) > H\omega^{\kappa+1},$$

for values of ω exceeding all limit.

Suppose $\xi > \omega$. Then

(3) $$B^{\kappa}(\xi)-B^{\kappa}(\omega) = \kappa\int_{\omega}^{\xi}B^{\kappa-1}(u)\,du.$$

Also
$$|B^{\kappa-1}(u)| = \Big|\sum_{\lambda_n\leqslant u}(u-\lambda_n)^{\kappa-1}b_n\Big|$$
$$< K\sum_{\lambda_n\leqslant u}(u-\lambda_n)^{\kappa-1}(\lambda_n-\lambda_{n-1})$$
$$< K\int_0^u (u-w)^{\kappa-1}\,dw = \frac{K}{\kappa}u^{\kappa}.$$

Hence

(4) $$|B^{\kappa}(\xi)-B^{\kappa}(\omega)| < K\int_{\omega}^{\xi}u^{\kappa}\,du = \frac{K}{\kappa+1}(\xi^{\kappa+1}-\omega^{\kappa+1}) < \frac{K}{\kappa+1}(\xi-\omega)\xi^{\kappa}.$$

Let
$$\Omega = (1+\rho)\omega,$$
where ρ is a positive constant to be chosen later. Then, for $\omega < \xi < \Omega$,

we have, by (2) and (4),

$$B^\kappa(\xi) \geqslant B^\kappa(\omega) - |B^\kappa(\xi) - B^\kappa(\omega)| > H\omega^{\kappa+1} - \frac{K}{\kappa+1}(\xi-\omega)\xi^\kappa$$

$$> \left\{ H - \frac{K\rho(1+\rho)^\kappa}{\kappa+1} \right\} \omega^{\kappa+1}.$$

But, when H and K are given, we can evidently choose ρ so that

$$\frac{K\rho(1+\rho)^\kappa}{\kappa+1} < \tfrac{1}{2}H.$$

Then $$B^\kappa(\xi) > \tfrac{1}{2}H\omega^{\kappa+1},$$

for $$\omega < \xi < \Omega,$$

and so $$\int_\omega^\Omega \frac{B^\kappa(\xi)}{\xi^{\kappa+2}}\, d\xi > \tfrac{1}{2}H(\Omega-\omega)\frac{\omega^{\kappa+1}}{\Omega^{\kappa+2}} = \frac{H\rho}{2(1+\rho)^{\kappa+2}}.$$

That this should hold for values of ω exceeding all limit is a contradiction of the hypothesis (β). The theorem is therefore established.

4. I add some remarks relating to the case in which the increase of the λ's is *rapid and fairly regular*.

Suppose that $$\lambda_{n+1}/\lambda_n \geqslant 1+\delta > 1,$$

where δ is a constant. Then the condition (1) of the theorem reduces to

$$c_n = O(1).$$

But, in this case, *more* is true than is asserted by the theorem. It is, in fact, true that *no* series can be summable (R, λ, κ) unless it is convergent; in other words, for such forms of λ_n, Riesz's methods are completely trivial; they sum convergent series and convergent series only. This can be deduced from another of Riesz's theorems, viz., that* *if* Σc_n *is summable* (R, λ, κ) *to sum* C, *then*

$$C_n - C = o\left(\frac{\lambda_{n+1}}{\lambda_{n+1}-\lambda_n}\right)^\kappa.$$

This conclusion, so far as it goes, bears out Mr. Littlewood's conjec-

* A proof of this theorem also will be given in the *Tract*.

ture* that, for such "regular high indices" λ_n, the existence of the limit

$$\lim_{x\to 0} \Sigma c_n e^{-\lambda_n x},$$

always involves the convergence of Σc_n.

5. Landau has generalised the "general Cesàro-Tauber" theorem by substituting for the hypothesis

$$c_n = O(1/n),$$

the less exacting hypothesis $c_n = O_L(1/n)$,

that is to say, $c_n > -K/n$.†

There is, of course, a corresponding extension of Theorem 1, which runs as follows.

THEOREM 2.—*A series* Σc_n, *for which*

$$c_n = O_L\left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right) \tag{1}$$

cannot be summable by Riesz's means unless it is convergent.

The proof of this theorem requires merely a slight modification of the proof of Theorem 1. We have

$$c_n > -K(\lambda_n - \lambda_{n-1})/\lambda_n. \tag{1}$$

If

$$B^\kappa(\omega) > H\omega^{\kappa+1} \tag{2}$$

for values of ω exceeding all limit, we take

$$\Omega = (1+\rho)\omega,$$

and show, by substantially the same argument‡ as in § 3, that

$$B^\kappa(\xi) > \tfrac{1}{2}H\omega^{\kappa+1} \quad (\omega < \xi < \Omega).$$

We thus prove, as in § 3, that

$$\lim_{\omega\to\infty} \frac{B^\kappa(\omega)}{\omega^{\kappa+1}} = 0.$$

---

* Littlewood, *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 446.

† *Prac Matematyczno-Fizycznych*, Vol. 21, p. 97. In Landau's form of the theorem c_n must obviously be supposed to be real, or it must be asserted explicitly that both the real and the imaginary parts of c_n satisfy his condition.

‡ The only difference is that we use algebraical inequalities instead of inequalities between absolute values.

In order to prove that the lower limit of indetermination is also zero, that is to say, that the inequality

$$B^{\kappa}(\omega) < -H\omega^{\kappa+1} \quad (H > 0);$$

cannot hold for values of ω exceeding all limit, we need only modify the argument to the extent of considering an interval

$$(1-\rho)\,\omega = \Omega < \xi < \omega,$$

instead of an interval $\omega < \xi < \Omega = (1+\rho)\,\omega$.*

* *Either* interval would have served our purposes in proving Theorem 1.

THE EQUATIONS OF THE THEORY OF ELECTRONS TRANSFORMED RELATIVE TO A SYSTEM IN ACCELERATED MOTION

By H. R. HASSÉ.

[Received November 12th, 1912.—Read December 12th, 1912.]

THE possibility of the exact transformation of the æthereal equations of electrodynamics from axes moving with a constant velocity to axes at rest was first proved by Larmor.* The extension to the case of accelerated motion has been rendered possible by recent investigations of Cunningham and Bateman, to the latter of whom is due the general theory of the transformation. So far the only rigorous mathematical investigation of the accelerated motion of a deformable electron, *i.e.*, an electron whose shape depends on its motion, has been given by Born,† who deals with a special case (*Hyperbelbewegungen*) of the motion discussed below. In the following paper the form of the transformation for a certain class of accelerated motions is investigated; but the applications are, for the most part, confined to the case of quasi-stationary motion, *i.e.*, motion which is very nearly uniform and in a straight line.

This is of considerable importance, as it enables us to verify and extend results obtained from the theory of quasi-stationary motion as developed by Abraham.‡

This latter theory is unsatisfactory, inasmuch as all the quantities involved are calculated for the uniform motion, and then differentiated with respect to the time in order to obtain their rates of increase for the accelerated motion. Further, from this theory Abraham has deduced the necessity of postulating for a deformable electron an energy other than electromagnetic, its value depending on the velocity of the electron.

---

* Larmor, *Æther and Matter*, Chap. XI.

† M. Born, "Die Theorie des Starren Elektrons in der Kinematik des Relativitätsprinzips," *Annalen der Physik*, Vol. 30, 1909.

‡ Abraham, "Die Dynamik des Elektrons," *Ann. der Physik*, 1903; *Théorie der Elektrizität*, Vol. II, 1st ed., pp. 136–214.

The necessity of introducing this extra energy has been admitted by Lorentz * and Poincaré, and is considered by the former to be an important objection to the hypothesis of a deformable electron.

The constants of the general transformation which have a kinematical interpretation include a constant δ, representing a uniform dilatation round the origin, † besides those interpretable in terms of the velocity and acceleration of the centre of the electron.

The exact theory of the transformation enables us to obtain for quasi-stationary motion the values of the electromagnetic masses, longitudinal and transverse, of a moving electron.

When δ is zero these values agree with those given by Lorentz for the two cases of volume and surface distribution of charge, an electron at rest being assumed to be a sphere and its charge to be distributed uniformly over its volume and surface respectively.

When the constant δ is so determined that $\delta + u\alpha$ is zero, where u is the velocity of the centre of the electron and α proportional to its acceleration in the direction of motion, then we have a special case where there is no distinction between the longitudinal and transverse mass, and, further, for the general motion, the accurate values of the forces due to the reaction of the æther can be obtained.

Again, according to the transformation theory, it is shown that there is no need to assume the existence of an energy other than electromagnetic, the equation of energy being of necessity satisfied when the electron is in motion.

The type of motion considered in this paper is characterized by the fact that the path of the centre of the electron is always a conic, which cannot be either an ellipse or a circle. It is, however, the simplest case of accelerated motion of a deformable electron for which the electromagnetic disturbance can be accurately calculated, and as such is of considerable importance.

It must be noticed that this paper does not deal with any question involving the "equations of motion" of an electron. The motion is determined beforehand solely by the transformation, and the forces acting on the electron are then calculated.

The converse investigation, as to how an electron moves under the action of a given force, constant or otherwise, is not attempted in this paper. As at present developed, the transformation theory, or Theory of Relativity, does not appear to be sufficiently complete to deal with this

* Lorentz, *The Theory of Electrons*, Leipzig, 1909, p. 213.

† Used by Lorentz, *Amsterdam Proceedings*, 1901.

question, which has, for rigid electrons, been treated with various assumptions by G. W. Walker, Schott, and others.

Nor is there any necessary inconsistency between the results of this paper and those obtained by Schott * for deformable electrons, inasmuch as the assumptions lying as the basis of the work are in many cases quite different.

In fact, the results of any electron theory as at present developed depend, to a very large extent, on the assumptions, mathematical as well as physical, made in the course of the work, and these it is not always easy to detect.

In this paper it is assumed (i) that the Principle of Relativity, or (mathematically speaking) the transformation theory of the electrodynamical equations, holds for accelerated motion as well as for uniform motion †; (ii) that an electron *at rest* is a sphere with a uniform volume or surface distribution of charge, leaving out of account the question as to how this charge can exist; (iii) that the equations of the electromagnetic field are those used by Larmor and Lorentz; (iv) that the force per unit volume due to the reaction of the æther is given by the expression $\rho(E+[v, H])$, where the symbols are defined in § 1.

1. *General Theory of the Transformation.*

It has been shown by Hargreaves ‡ that the equations of the theory of electrons can be expressed in the form of integrals in which use is made of the generalized form of Stokes' Theorem for four dimensions.

The fundamental equations of the theory of electrons are, using the vector notation,

$$\left.\begin{aligned} \text{curl } H &= \frac{\partial E}{\partial t} + \rho w \\ \text{curl } E &= -\frac{\partial H}{\partial t} \\ \text{div } E &= \rho, \quad \text{div } H = 0 \end{aligned}\right\}, \tag{1}$$

* G. A. Schott, "Electromagnetic Radiation," Appendixes F and G, and *passim*.

† From a mathematical point of view this appears to be absolutely necessary, since it is the variable motion of an electron which is of interest.

‡ "Integral Forms and their connexion with Physical Equations," *Camb. Phil. Trans.*, Vol. 21, 1908, p. 107.

whence it follows that $$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho w = 0,$$

which is the equation of continuity.

The electric units are those adopted by Lorentz, while the units of length and time have been so chosen that the velocity of radiation shall be unity.

We take the axes to form a right-handed system, so that curl A represents a vector whose components are

$$\frac{\partial Az}{\partial y} - \frac{\partial Ay}{\partial z}, \quad \ldots, \quad \ldots .$$

Also H is the magnetic force, E the electric force, ρ the volume density of electricity, and ρw the convection current.

These equations may be replaced by the two integral forms

$$\iint (H_x dy\,dz + H_y dz\,dx + H_z dx\,dy + E_x dx\,dt + E_y dy\,dt + E_z dz\,dt) = 0,$$

$$\iint (E_x dy\,dz + E_y dz\,dx + E_z dx\,dy - H_x dx\,dt - H_y dy\,dt - H_z dz\,dt)$$
$$= \iiint (\rho\, dx\,dy\,dz - \rho w_x dy\,dz\,dt - \rho w_y dz\,dx\,dt - \rho w_z dx\,dy\,dt),$$

a result which follows by the rule given by Hargreaves, the equation of continuity

$$\frac{\partial \rho}{\partial t} + \operatorname{div} (\rho w) = 0$$

being the condition that the above triple integral can be expressed as a double integral.

These forms of the equations are specially suitable for the theory of transformation, which is limited by the condition that the total charge on the system of particles remains constant.*

If it is possible under this limitation to transform the equations so as to leave them in the same form when referred to another system of axes x_1, y_1, z_1, t_1, then we must have

$$E_x dy\,dz + \ldots - H_x dx\,dt - \ldots = E_{1x} dy_1 dz_1 + \ldots - H_{1x} dx_1 dt_1 - \ldots,$$

and $$H_x dy\,dz + \ldots + E_x dx\,dt + \ldots$$
$$= \theta [H_{1x} dy_1 dz_1 + \ldots + E_{1x} dx_1 dt_1 + \ldots],$$

where θ is a constant.

---

* Bateman, "The Transformation of the Electrodynamical Equations," *Proc. London Math. Soc.*, Ser. 2, Vol. 8, 1910.

These two relations give two sets of equations connecting the vectors E, H with the vectors E_1, H_1, viz.,

$$\begin{aligned} E_x = {} & E_{1x}\frac{\partial(y_1, z_1)}{\partial(y, z)} + E_{1y}\frac{\partial(z_1, x_1)}{\partial(y, z)} + E_{1z}\frac{\partial(x_1, y_1)}{\partial(y, z)} \\ & - H_{1x}\frac{\partial(x_1, t_1)}{\partial(y, z)} - H_{1y}\frac{\partial(y_1, t_1)}{\partial(y, z)} - H_{1z}\frac{\partial(z_1, t_1)}{\partial(y, z)}, \\ -H_x = {} & E_{1x}\frac{\partial(y_1, z_1)}{\partial(x, t)} + E_{1y}\frac{\partial(z_1, x_1)}{\partial(x, t)} + E_{1z}\frac{\partial(x_1, y_1)}{\partial(x, t)} \\ & - H_{1x}\frac{\partial(x_1, t_1)}{\partial(x, t)} - H_{1y}\frac{\partial(y_1, t_1)}{\partial(x, t)} - H_{1z}\frac{\partial(z_1, t_1)}{\partial(x, t)}, \end{aligned}$$

and

$$\begin{aligned} H_x = \theta\Big[& H_{1x}\frac{\partial(y_1, z_1)}{\partial(y, z)} + H_{1y}\frac{\partial(z_1, x_1)}{\partial(y, z)} + H_{1z}\frac{\partial(x_1, y_1)}{\partial(y, z)} \\ & + E_{1x}\frac{\partial(x_1, t_1)}{\partial(y, z)} + E_{1y}\frac{\partial(y_1, t_1)}{\partial(y, z)} + E_{1z}\frac{\partial(z_1, t_1)}{\partial(y, z)}\Big], \\ E_x = \theta\Big[& H_{1x}\frac{\partial(y_1, z_1)}{\partial(x, t)} + H_{1y}\frac{\partial(z_1, x_1)}{\partial(x, t)} + H_{1z}\frac{\partial(x_1, y_1)}{\partial(x, t)} \\ & + E_{1x}\frac{\partial(x_1, t_1)}{\partial(x, t)} + E_{1y}\frac{\partial(y_1, t_1)}{\partial(x, t)} + E_{1z}\frac{\partial(z_1, t_1)}{\partial(x, t)}\Big]. \end{aligned}$$

In order that these two sets of equations may be consistent, we must have $\theta^2 = +1$, and eighteen relations of the type

$$\frac{\partial(y_1, z_1)}{\partial(x, y)} = \theta\frac{\partial(x_1, t_1)}{\partial(z, t)},$$

$$\frac{\partial(y_1, z_1)}{\partial(x, t)} = -\theta\frac{\partial(x_1, t_1)}{\partial(y, z)}.$$

Bateman,* to whom this transformation theory is due, has shown that the two systems of coordinates must satisfy the equation

$$dx_1^2 + dy_1^2 + dz_1^2 - dt_1^2 = \lambda^2(dx^2 + dy^2 + dz^2 - dt^2), \qquad (2)$$

where

$$\lambda^4 = \frac{\partial(x_1, y_1, z_1, it_1)}{\partial(x, y, z, it)},$$

a relation implied in the equation (2), and has further proved that, when the systems of axes are both right-handed and $\frac{\partial t_1}{\partial t} > 0$, θ must be positive so that its value is $+1$.

---

* *Loc. cit.*, p. 235.

Certain results which will be needed afterwards and which will enable us to change from differential coefficients $\frac{\partial x_1}{\partial x}$, ..., to differential coefficients $\frac{\partial x}{\partial x_1}$, ... may, perhaps, be best proved at this stage.

For any quantity V which can be expressed either as a function of x, y, z, t or of $x_1, y_1 z_1, t_1$ we have

$$\frac{\partial V}{\partial x_1} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial x_1} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial x_1} + \frac{\partial V}{\partial z}\frac{\partial z}{\partial x_1} + \frac{\partial V}{\partial t}\frac{\partial t}{\partial x_1},$$

$$\frac{\partial V}{\partial y_1} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial y_1} + \ldots,$$

$$\frac{\partial V}{\partial z_1} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial z_1} + \ldots,$$

$$\frac{\partial V}{\partial t_1} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial t_1} + \ldots,$$

whence

$$\frac{\partial V}{\partial t}\frac{\partial (x, y, z, t)}{\partial (x_1, y_1, z_1, t_1)} = \frac{\partial (x, y, z, V)}{\partial (x_1, y_1, z_1, t_1)}, \tag{3}$$

with corresponding expressions for $\frac{\partial V}{\partial x}$, $\frac{\partial V}{\partial y}$, $\frac{\partial V}{\partial z}$.

Now, from the transformation theory,

$$\frac{\partial (x, y, z, t)}{\partial (x_1, y_1, z_1, t_1)} = \frac{1}{\lambda^4}.$$

Also by reversing the process used by Bateman* to prove

$$\frac{\partial (x_1, y_1, z_1)}{\partial (x, y, z)} = \lambda^2 \frac{\partial t_1}{\partial t},$$

and formulæ analogous to this, it can be shown that

$$\frac{\partial (x, y, z)}{\partial (x_1, y_1, z_1)} = \frac{1}{\lambda^2}\frac{\partial t}{\partial t_1}. \tag{4}$$

utting $V = t_1$ in (3), we find

$$\frac{\partial t_1}{\partial t} = \lambda^4 \frac{\partial (x, y, z)}{\partial (x_1, y_1, z_1)} = \lambda^2 \frac{\partial t}{\partial t_1},$$

by using equation (4).

* *Loc. cit.*, p. 237, equation (A).

In a similar way it may be shown that

$$\frac{\partial x_1}{\partial t} = -\lambda^2 \frac{\partial t}{\partial x_1}, \quad \frac{\partial t_1}{\partial y} = -\lambda^2 \frac{\partial y}{\partial t_1}, \quad \frac{\partial x_1}{\partial y} = \lambda^2 \frac{\partial y}{\partial x_1}, * \tag{5}$$

a minus sign occurring whenever the time t or t_1 occurs *once* on each side of the equation.

2. *The General Transformation.*

The general solution of (2) has been given by Darboux,† and may be written in the form

$$x = a + \frac{k}{\lambda'} \{l_{11}(x_1-a)+l_{12}(y_1-\beta)+l_{13}(z_1-\gamma)+l_{14}(t_1-\delta)\},$$

$$y = b + \frac{k}{\lambda'} \{l_{21}(x_1-a)+l_{22}(y_1-\beta)+l_{23}(z_1-\gamma)+l_{24}(t_1-\delta)\},$$

$$z = c + \frac{k}{\lambda'} \{l_{31}(x_1-a)+l_{32}(y_1-\beta)+l_{33}(z_1-\gamma)+l_{34}(t_1-\delta)\},$$

$$t = d + \frac{k}{\lambda'} \{l_{41}(x_1-a)+l_{42}(y_1-\beta)+l_{43}(z_1-\gamma)+l_{44}(t_1-\delta)\},$$

where $\lambda' = x_1^2+y_1^2+z_1^2-t_1^2-2(ax_1+\beta y_1+\gamma z_1-\delta t_1)+(a^2+\beta^2+\gamma^2-\delta^2)$.

Also the quantities l form an orthogonal linear substitution in four dimensions, the coordinates being x, y, z, it and x_1 y_1, z_1 it_1, and therefore contain six arbitrary constants, so that the above solution contains fifteen constants, reducing to eleven if we make $x_1 = y_1 = z_1 = t_1 = 0$ when $x = y = z = t = 0$.

If we keep x_1, y_1, z_1 constant the path of a point in the x, y, z space is obtained by eliminating t_1 from the first three of the above equations. From the way in which t_1 occurs it is clear that the path is always a conic, which cannot be an ellipse, since we can always obtain values of the parameter t_1 for which x, y, z are infinite.

To prove this we consider the value of λ' treated as a quadratic in t_1; the roots of the equation $\lambda' = 0$ are real if

$$\delta^2-2(ax_1+\beta y_1+\gamma z_1)+r_1^2+(a^2+\beta^2+\gamma^2-\delta^2)$$

is positive.

---

* Cf. Darboux, *Leçons sur les systèmes orthogonaux et les coordonnées curvilignes*, 2nd ed., Vol. I, § 91, p. 159.

† *Op. cit.*, Vol. I, § 96, p. 167. The notation has been slightly changed in order to conform to the electromagnetic notation.

This expression can be written in the form

$$\frac{1}{r_1^2}[r_1^2-(ax_1+\beta y_1+\gamma z_1)]^2+\frac{1}{r_1^2}[r_1^2(a^2+\beta^2+\gamma^2)-(ax_1+\beta y_1+\gamma z_1)^2],$$

which is never negative, becoming zero if

$$x_1=a, \quad y_1=\beta, \quad z_1=\gamma.$$

We see, therefore, that we are dealing with a particular type of accelerated motion in which the path described by any point of an electron is a conic which cannot be either an ellipse or a circle. For our purpose, however, we may build up the transformation in the following way :—

Consider the transformation*

$$x'=\frac{kx''}{r''^2-t''^2}, \quad y'=\frac{ky''}{r''^2-t''^2}, \quad z'=\frac{kz''}{r''^2-t''^2}, \quad t'=\frac{kt''}{r''^2-t''^2},$$

change x'' into $\bar{x}-a$, y'' into $\bar{y}-\beta$, z'' into $\bar{z}-\gamma$, and t'' into $\bar{t}-\delta$, and we obtain

$$x'=\frac{k(\bar{x}-a)}{\bar{r}^2-\bar{t}^2-2(a\bar{x}+\beta\bar{y}+\gamma\bar{z}-\delta\bar{t})+a^2+\beta^2+\gamma^2-\delta^2}, \quad \ldots, \quad \ldots,$$

whence $$x=a+l_{11}x'+l_{12}y'+l_{13}z'+l_{14}t',$$

and similar equations give Darboux's solution of the equation

$$d\bar{x}^2+d\bar{y}^2+d\bar{z}^2-d\bar{t}^2=\lambda^2(dx^2+dy^2+dz^2-dt^2),$$

$k\lambda$ being equal to

$$\bar{r}^2-\bar{t}^2-2(a\bar{x}+\beta\bar{y}+\gamma\bar{z}-\delta\bar{t})+a^2+\beta^2+\gamma^2-\delta^2.$$

If we proceed further, and put

$$\bar{x}=\frac{x_1}{r_1^2-t_1^2}, \quad \ldots, \quad \ldots,$$

we find $$x'=\frac{k[x_1-a(r_1^2-t_1^2)]}{1-2(ax_1+\beta y_1+\gamma z_1-\delta t_1)+(a^2+\beta^2+\gamma^2-\delta^2)(r_1^2-t_1^2)}, \qquad (6)$$

and similar expressions for $\bar{y}$, $\bar{z}$, and $\bar{t}$, as the solution of the equation

$$dx_1^2+dy_1^2+dz_1^2-dt_1^2=\lambda^2(dx'^2+dy'^2+dz'^2-dt'^2),$$

* Used by E. Cunningham, "The Principle of Relativity in Electrodynamics, and an extension thereof," *Proc. London Math. Soc.*, Ser. 2, Vol. 8, 1910.

where $k\lambda$ is now equal to

$$1-2(\alpha x_1+\beta y_1+\gamma z_1-\delta t_1)+(\alpha^2+\beta^2+\gamma^2-\delta^2)(r_1^2-t_1^2).$$

This transformation is of a different type to the one used by Cunningham, reducing to the form $\bar{x}' = kx_1, \ldots$, when $\alpha=\beta=\gamma=\delta=0$.

In this paper this latter type of transformation of the form given by equation (6) is alone considered.

The general transformation of this type may now be written in the form

$$x=\frac{k}{D_1}\{l_{11}[x_1-\alpha(r_1^2-t_1^2)]+l_{12}[y_1-\beta(r_1^2-t_1^2)]+l_{13}[z_1-\gamma(r_1^2-t_1^2)] \\ +l_{14}[t_1-\delta(r_1^2-t_1^2)]\},$$

with similar expressions for z, y, t, where

$$D_1\equiv 1-2(\alpha x_1+\beta y_1+\gamma z_1-\delta t_1)+(\alpha^2+\beta^2+\gamma^2-\delta^2)(r_1^2-t_1^2),$$

which contains eleven constants, the remaining four having been chosen so as to make $x=y=z=t=0$ when $x_1=y_1=z_1=t_1=0$.

The interpretation of these eleven constants can be best obtained by considering the infinitesimal transformation. If we suppose that the various constants, with the exception of k, are all of an order of magnitude whose square may be neglected, the values of the quantities l become

$$\begin{array}{llll} l_{11}=1, & l_{12}=+2a, & l_{13}=+2b, & l_{14}=2f, \\ l_{21}=-2a, & l_{22}=1, & l_{23}=+2c, & l_{24}=2g, \\ l_{31}=-2b, & l_{32}=-2c, & l_{33}=1, & l_{34}=2h, \\ l_{41}=2f, & l_{42}=2g, & l_{43}=2h, & l_{44}=1, \end{array}$$

and the infinitesimal transformation is

$$x=k\{x_1+\alpha(x_1^2-y_1^2-z_1^2+t_1^2)+2\beta y_1x_1+2\gamma z_1x_1-2\delta t_1x_1+2ay_1+2bz_1+2ft_1\},$$
$$y=k\{y_1+2\alpha x_1y_1+\beta(y_1^2-x_1^2-z_1^2+t_1^2)+2\gamma y_1z_1-2\delta t_1y_1-2ax_1+2cz_1+2gt_1\},$$
$$z=k\{z_1+2\alpha x_1z_1+2\beta y_1z_1+\gamma(z_1^2-x_1^2-y_1^2-t_1^2)-2\delta t_1z_1-2bx_1-2cy_1+2ht_1\},$$
$$t=k\{t_1+2\alpha x_1t_1+2\beta y_1t_1+2\gamma z_1t_1-\delta(x_1^2+y_1^2+z_1^2+t_1^2)-2fx_1-2gy_1-2hz_1\}.$$

We may replace t_1 by t/k in the first three equations, whence, keeping $x_1y_1z_1$ constant,

$$\frac{dx}{dt}=\left[\frac{2\alpha}{k}t-2\delta x_1+2f\right]=2\left[\frac{\alpha t}{k}-\frac{\delta x}{k}+f\right],$$

with similar expressions for dy/dt and dz/dt.

Hence to a fixed point in the x_1, y_1, z_1 space there corresponds a point in motion in the x, y, z space, the motion being such that any point (x, y, z) has the velocities given above. The origin, therefore, moves with a velocity whose components at time $t = 0$ are $2f$, $2g$, $2h$, and with an acceleration whose components are $2a/k$, $2\beta/k$, $2\gamma/k$.

The motion of a point (x, y, z) relative to the origin is given by the velocities $-2x\delta/k$, $-2y\delta/k$, and $-2z\delta/k$, so that the constant δ may be said to express an expansion or contraction relative to the origin such that

$$-2\delta = kV/r,$$

where V is the radial velocity relative to the origin at a distance r.

The constant k clearly expresses a uniform change of volume, while the remaining constants a, b, c have no kinematical significance and correspond to a rotation in the theory of small deformations. The infinitesimal transformation, in fact, may be divided into two parts, a conformal transformation in three dimensions given by

$$x = k\{x' + a(x'^2 - y'^2 - z'^2) + 2\beta y'x' + 2\gamma z'x' + 2ay' + 2bz'\},$$

$$y = k\{y' + 2ax'y' + \beta(y'^2 - x'^2 - z'^2) + 2\gamma z'y' - 2ax' + 2cz'\},$$

$$z = k\{z' + 2ax'z' + 2\beta y'z' + \gamma(z'^2 - x'^2 - y'^2) - 2bx' - 2cy'\},$$

and a kinematical part given by

$$x' = x_1 + at_1^2 - 2\delta t_1 x_1 + 2ft_1,$$

$$y' = y_1 + \beta t_1^2 - 2\delta t_1 y_1 + 2gt_1,$$

$$z' = z_1 + \gamma t_1^2 - 2\delta t_1 z_1 + 2ht_1.$$

It may further be noticed that the transformation for the time t contains only those constants which have a kinematical interpretation. If we retain only this class of constants and, therefore consider a, b, and c as zero, we may obtain the general transformation for a constant velocity by putting $\alpha = \beta = \gamma = \delta = 0$. In this case the values of the quantities l are given by

$$Bl_{11} = 1 + f^2 - g^2 - h^2, \qquad Bl_{12} = 2fg,$$

$$Bl_{21} = 2fg, \qquad Bl_{22} = 1 + g^2 - f^2 - h^2,$$

$$Bl_{31} = 2fh, \qquad Bl_{32} = 2gh,$$

$$Bl_{41} = 2f, \qquad Bl_{42} = 2g,$$

$$Bl_{13} = 2fh, \qquad Bl_{11} = 2f,$$
$$Bl_{23} = 2gh, \qquad Bl_{24} = 2g,$$
$$Bl_{33} = 1+h^2-f^2-g^2, \qquad Bl_{34} = 2h,$$
$$Bl_{43} = 2h, \qquad Bl_{44} = 1+f^2+g^2+h^2,$$

where $$B = -f^2-g^2-h^2.$$

If we put
$$1+f^2+g^2+h^2 = 2f/u = 2g/v = 2h/w,$$
$$s^2 = u^2+v^2+w^2, \qquad \epsilon^2 = (1-s^2)^{-1},$$

we find that, when k is unity,

$$x = x_1+\frac{\epsilon^2 u}{1+\epsilon}(ux_1+vy_1+wz_1)+u\epsilon t_1,$$
$$y = y_1+\frac{\epsilon^2 v}{1+\epsilon}(ux_1+vy_1+wz_1)+v\epsilon t_1,$$
$$z = z_1+\frac{\epsilon^2 w}{1+\epsilon}(ux_1+vy_1+wz_1)+w\epsilon t_1,$$
$$t = \epsilon(ux_1+vy_1+wz_1)+\epsilon t_1,$$

which is the general transformation for a point moving with a constant velocity whose components are u, v, w.* If v and w are zero, so that the point moves along the axis of x with a constant velocity u, the above transformation reduces to the well known equations given by Larmor and Lorentz.

3. *A Special Type of Transformation.*

We will now consider in detail the transformation in which the constants a, b, c are zero, the other constants being retained in the greater part of the work.

If we have a transformation

$$x = x', \quad y = y', \quad z = z', \quad t = t'$$

which satisfies equation (2), x', y', z', t' being functions of x_1, y_1, z_1, t_1, it it can be easily shown that the transformation

$$x = \epsilon(x'+ut'), \quad y = y', \quad z = z', \quad t = \epsilon(t'+ux')$$

* Recently given by G. Herglotz, "Über die Mechanik des deformierbaren Körpers vom Standpunkte der Relativitätstheorie," *Annalen der Physik*, Vol. 36, 1911, p. 497.

where $\epsilon^2 \equiv (1-u^2)^{-1}$ also satisfies equation (2), the Jacobian λ^4 being independent of u.

Using the values of x', y', z', t' given in equation (6), we finally obtain the transformation

$$\left.\begin{aligned} x &= \frac{\epsilon k}{D_1}\{x_1-\alpha(r_1^2-t_1^2)+u[t_1-\delta(r_1^2-t_1^2)]\} \\ t &= \frac{\epsilon k}{D_1}\{t_1-\delta(r_1^2-t_1^2)+u[x_1-\alpha(r_1^2-t_1^2)]\} \\ y &= \frac{k}{D_1}[y_1-\beta(r_1^2-t_1^2)], \quad z = \frac{k}{D_1}[z_1-\gamma(r_1^2-t_1^2)] \end{aligned}\right\}, \tag{7}$$

where we have now to interpret the constants α, β, γ, δ, u.

Consider the origin of a system of moving axes which at time $t=0$ coincides with a system of fixed axes x, y, z.

Then the component velocities of the origin of the moving axes are the values of

$$\frac{dx}{dt}, \quad \frac{dy}{dt}, \quad \frac{dz}{dt} \quad \text{for} \quad x=y=z=t=0,$$

and its accelerations the values of

$$\frac{d^2x}{dt^2}, \quad \frac{d^2y}{dt^2}, \quad \frac{d^2z}{dt^2} \quad \text{for} \quad x=y=z=t=0.$$

In finding the velocity and acceleration, x, y, z are to be considered as expressed in terms of x_1, y_1, z_1, and t; further x_1, y_1, z_1 are to be kept constant, since we wish to transform from a system at rest in the x_1, y_1, z_1 space to a system in motion in the x, y, z space.

Hence $$\frac{dx}{dt} = \frac{\partial x}{\partial t_1}\Big/\frac{\partial t}{\partial t_1},$$

and when $\quad x=y=z=t=0, \quad x_1=y_1=z_1=t_1=0.$

Now

$$\frac{\partial x}{\partial t_1} = \frac{\epsilon k}{D_1^2}\{D_1[u+2(\alpha+u\delta)t_1] - 2[x_1+ut_1-(\alpha+u\delta)(r_1^2-t_1^2)][\delta-(\alpha^2+\beta^2+\gamma^2-\delta^2)t_1]\},$$

$$\frac{\partial t}{\partial t_1} = \frac{\epsilon k}{D_1^2}\{D_1[1+2(\delta+u\alpha)t_1] - 2[t_1+ux_1-(\delta+u\alpha)(r_1^2-t_1^2)][\delta-(\alpha^2+\beta^2+\gamma^2-\delta^2)t_1]\},$$

so that $$\frac{dx}{dt} = u, \quad \text{for} \quad x_1 = y_1 = z_1 = t_1 = 0.$$

Similarly $$\frac{dy}{dt} = \frac{dz}{dt} = 0, \quad \text{for} \quad x_1 = y_1 = z_1 = t_1 = 0.$$

Again, $$\frac{d^2x}{dt^2} = \left(\frac{\partial}{\partial t_1}\right)^{-1} \frac{\partial}{\partial t_1}\left(\frac{dx}{dt}\right) = \frac{2\alpha(1-u^2)}{k\epsilon},$$

for the origin of the moving axes, and similarly

$$\frac{d^2y}{dt^2} = 2\beta/k\epsilon^2, \quad \frac{d^2z}{dt^2} = 2\gamma/k\epsilon^2.$$

Therefore, if p, q, r be the component accelerations of the origin,

$$2\alpha = kp\epsilon^3, \quad 2\beta = kq\epsilon^2, \; 2\gamma = kr\epsilon^2.$$

Hence the above transformation (7) gives the motion of a point which at time $t = 0$ has a velocity $(u, 0, 0)$ and an acceleration (p, q, r). As regards the constant δ, it is not difficult to show that when u is zero δ is defined by

$$-2\delta = k \lim_{r \to 0}\left(\frac{V}{r}\right),$$

where V is the radial velocity relative to the origin at a distance r from the origin, giving an interpretation to the constant δ of a nature similar to that adopted in the case of the infinitesimal transformation, when δ, α, β, γ were all small.

When u is not zero, *i.e.*, when the centre of the electron is not instantaneously at rest, we have to distinguish between the velocity of a point relative to the centre in different directions, and it is the velocity of a point relative to the centre in its direction of motion which enables us to give an interpretation to the constant δ, which now, however, is grouped together with other constants in the form $\delta + u\alpha$.

It is, in fact, not difficult to prove that

$$-2(\delta + u\alpha) = \epsilon k \lim\left(\frac{V_x}{x}\right),$$

where V_x is the velocity relative to the centre of the electron in the direction of motion, x is measured from the centre in the direction of motion, and in performing the limiting process indicated we first make y and z tend to zero, and then make x tend to zero.

Thus the motion in which $\delta + u\alpha$ vanishes is a special type character-

ized by the fact that there is no relative expansion or contraction in the direction of motion.

When the constants are such that $\delta+u\alpha=0$ we shall say that we are dealing with the motion of a non-expanding electron.

For some purposes it is useful to express x_1, y_1, z_1, t_1 in terms of x, y, z, t, which can be done by reversing the equations (7), giving

$$\left.\begin{aligned} x_1 &= \frac{\epsilon k(x-ut)+\alpha(r^2-t^2)}{D} \\ t_1 &= \frac{\epsilon k(t-ux)+\delta(r^2-t^2)}{D} \\ y_1 &= \frac{ky+\beta(r^2-t^2)}{D}, \qquad z_1 = \frac{kz+\gamma(r^2-t^2)}{D} \end{aligned}\right\}, \tag{8}$$

where

$$D \equiv k^2+2k[\beta y+\gamma z+\alpha\epsilon(x-ut)-\delta\epsilon(t-ux)]+(\alpha^2+\beta^2+\gamma^2-\delta^2)(r^2-t^2).$$

If we consider an electron at rest in the x_1, y_1 z_1 space to be bounded by the sphere $x_1^2+y_1^2+z_1^2=a^2$, we see that at time $t=0$ its boundary when in motion will be given by

$$a^2[k^2+2k(\beta y+\gamma z+\alpha\epsilon x+\delta\epsilon ux)+(\alpha^2+\beta^2+\gamma^2-\delta^2)r^2]^2$$
$$= k^2(\epsilon^2x^2+y^2+z^2)+(\alpha^2+\beta^2+\gamma^2)r^4+2kr^2(\alpha\epsilon x+\beta y+\gamma z).$$

If the electron is moving with a uniform velocity, this reduces to the spheroid

$$\epsilon^2x^2+y^2+z^2=k^2a^2,$$

giving the Fitzgerald-Lorentz contraction in the direction of motion.*

To discuss the particular type of motion given by (7) it is convenient to take the centre of the electron at rest in the x_1, y_1, z_1 space to be the origin in that space and also to choose the axes of y and z in the x, y, z space so that the acceleration of the centre of the electron at the time $t=0$ shall be in the plane of xy. Under these circumstances $x_1=y_1=z_1=\gamma=0$, so that the path of the centre is given by eliminating t_1 from the first two of the equations

$$x = \frac{\epsilon k}{D_1}\{ut_1+(\alpha+u\delta)t_1^2\}, \qquad y = \frac{k\beta t_1^2}{D_1},$$

$$t = \frac{\epsilon k}{D_1}\{t_1+(\delta+u\alpha)t_1^2\}, \qquad z = 0,$$

* It is important to notice that this contraction is only defined for the case of uniform motion.

where $$D_1 = 1+2\delta t_1-(a^2+\beta^2-\delta^2)\, t_1^2.$$

The path is, therefore, the hyperbola

$$\beta^2x^2+y^2(a^2-\beta^2u^2\epsilon^2)-2a\beta\epsilon xy-k\beta\epsilon^2u^2y = 0,$$

and the centre of the electron describes that branch which passes through the origin, the tangent at this latter point being the axis of x.

The values of x and y in terms of the time t are given by

$$a^2(t^2-x^2)-\beta^2\epsilon^2(x-ut)^2-\epsilon ka(x-ut) = 0,$$

$$y^2(a^2+\epsilon^2\beta^2)+2\beta\epsilon uaty+k\beta\epsilon^2y-t^2\beta^2 = 0,$$

δ being automatically eliminated, from which the position of the centre of the electron can be found for different values of t, that branch of the hyperbola being taken for which $x = y = 0$ when $t = 0$.

A particular case is obtained by making $\beta = 0$; this gives $y = 0$, so the centre moves along the axis of x. If we make it pass through the origin

$$x = \frac{1}{2a}\left\{\sqrt{(\epsilon^2k^2+4\epsilon kuat+4a^2t^2)}-\epsilon k\right\}.$$

Tracing the position of the centre of the electron from $t = -\infty$ to $t = +\infty$, we find that it starts with a velocity equal to that of light from $x = +\infty$, reaching the origin at a time $-\epsilon uk/a$ with a velocity $-u$; it comes to rest at a time $-\epsilon uk/2a$, and then retraces its path, passing through the origin again at time $t = 0$, when it has a velocity u and an acceleration $2a/k\epsilon^3$, its greatest distance from the origin in the negative direction being $k(\epsilon-1)/2a$.

4. *Expressions for the Forces acting on an Electron.*

For the purpose of calculating the forces on an electron it will be easiest to avail ourselves of the invariant theory of the general transformation as developed by Bateman, though the results can be obtained by a calculation of the electric and magnetic vectors from the formulæ connecting their values in the two spaces.

The following invariants of the theory will be found useful:—

$$\lambda^2(dx^2+dy^2+dz^2-dt^2) = dx_1^2+dy_1^2+dz_1^2-dt_1^2, \qquad (2)$$

$$\lambda^4\, dx\, dy\, dz\, dt = dx_1\, dy_1\, dz_1\, dt_1, \qquad (9)$$

$$H_x dy\,dz + H_y dz\,dx + H_z dx\,dy + E_x dx\,dt + E_y dy\,dt + E_z dz\,dt, \quad (10)$$

$$E_x dy\,dz + E\ dz\,dx + E_z dx\,dy - H\ dx\,dt - H_y dy\,dt - H_z dz\,dt, \quad (11)$$

$$\rho w_x dy\,dz\,dt + \rho w_y dz\,dx\,dt + \rho w_z dx\,dy\,dt - \rho\, dx\,dy\,dz. \quad (12)$$

From these invariants others can be derived by processes analogous to those of multiplication and reciprocation.

In this way we may deduce the following :—

$$\frac{1}{\lambda^2}(\rho w_x dx + \rho w_y dy + \rho w_z dz - \rho\, dt), \quad (13)$$

$$\frac{\rho^2}{\lambda^2}(1-w^2)\, dx\,dy\,dz\,dt, \quad (14)$$

$$\rho\, dx\,dy\,dz\,dt\,[(E_x + w_y H_z - w_z H_y)\delta x + \ldots + \ldots - (w_x E_x + w_y E_y + w_z E_z)\delta t], \quad (15)$$

where δx, δy, δz, δt are arbitrary increments of the variables. For our purpose we seek to express the various quantities involved in terms of x_1, y_1, z_1, and t, where x_1, y_1, z_1 are independent of the time t, and to integrate always over the x_1, y_1, z_1 space where the electron is assumed to be spherical and the integrations are simple.

In this case we have

$$\frac{dx}{dt} = w_x, \ \ldots, \ \frac{dx_1}{dt} = 0, \ \ldots,$$

so that (2) becomes $\lambda^2(1-w^2)dt^2 = dt_1^2$,

or $$\lambda\sqrt{(1-w^2)}\,dt = dt_1, \quad (16)$$

which is an important result, since from (9), (14), and (16), we now deduce that

$$\rho\, dx\,dy\,dz = \rho_1 dx_1 dy_1 dz_1. \quad (17)$$

This expresses the condition, already assumed, that the total charge is invariant and can be verified by using Jacobians, only that we must take care to express x, y, z in terms of x_1, y_1, z_1, and the ordinary time t.

Since the electron is at rest in the x_1, y_1, z_1 space $H_1 = 0$, while E_1 is simply the electrostatic force due to the electron itself.

---

* These and other invariants are given by Bateman, *loc. cit.*, p. 252, together with an account of the processes used in obtaining them.

From (15) and (16) we therefore deduce

$$\rho\,dx\,dy\,dz[(E_x+w_yH_z-w_zH_y)\delta x+\ldots+\ldots-(w_xE_x+w_yE_y+w_zE_z)\delta t]$$
$$=\rho_1 dx_1 dy_1 dz_1 \lambda\sqrt{(1-w^2)}[E_{1x}\delta x_1+E_{1y}\delta y_1+E_{1z}\delta z_1],$$

so that

$$\rho\,dx\,dy\,dz\,F_x=\rho_1 dx_1 dy_1 dz_1\lambda\sqrt{(1-w^2)}\left[E_{1x}\frac{\partial x_1}{\partial x}+E_{1y}\frac{\partial y_1}{\partial x}+E_{1z}\frac{\partial z_1}{\partial x}\right],$$

$$\rho\,dx\,dy\,dz\,F_y=\rho_1 dx_1 dy_1 dz_1\lambda\sqrt{(1-w^2)}\left[E_{1x}\frac{\partial x_1}{\partial y}+E_{1y}\frac{\partial y_1}{\partial y}+E_{1z}\frac{\partial z_1}{\partial y}\right],$$

$$\rho\,dx\,dy\,dz\,F_z=\rho_1 dx_1 dy_1 dz_1\lambda\sqrt{(1-w^2)}\left[E_{1x}\frac{\partial x_1}{\partial z}+E_{1y}\frac{\partial y_1}{\partial z}+E_{1z}\frac{\partial z_1}{\partial z}\right],$$

$$-\rho\,dx\,dy\,dz(w,F)=\rho_1 dx_1 dy_1 dz_1\lambda\sqrt{(1-w^2)}\left[E_{1x}\frac{\partial x_1}{\partial t}+E_{1y}\frac{\partial y_1}{\partial t}+E_{1z}\frac{\partial z_1}{\partial t}\right],$$

where $F\equiv E+[w,H]$ is the force per unit charge acting on the electron due to the reaction of the æther.

By integration on both sides of these equations we obtain from the first three the components of the force on the electron, and from the last equation the rate of work done by the electromagnetic forces. Calling these X, Y, Z and dA/dt respectively, we shall calculate, as far as possible, their values at time $t=0$.

Replacing w_x by $\dfrac{\partial x}{\partial t_1}\Big/\dfrac{\partial t}{\partial t_1}$, ..., and remembering that

$$\lambda^2\left[\left(\frac{\partial t}{\partial t_1}\right)^2-\left(\frac{\partial x}{\partial t_1}\right)^2-\left(\frac{\partial y}{\partial t_1}\right)^2-\left(\frac{\partial z}{\partial t_1}\right)^2\right]=1,$$

we can substitute $\left(\dfrac{\partial t}{\partial t_1}\right)^{-1}$ for $\lambda\sqrt{(1-w^2)}$, giving a more convenient form for our purpose.

We can also replace $\dfrac{\partial x_1}{\partial x}$, $\dfrac{\partial y_1}{\partial x}$, $\dfrac{\partial z_1}{\partial x}$ by $\lambda^2\dfrac{\partial x}{\partial x_1}$, $\lambda^2\dfrac{\partial x}{\partial y_1}$, $\lambda^2\dfrac{\partial x}{\partial z_1}$ from equations (5).

Again, at time $t=0$, we must have, from equations (7),

$$t_1+ux_1-(\delta+ua)(r_1^2-t_1^2)=0,$$

whence $$t_1=\frac{1}{2(\delta+ua)}\{\sqrt{[1-4(\delta+ua)ux_1+4(\delta+ua)^2r_1^2]}-1\},\qquad(18)$$

the positive sign being given to the square root, since $t_1=0$ for

$$x_1=y_1=z_1=0.$$

Further, for $t = 0$, we have

$$\frac{\partial t}{\partial t_1} = \frac{k\epsilon}{D_1}\left[1+2(\delta+u\alpha)\,t_1\right] = \frac{k\epsilon}{D_1}\sqrt{\left[1-4(\delta+u\alpha)ux_1+4(\delta+u\alpha)^2 r_1^2\right]},$$

using the last equation.

We are now in a position to calculate the forces acting on the electron, which we shall consider to be when at rest a spherical shell of radii b and a, the charge e_1 being uniformly distributed over its volume. We can pass from this case to the limiting cases of volume and surface distribution by making $b = 0$ and $b = a$ respectively.

Hence $\quad E_{1x} = \frac{1}{3}\rho_1 x_1\left(1-\frac{b^3}{r_1^3}\right), \qquad E_{1y} = \frac{1}{3}\rho_1 y_1\left(1-\frac{b^3}{r_1^3}\right),$

$$E_{1z} = \tfrac{1}{3}\rho_1 z_1\left(1-\frac{b^3}{r_1^3}\right),$$

where r_1 is the distance of the point (x_1, y_1, z_1) from the origin.

Making use of these simplifications, we finally obtain

$$X = \iiint \rho F_x\,dx\,dy\,dz$$

$$= \iiint \rho_1\,dx_1\,dy_1\,dz_1\lambda^2\left(\frac{\partial t}{\partial t_1}\right)^{-1}\left[E_{1x}\frac{\partial x}{\partial x_1}+E_{1y}\frac{\partial x}{\partial y_1}+E_{1z}\frac{\partial x}{\partial z_1}\right]$$

$$= \iiint \tfrac{1}{3}\frac{\rho_1^2\,dx_1\,dy_1\,dz_1 D_1^3\,(1-b^3/r_1^3)}{\epsilon k^3\sqrt{\left[1-4(\delta+u\alpha)ux_1+4(\delta+u\alpha)^2 r_1^2\right]}}\left[x_1\frac{\partial x}{\partial x_1}+y_1\frac{\partial x}{\partial y_1}+z_1\frac{\partial x}{\partial z_1}\right]$$

$$= \iiint \tfrac{1}{3}\rho_1^2\left(1-\frac{b^3}{r_1^3}\right)\frac{D_1\,dx_1\,dy_1\,dz_1 N_x}{k^2\sqrt{\left[1-4(\delta+u\alpha)ux_1+4(\delta+u\alpha)^2 r_1^2\right]}},$$

where $N_x \equiv x_1+2x_1t_1\delta-x_1\kappa^2(r_1^2+t_1^2)-2(\alpha+u\delta)r_1^2-2ut_1\kappa^2 r_1^2$

$$-4\delta t_1(\alpha+u\delta)r_1^2+2ut_1(\alpha x_1+\beta y_1+\gamma z_1)$$

$$+2(\alpha+u\delta)(r_1^2+t_1^2)(\alpha x_1+\beta y_1+\gamma z_1),$$

and $\qquad \kappa^2 \equiv \alpha^2+\beta^2+\gamma^2-\delta^2.$

Similarly,

$$Y = \iiint \tfrac{1}{3}\rho_1^2\left(1-\frac{b^3}{r_1^3}\right)\frac{D_1\,dx_1\,dy_1\,dz_1 N_y}{\epsilon k^2\sqrt{\left[1-4(\delta+u\alpha)ux_1+4(\delta+u\alpha)^2 r_1^2\right]}},$$

$$Z = \iiint \tfrac{1}{3}\rho_1^2\left(1-\frac{b^3}{r_1^3}\right)\frac{D_1\,dx_1\,dy_1\,dz_1 N_z}{\epsilon k^2\sqrt{\left[1-4(\delta+u\alpha)ux_1+4(\delta+u\alpha)^2 r_1^2\right]}},$$

where
$$N_y \equiv y_1[1-\kappa^2(r_1^2+t_1^2)]-2\beta r_1^2-4\beta r_1^2 t_1\delta+2y_1 t_1\delta$$
$$+2\beta(ax_1+\beta y_1+\gamma z_1)(r_1^2+t_1^2),$$
$$N_z \equiv z_1[1-\kappa^2(r_1^2+t_1^2)]-2\gamma r_1^2-4\gamma r_1^2 t_1\delta+2z_1 t_1\delta$$
$$+2\gamma(ax_1+\beta y_1+\gamma z_1)(r_1^2+t_1^2),$$

and in all cases t_1 must be expressed in terms of x_1, y_1, z_1 by equation (18) before the integration is performed.

5. *Calculations of the Forces.*

It does not seem possible to perform the integrations in the general case, so that we shall be content with discussing some simple cases.

I. $\delta = 0$. *Lorentz Electrons.*

The case of an electron whose shape depends on its motion was first investigated by Lorentz. This type is an obvious extension of his idea, and gives, as we shall see, for the case of quasi-stationary motion the same values of the electromagnetic masses as deduced by him, proving that in this case the application of the theory of quasi-stationary motion is justified.

There are in this type of electron two special cases where the integration can be easily effected.

The first case is when $u = 0$, *i.e.*, the centre of the electron is instantaneously at rest at the instant when the force is calculated; t is then zero from (18), so that the integrals are all of the same kind, and we find for a volume distribution ($b = 0$)

$$X, Y, Z = -\frac{e_1^2}{\pi k^2}(2a, 2\beta, 2\gamma)\left[\frac{1}{5a}+\frac{1}{7}(a^2+\beta^2+\gamma^2)a\right]$$

and for the case of a surface distribution

$$X, Y, Z = -\frac{e_1^2}{6\pi k^2}(2a, 2\beta, 2\gamma)\left[\frac{1}{a}+(a^2+\beta^2+\gamma^2)a\right] \quad (19)$$

In terms of the acceleration (p, q, r) the forces are

$$\left.\begin{aligned} X, Y, Z &= -(p, q, r)\,\frac{e_1^2}{\pi k}\left[\frac{1}{5a}+\frac{k^2}{28}(p^2+q^2+r^2)\,a\right]* \\ &\text{for a volume distribution of charge, and} \\ &= -(p, q, r)\,\frac{e_1^2}{6\pi k}\left[\frac{1}{a}+\frac{k^2}{4}(p^2+q^2+r^2)\,a\right] \\ &\text{for a surface distribution.} \end{aligned}\right\}. \qquad (19)$$

The second case in which the integration can be easily performed is when $a = 0$, *i.e.*, the acceleration of the centre of the electron in the direction of motion is instantaneously zero. In this case we find from (18) that $t_1 = -ux_1$, and it follows that X is zero, while

$$\left.\begin{aligned} Y, Z &= -(2\beta, 2\gamma)\,\frac{e_1^2}{6\pi\epsilon k^2}\left[\frac{6}{5a}+(6-u^2)(\beta^2+\gamma^2)\,\frac{a}{7}\right] \\ &\text{for a volume distribution, and} \\ Y, Z &= -(2\beta, 2\gamma)\,\frac{e_1^2}{4\pi\epsilon k^2}\left[\frac{2}{3}a+(6-u^2)(\beta^2+\gamma^2)\,\frac{a}{9}\right] \\ &\text{for a surface distribution.} \end{aligned}\right\}. \qquad (20)$$

In terms of the accelerations q, r the forces are

$$\left.\begin{aligned} Y, Z &= -(q, r)\,\frac{e_1^2\epsilon}{6\pi k}\left[\frac{6}{5a}+\frac{(6-u^2)}{28}(q^2+r^2)\,k^2\epsilon^4 a\right] \\ Y, Z &= -(q, r)\,\frac{e_1^2\epsilon}{4\pi k}\left[\frac{2}{3}a+\frac{(6-u^2)}{36}(q^2+r^2)\,k^2\epsilon^4 a\right] \end{aligned}\right\}, \qquad (20a)$$

for volume and surface distribution respectively.

II. $\delta+ua=0$. *Non-expanding Electrons.*

This case has some features of special interest, the first being that the values of the forces can be accurately worked out, since, when $\delta+ua=0$,

---

* The special case of $q = r = 0$ is treated by Born, "Die Theorie des Starren Elektrons in der Kinematik des Relativitäts prinzips," *Annalen der Physik*, Vol. 30, 1909. He finds the first term $-\frac{e_1^2 p}{5\pi a}$, $(k = 1)$, and shows that there is no term in p^2.

we have, from (18), $t_1 = -ux_1$, and the integration is easy, the results being

$$\left.\begin{aligned} X &= -\frac{2a(1-u^2)e_1^2}{\pi k^2}\left\{\frac{1}{5a}+\frac{a}{7.3}\left[a^2(1-u^2)(3-2u^2)+(\beta^2+\gamma^2)(3+u^2)\right]\right\} \\ Y,Z &= -\frac{(2\beta,\,2\gamma)e_1^2}{\pi\epsilon k^2}\left\{\frac{1}{5a}+\frac{a}{7.6}\left[a^2(1-u^2)(6-7u^2)+(\beta^2+\gamma^2)(6-u^2)\right]\right\} \end{aligned}\right\}, \tag{21}$$

for volume distribution, while for a surface distribution of charge the factors $\frac{1}{5}$ and $\frac{1}{7}$ are both replaced by $\frac{1}{6}$.

When either u or a is zero the above values of the forces agree with those given in equations (19) and (20).

III. *Approximations, δ and a small.*

If in the general case δ and a are both small, we may expand in terms of these quantities and obtain the following results.

Let the expansions have the following form

$$X = X_0 + aX_1 + \delta X' + \tfrac{1}{2}(a^2X_2 + 2a\delta\bar{X} + \delta^2X'') + \ldots,$$

with similar expressions for Y and Z.

Then X_0 is zero,

$$\left.\begin{aligned} X_1 &= -\frac{2e_1^2}{\pi k^2}\left\{\frac{1}{5a}+\frac{a}{7}\,\frac{3+4u^2-3u^4}{3}(\beta^2+\gamma^2)\right. \\ &\qquad \left.+\frac{a^3}{9}\,\frac{u^2(3-u^2)(1-u^2)}{3}(\beta^2+\gamma^2)^2\right\} \\ X' &= -\frac{2ue_1^2}{\pi k^2}\left\{\frac{1}{5a}+\frac{a}{7}\,\frac{6-2u^2}{3}(\beta^2+\gamma^2)\right. \\ &\qquad \left.+\frac{a^3}{9}\,\frac{(3-u^2)(1-u^2)}{3}(\beta^2+\gamma^2)^2\right\} \end{aligned}\right\}, \tag{22}$$

for volume distribution of charge, while for a surface distribution the factors $\frac{1}{5}$, $\frac{1}{7}$, $\frac{1}{9}$ are each replaced by $\frac{1}{6}$.

Further, the terms of the second order X_2, $\bar{X}$, X'' all vanish. As regards Y and Z, Y_0 and Z_0 are given by their values in (20), Y_1, Y', Z_1, Z' all vanish, and

$$\left.\begin{aligned} Y_2 &= -\frac{2\beta e_1^2}{\pi k^2\epsilon}\left\{\frac{6-19u^2+10u^4}{3}\,\frac{a}{7}-\frac{2u^2(1-u^2)(9-2u^2)}{3}\,\frac{a^3}{9}(\beta^2+\gamma^2)\right\} \\ \bar{Y} &= -\frac{2u\beta e_1^2}{\pi k^2\epsilon}\left\{\frac{u^2}{3}\,\frac{a}{7}-\frac{2(1-u^2)(6-u^2)}{3}\,\frac{a^3}{9}(\beta^2+\gamma^2)\right\} \\ Y'' &= -\frac{2\beta e_1^2}{\pi k^2\epsilon}\left\{\frac{6-u^2}{3}\,\frac{a}{7}-\frac{6(1-u^2)}{3}\,\frac{a^3}{9}(\beta^2+\gamma^2)\right\} \end{aligned}\right\}, \tag{23}$$

with similar expressions for the coefficients in the expansion of Z.

Again, for a surface distribution of charge the numerical factors are changed as before.

If we put $\delta = -ua$ in the expansions for X, Y, and Z we obtain the values given in (21), thus affording a verification of the various calculations.

6. *Quasi-Stationary Motion and Electromagnetic Mass.*

The most interesting application of the above results is to the case of quasi-stationary motion; in this case the accelerations (p, q, r) are small, and the values of X, Y, and Z for Lorentz electrons ($\delta = 0$) are given by

$$Y, Z = -(q, r)\frac{e_1^2\epsilon}{5\pi ka} \quad \text{[from (20)]},$$

and

$$X = aX_1 = -\frac{2ae_1^2}{5\pi k^2 a} \quad \text{[from (22)]}$$

$$= -\frac{pe_1^2\epsilon^3}{5\pi ka},$$

in terms of the acceleration (p, q, r).

The external force necessary in order to make the electron describe its path must be equal and opposite to the force exerted on the electron by the æther. If X_e, Y_e, Z_e are the components of this external force,

$$X_e = -X, \quad Y_e = -Y, \quad Z_e = -Z.$$

Hence

$$X_e = \frac{e_1^2}{5\pi ka}p\epsilon^3, \qquad Y_e, Z_e = (q, r)\frac{e_1^2\epsilon}{5\pi ka},$$

so that the external force is *to this order* proportional to the acceleration, the coefficients

$$\frac{e_1^2}{5\pi ka}(\epsilon^3, \epsilon, \epsilon) \quad [\epsilon^2 \equiv (1-v^2)^{-1}],$$

being, therefore, interpreted as electromagnetic masses. When $k = 1$ these values agree with those given by Lorentz and Abraham.

Lorentz at first retains the constant k, afterwards seeking to show by the comparison of two expressions for the electromagnetic mass that it has the value unity. This proof cannot be considered as necessarily correct, since it rests on the general quasi-stationary principle that the quantities

involved, in this case the "electromagnetic momentum," can be calculated for the uniform motion and not for the variable motion.*

It has been suggested by Bucherer that an electron has the same volume whether at rest or in motion. For *uniform* motion this would give $k = \epsilon^{\frac{1}{3}}$, and, using this value of k, we obtain the values for the electromagnetic masses given by him.†

It does, not, however, appear possible to extend this idea of constant volume to the general case of accelerated motion. If $dx\,dy\,dz = dx_1 dy_1 dz_1$ we must have $\rho = \rho_1$ from (17), and hence from (9) and (14) $(1-u^2)^{\frac{1}{2}} = \lambda^3$, or $\lambda^4 \dfrac{\partial t}{\partial t_1} = 1$, which cannot be satisfied for all values of x_1, y_1, z_1, t_1, even for the infinitesimal transformation of § 2, in which $\lambda \dfrac{\partial t}{\partial t_1} = 1$.

For surface distribution of charge the factor $\frac{1}{5}$ in the electromagnetic mass must be replaced by $\frac{1}{6}$. For higher powers of the accelerations each component of the external force will still be proportional to the acceleration in its own direction, the coefficients being, however, functions of the accelerations in all three directions of the axes.

In the case of the non-expanding electrons defined by $\delta + ua = 0$ we obtain from (21) the accurate values of X, Y, Z. In terms of the acceleration (p, q, r) we therefore have

$$\left.\begin{aligned} X_e &= \frac{p\epsilon e_1^2}{\pi k}\left\{\frac{1}{5a} + \frac{a}{7}\,\frac{\epsilon^4}{12}\left[(3-2u^2)p^2 + (3+u^2)(q^2+r^2)\right]\right\} \\ Y_e,\ Z_e &= (q, r)\,\frac{\epsilon e_1^2}{\pi k}\left\{\frac{1}{5a} + \frac{a\epsilon^4}{7.24}\left[(6-7u^2)p^2 + (6-u^2)(q^2+r^2)\right]\right\} \end{aligned}\right\}. \quad (24)$$

So that to the *first order* in (p, q, r) the external force is again proportional to the acceleration, the coefficients for the three components being, however, in this case the same, viz.,

$$\frac{e_1^2\epsilon}{5\pi a k},$$

for a volume distribution of charge.

There is, therefore, for quasi-stationary motion no distinction in the

---

* Lorentz, *Theory of Electrons*, pp. 205 ff. Also §§ 27 and 37.

† *Mathematische Einführung in die Elektronentheorie*, Leipzig, 1904, pp. 57, 58. See also Lorentz, *op. cit.*, pp. 205 ff.

case of non-expanding electrons ($\delta+ua=0$) between longitudinal and transverse mass, as is the case for Lorentz electrons ($\delta=0$); this is another feature of special interest attaching to the electrons in which $\delta+ua$ vanishes.

In the general case from (24) we see that the components of the external forces are still proportional to the component accelerations; but the coefficients depend on the acceleration and are different along the direction of motion and perpendicular to it.

7. *The Equation of Energy and the Rate of Work done by the Electromagnetic Forces.*

It seems worth while to notice that, in the treatment of the theory of electrons as given by Lorentz, Abraham, and others, where the circuital relations form the basis of the theory, together with an expression for the electrodynamic force on a unit charge, the energy and the radiation are subsidiary quantities. In fact, the rate at which work is done by the electrodynamical forces is expressed by means of the circuital relations as the sum of a certain surface integral and the rate of decrease of a certain volume integral. The former integral is interpreted as rate of radiation of energy, and the latter as electromagnetic energy.

This point is clearly brought out in the above transformation theory, where it is found to be impossible to obtain invariants which separate the electromagnetic energy from the radiation.

The equation of energy in the usual form, viz.,

$$\iiint \rho\,(w, F)\,dx\,dy\,dz = -\frac{d}{dt}\iiint \tfrac{1}{2}\,(E^2+H^2)\,dx\,dy\,dz - \iint (lS_x+mS_y+nS_z)\,d\sigma,$$

where S_x, S_y, S_z are the components of the Poynting vector $[E, H]$, is a direct consequence of the circuital relations and the definition of F. We can add on to the Poynting vector dU/dt, where U is any vector, and subtract $\frac{d}{dt}$ div U from the electromagnetic energy $\frac{1}{2}(E^2+H^2)$ without altering the whole equation. In particular, if $U=[A, H]$, where curl $A \equiv H$, we should find Macdonald's expression for the flow of energy based on Maxwell's first expression for the electromagnetic energy.*

* Macdonald, *Electric Waves*, Chapter viii.

Similar considerations apply to the total force on a system of electrons expressed in terms of the Maxwell stress system and the rate of decrease of a volume integral, this latter being interpreted as a momentum.

When the circuital relations are satisfied in the $x_1, y_1 z_1 t_1$ space the corresponding relations in the x, y, z, t space are also satisfied, so that the energy equation in this latter space is also true. Since for a system at rest in the former space, as in the case here contemplated, the energy equation is certainly true, it follows that for the moving system in the x, y, z, t space the equation is also satisfied. Thus there is no need to assume the existence of any energy other than electromagnetic, as done by Abraham and Lorentz. This general conclusion could be verified in any special case by a calculation of the values of the electromagnetic energy, rate of radiation, and rate of work done by the electromagnetic forces, though, except in special cases, it would be a long piece of work to calculate the two former quantities. Here we shall restrict ourselves to the calculation of the rate of work done by the electromagnetic forces, which we called $\frac{dA}{dt}$ and which is given by the formulæ of § 3, deduced from the theory of integral invariants.

We have

$$\frac{dA}{dt} = \iiint \rho\, dx\, dy\, dz\, (w, F)$$

$$= -\iiint \rho_1 dx_1 dy_1 dz_1 \left(\frac{\partial t}{\partial t_1}\right)^{-1} \left[E_{1x}\frac{\partial x_1}{\partial t} + E_{1y}\frac{\partial y_1}{\partial t} + E_{1z}\frac{\partial z_1}{\partial t}\right]$$

$$= \iiint \frac{\rho_1^2}{3}\, \frac{dx_1 dy_1 dz_1 D_1 \left(1 - \frac{b^3}{r_1^3}\right) \lambda^2 \left(x_1\frac{\partial t}{\partial x_1} + y_1\frac{\partial t}{\partial y_1} + z_1\frac{\partial t}{\partial z_1}\right)}{k\epsilon\sqrt{\{1 - 4(\delta + u\alpha)\, u x_1 + 4(\delta + u\alpha)^2 r_1^2\}}}.$$

Now $$\lambda^2 \left(x_1\frac{\partial t}{\partial x_1} + y_1\frac{\partial t}{\partial y_1} + z_1\frac{\partial t}{\partial z_1}\right) \quad \text{for} \quad t = 0$$

is equal to $$\frac{\epsilon D_1}{k}\left[u x_1 - 2(\delta + u\alpha) r_1^2\right],$$

since $$k\lambda = D_1,$$

so that

$$\frac{dA}{dt} = \iiint \frac{\rho_1^2}{3}\, \frac{dx_1 dy_1 dz_1 D_1^2 \left(1 - \frac{b^3}{r_1^3}\right) \left[u x_1 - 2(\delta + u\alpha) r_1^2\right]}{k^2\sqrt{\{1 - 4(\delta + u\alpha)\, u x_1 + 4(\delta + u\alpha)^2 r_1^2\}}},$$

where, again, t_1 is to be expressed in terms of x_1, y_1, z_1 by (18) before the integration is performed.

For non-expanding electrons ($\delta + ua = 0$) the integration is quite simple, giving in terms of a, β, γ

$$\frac{dA}{dt} = -\frac{e_1^2}{\pi k^2} ua(1-u^2)\left[\frac{1}{5a} + \frac{a}{7}\,\frac{(3-u^2)}{3}\,\{\beta^2+\gamma^2+a^2(1-u^2)\}\right] \quad (25)$$

for a volume distribution of charge.

In the general case we may expand in powers of a and δ, so that

$$\frac{dA}{dt} = a\,\frac{dA_1}{dt} + \delta\,\frac{dA'}{dt} + \tfrac{1}{2}\left(a^2\,\frac{dA_2}{dt} + 2a\delta\,\frac{d\bar{A}}{dt} + \delta^2\,\frac{dA''}{dt}\right) + \ldots,$$

and we find

$$\left.\begin{aligned}
\frac{dA_1}{dt} &= -\frac{ue_1^2}{2\pi k^2}\left[\frac{(5-u^2)}{5a} + \frac{4}{3}\,(\beta^2+\gamma^2)(3-u^2)^2\frac{a}{7}\right.\\
&\qquad\left. + \frac{1}{3}(\beta^2+\gamma^2)^2(1-u^2)^2(3-u^2)^2\frac{a^3}{9}\right]\\
\frac{dA'}{dt} &= -\frac{e_1^2}{2\pi k^2}\left[\frac{(3+u^2)}{5a} + \frac{2}{3}\,(\beta^2+\gamma^2)(3-u^2)(5-u^2)\,\frac{a}{7}\right.\\
&\qquad\left. + \frac{1}{3}(\beta^2+\gamma^2)^2(3-u^2)^2(1-u^2)\,\frac{a^3}{9}\right]
\end{aligned}\right\}, \quad (26)$$

while $$\frac{dA_2}{dt} = \frac{d\bar{A}}{dt} = \frac{dA''}{dt} = 0.$$

If we make $\delta = -ua$ in the expansion for $\dfrac{dA}{dt}$, we may verify the result of (25) up to the first order in a.

It will be noticed that in all the values found for the forces X, Y, Z and the rate of work done $\dfrac{dA}{dt}$ there is no term independent of the radius a of the electron.

There is thus no term in the expression for the force corresponding to the radiation pressure as obtained by Abraham, nor a term in the expression for the rate of work done corresponding to energy lost by radiation, since both these terms are independent of the dimensions of the electron and of the distribution of charge.*

These results, of course, apply only to the particular case of accelerated motion considered, but suggest that there are fundamental differences between accelerated motion of a deformable electron, *i.e.*, accelerated motion according to the Theory of Relativity, and accelerated motion treated from other standpoints.

* *Cf.* Schott, *Electromagnetic Radiation*, Appendix D, §§ 225, 231, 235.

ON DERIVATES AND THEIR PRIMITIVE FUNCTIONS

By W. H. YOUNG, Sc.D., F.R.S.

[Received November 28th, 1912.—Read December 12th, 1912.]

1. The most general of Lebesgue's already classical results in connexion with the fundamental theorem of the Integral Calculus * is that which states that, if one of the derivates of a continuous function is finite and summable, its Lebesgue integral is the function itself. It was pointed out by Beppo Levi † that, in the case of an analogous result, it followed by simple Cantor induction that that result is still true if the value $+\infty$, or $-\infty$, or both, are assumed by the derivate at a reducible set of points only. By employing the notions of positive and negative variation rather more in the form originally given to them by Jordan, I showed ‡ that the theorem, equally in the Lebesgue or in the extended form, could be split up into two components, relating respectively to functions upper semi-continuous on the right and lower semi-continuous on the left, and functions upper semi-continuous on the left and lower semi-continuous on the right respectively. In a slightly more recent paper § I showed by an entirely different method that another result of Lebesgue's, almost as general as the one just referred to, namely, that a function of bounded variation is an integral if one of its derivates is finite, could be generalised in this sense that the assuming of the value infinity with determinate sign by the derivate was permissible at any countable set of points or, even more generally, at any set of points which contained no perfect sub-set.

Since then a careful examination of the proof given by myself of Lebesgue's former theorem has led me to the conclusion that that result,

---

* The first statement of this theorem is on p. 123 of Lebesgue's *Leçons sur l'intégration*, Paris, 1904. The first wholly satisfactory proof is to be found in Lebesgue's paper, "Sur les fonctions dérivées," 1906, *Rend. dei Lincei*, Ser. 2, Vol. xv, p. 5. The only correct proof in English, as far as I am aware, is that in my paper "On Functions of Bounded Variation," 1910, *Quarterly Journal of Mathematics*, Vol. XLII, p. 78.

† "Ancora alcune osservazioni sulle funzioni derivate," 1906, *Rend. dei Lincei*, Ser. 2, Vol. xv, pp. 359 *seq.*

‡ *Loc. cit.*, *supra*, first footnote.

§ "Note on the Fundamental Theorem of Integration," 1910, *Proc. Camb. Phil. Soc.*, Vol. XVI, pp. 35–38.

like the one just referred to, is capable of a similar extension, that we are, in fact, in the extended enunciation, at liberty to substitute for the word "reducible" the word "countable."

Combining this result with those corresponding to the two components for the two classes of non-continuous functions above described, I find that we get the following general statements, which include all others so far made :—

If a lower derivate of a function which is upper semi-continuous on the right and lower semi-continuous on the left assumes the value $+\infty$ *at a countable set of points only, and is summable over the set of points at which it is positive, the function is a lower semi-integral and is accordingly expressible as the sum of an integral and a monotone nowhere increasing function of* x. *Moreover, its positive variation is the integral of any one of its derivates over the set of points where the lower derivate in question is positive.*

If an upper derivate of a function which is lower semi-continuous on the right and upper semi-continuous on the left assumes the value $-\infty$ *at a countable set of points only, and is summable over the set of points where it is negative, the function is an upper semi-integral and is accordingly expressible as the sum of an integral and a nowhere decreasing function. Moreover, its negative variation is the integral of any one of its derivates over the set of points where the upper derivate in question is negative.*

As I think it would be difficult to exaggerate the importance in the theory of derivates of Lebesgue's original theorem, I propose to show in the present communication how these generalisations may be obtained. In the method of proof I have adopted I have kept in view the principle which actuated my wife and myself in a recent joint paper on the Existence of a Differential Coefficient, published in the *Proceedings* of this Society.* In that paper the fact that we had not succeeded to our own satisfaction in proving Lebesgue's general theorem without the use of Cantor's numbers was mentioned.

From one point of view we might, indeed, have claimed that we had been successful in doing so ; for it is only in the assumption of the truth of his lemma on the existence † of a chain of intervals that Lebesgue's

* Ser. 2, Vol. 9, pp. 325–335.

† Lebesgue also used Cantor's numbers to justify the use of such a chain in calculating the variation, "Encore une observation sur les fonctions dérivées," 1907, *Rend. dei Lincei*, Ser. 5, Vol. xv, p. 96. Beppo Levi has given an argument independent of Cantor's numbers, which serves the same purpose, "Ancora alcune osservazioni sulle funzioni derivate," 1906, *Rend. dei Lincei*, Ser. 2, Vol. xv, p. 361.

proof involves Cantor's numbers implicitly. Now one of the things we did in the paper quoted was to show how that lemma could be proved without Cantor's numbers. Our object was, however, to avoid not only the use of the numbers themselves, but also the use of what might be objected to as a transfinite construction. A proof is even more convincing if we are able to employ finite sets of intervals only. The contention, indeed, that it is only with a finite number of intervals that we ought to operate is not wholly ungrounded.

In the present paper, therefore, I have been at pains to avoid all trace of the transfinite construction.

In my process of analysis of Lebesgue's theorem into its component parts I have been able to take a further step. In the enunciations above given, the hypothesis of the summability of the derivate considered over either the whole interval or a set of points in that interval has to be made. Now I had succeeded in proving* without transfinite numbers the theorem that the integral of the derivates of an integral is the integral itself, another of the important results due to Lebesgue. Subtracting, so to speak, this theorem from the two above stated as to be proved, we find that the residual parts take the following form, from which all reference to the concept of summability has disappeared :

If a function is upper semi-continuous on the right and lower semi-continuous on the left, and one of its lower derivates is negative or zero except at a set of points of zero content, then either

(1) *that derivate assumes the value* $+\infty$ *at a more than countable set of points,*

Or (2) *the function is a monotone nowhere increasing function, so that all its derivates are everywhere negative or zero.*

Again,

If a function is lower semi-continuous on the right and upper semi-continuous on the left, and one of its upper derivates is positive or zero except at a set of points of zero content, then either

(1) *that derivate assumes the value* $-\infty$ *at a more than countable set of points, or*

---

* "Functions of Bounded Variation," *loc. cit.*, §§ 23 and 24, pp. 81 and 82.

(2) *the function is a monotone nowhere decreasing function, so that all its derivates are everywhere positive or zero.*

From these results those mentioned above follow.

2. I shall first prove the following extension of the lemma in the joint paper already cited.

LEMMA.—*If with each point of a closed interval* (a, b) *as left-hand end-point we are given an interval or several intervals, and* $e_1 > e_2 > \ldots$ *be any monotone descending sequence of constants with zero as limit, we can find a finite number of the given intervals, nowhere overlapping, such that the complementary intervals have lengths which are respectively less than different terms of the e-sequence.**

By the lemma proved in a preceding number of these *Proceedings* † we saw that we could choose out a finite number, say m of the given intervals, numbered from left to right, $d_1, d_2, \ldots, d_m$, nowhere overlapping, and such that the sum of the lengths of the complementary intervals or gaps is less than a quantity as small as we please, which we may take to be e_1. This may be said to constitute the first stage in the process of finding the required intervals.

Now the intervals were so chosen that there is a definite integer i, different for each gap, so that the chosen interval on the right of the gap in question is d_i, and a definite integer r_i, *not necessarily different for each gap*, such that

$$d_i \geqslant e_{r_i}, \tag{1}$$

while, if X is any point belonging to the gap, whether as internal or end-point, the length of each of the given intervals corresponding to it is less than e_{r_i-1}, say

$$H < e_{r_i-1}. \tag{2}$$

To render this quite clear it would be well, perhaps, to recall the mode in which the intervals $d_1, d_2, \ldots, d_m$ were chosen. We begin by elongating all the given intervals on the left by one n-th of their length. Starting

* More generally we have by similar reasoning the following lemma:—

LEMMA.—*If with each point of a closed set* S *as left-hand end-point, we are given an interval, or several intervals, and* $e_1 > e_2 > \ldots$ *be any monotone descending sequence of constants with zero as limit, we can find a finite number of the given intervals, nowhere overlapping, such that in the complementary intervals the points of* S *can be enclosed in a finite number of intervals whose lengths are respectively less than different terms of the e-sequence.*

† *Loc. cit., supra*, § 1, footnote §.

with the point a, we then determine the first integer r_1 such that there is at least one of the given intervals of original length $\geqslant e_{r_1}$, which in its elongated form reaches at least to a. The integer r_1 is accordingly such that there is no such interval of length $\geqslant e_{r_1-1}$. That such an integer must exist follows from the fact that the given intervals include intervals with a as left-hand end-point. We then selected one of these intervals of length $\geqslant e_{r_1}$, and called it d_1. If a is not the left-hand end-point of d_1, there is a gap between a and d_1 of length $\leqslant d_1/n$, and therefore $\leqslant e_{r_1-1}/n$. If a is the left-hand end-point of d_1 there is no gap at a. In any case we then proceed on the right of d_1 as before. The gaps left are all precisely similar in character. Let us suppose, therefore, for definiteness, that there was a gap formed at a, and that X is a point internal to the gap, while H is the length of any one of the given intervals corresponding to X. Then, since the distance of X from a is less than the width of the gap, and therefore $< e_{r_1-1}/n$, the interval would, if its length were $\geqslant e_{r_1-1}$, be an interval of length $\geqslant e_{r_1-1}$ reaching, when elongated, at least as far as a, contrary to our choice of the integer r_1. Thus, the inequality (2) holds for every point X internal to the gap. The argument applies as it stands if X is the right-hand end-point of the gap. That it is true when X is the left-hand end-point a follows from the choice of r_1; for in the contrary case such an interval would be an interval $\geqslant e_{r_1-1}$ and would reach, of course, to a; thus, r_1 would not be the lowest integer of the type considered.

Thus, the inequality (2) is true, as stated. Moreover, the integer r_i in that inequality has been determined in accordance with the inequality (1).

We have thus shown that the gaps and intervals constructed at what we may call the first stage possess a certain property, as defined by the inequalities (1) and (2). At the end of this first stage the extreme gap on the right is of length less than e_1. It may also happen that the remaining gaps, taken in order from right to left, are respectively less than $e_2, e_3, \ldots, e_{m+1}$.

If so, our process is complete; if not, we stop at the first gap, say G_s, in order from right to left, whose length is too great. The gaps to the right of this we call *permanent gaps*, and the others *transitory gaps at the first stage.*

The gap G_s which has arrested us we treat in precisely the same way as we treated the whole interval, taking the quantity e_s instead of e_1 and replacing the whole e-sequence by that part of it which begins with e_s.

At the end of the second stage we shall then have at least one permanent gap added to those already obtained, and may have more.

In the third stage, if it be needed, we have to deal with gaps not made permanent lying in G_s, if there be any, and the transitory gaps at the first

stage which remain transitory at the second stage. As we proceed steadily with our permanent gaps from right to left, the gaps other than G_s which were transitory at the first stage will necessarily remain transitory at the second stage, unless G_s has been replaced by permanent gaps only.

As our process proceeds we advance steadily from right to left, always adding at least one permanent gap to those already obtained and leaving no transitory gaps behind us. We must, then, in this way either obtain a finite number of permanent gaps which answer to our requirements or we must approach a limiting point X, say, of the permanent gaps, our approach being, of course, from the right.

This point X cannot be internal to any of the intervals chosen at any any one of the stages as we approach X; for if so that chosen interval would contain one of the permanent gaps of which X is the limit, and this is a contradiction in terms.

At each stage of our approach to X, therefore, X must belong to a transitory gap, either as internal or end-point. Now this is easily seen to be inconsistent with the inequalities (1) and (2).

In fact, the quantity H in inequality (2) may be supposed to be the length of an interval $(X, X+H)$ of the given set. This inequality shows that H can only be less than an infinite number of the quantities e_{r_i-1} if an infinite number of them coincide. In other words, there is an infinite succession of integers i for which r_i is the same. But to each such integer i we have, as the inequality (1) shows, a distinct interval d_i of length not less than the value assumed by an infinite number of the e_{r_i}'s. Thus, such d's must overlap. But it follows, from our mode of construction, that they do not overlap; thus, by a *reductio ad absurdum*, such a limiting point X cannot exist, and, therefore, at one stage all the gaps have been replaced by a finite number of permanent gaps answering to our requirements. Thus our lemma has been proved to be true.

3. In the preceding proof I have avoided the use of Cantor's numbers. The method of transfinite numbers is a powerful instrument of research. I propose to illustrate this fact by now deducing the lemma of the preceding article from the existence of a Cantor chain of intervals postulated by Lebesgue.

We suppose, then, a chain of intervals to stretch in the usual manner from the left-hand extremity of the interval of integration to the right-hand end, and we wish to show that we may omit all but a finite number of these intervals in such a manner as to obtain gaps of the lengths required by the lemma.

Starting from the right-hand end b, we can clearly take a gap of length less than e_1, and having its left-hand extremity coinciding with one of the right-hand end-points of the intervals of the chain. Since the semi-external points of the chain form a closed set, there is one of them which is nearest to the gap just formed of those on the left of the gap; this point is, moreover, not the left-hand extremity of that gap. We treat it in precisely the same way as we treated b, taking e_2 instead of e_1. Between the two gaps there is only a finite number of the intervals, because between them there is no limiting point of the original chain. Moreover, if we continue the process thus started, it is bound, after a finite number of steps, to come to an end. For if not the gaps so formed would necessarily have a limiting point to the left, which they would approach indefinitely near. This point would be a limiting point of the left-hand end-points of the gaps; that is, of end-points of intervals of the chain lying on its right. But the chain has only limiting points which are approached from the left, so that this is impossible. Thus, after a finite number of steps, we obtain the system of gaps and intervals required.*

4. We now pass to the following theorem on the nature of a derivate.

THEOREM.—*If $F(x)$ is a function which is upper semi-continuous on the right and lower semi-continuous on the left and has a derivate which is $\leqslant 0$ at a set whose complementary set has zero content, then either*

(i) *the derivate has the value $+\infty$ at a more than countable set of points, or*

(ii) *its values at all the points of content zero are also $\leqslant 0$, so that the function is a monotone nowhere increasing function, all of whose derivates are $\leqslant 0$.*

Suppose for definiteness that it is the lower right-hand derivate $F_+(x)$ which is known to be $\leqslant 0$, except at a set G of zero content, about which we only know that at all but a countable set of points $P_1, P_2, \ldots$, of G $F_+(x)$ is finite. At the points P_n the value is quite unknown, and may *a priori be* $+\infty$.

Let us attach to each point P_n an interval $(x, x+h)$ of length less than d,†

* The argument is a modification of that of Heine, used by Beppo Levi in dealing with the variation of a function, *loc. cit.*

† We may of course also secure that the *sum* of all these intervals is less than d. This is, however, not required in the present argument.

with P_n as left-hand end-point, and so small that

$$F(x+h)-F(x) \leqslant 2^{-n}e.$$

This is possible since $F(x)$ is upper semi-continuous on the right. Then the sum of the positive increments of $F(x)$ over any non-overlapping set of these intervals is less than e.

Next let us take any small positive quantity k, and let x be a point at which

$$(i-1)k < F_+(x) \leqslant ik,$$

where i is any positive integer.

Since these points form a sub-set of the exceptional set of zero content, we can enclose them all in a set of intervals of content less than $2^{-i}e/(i+1)k$. To each such point x let us attach an interval $(x, x+h)$ of length less than d, with x as left-hand end-point, contained in an interval of the enclosing set just constructed, and such that

$$\frac{F(x+h)-F(x)}{h} < (i+1)k.$$

Then the sum of the positive increments of $F(x)$ over any non-overlapping set of these intervals is less than $(i+1)k$ times the sum of their lengths; that is, it is less than $2^{-i}e$, since these intervals all lie inside the enclosing intervals of content less than $2^{-i}e/(i+1)k$.

Let us do this for every positive integer i.

Finally, let x be a point at which $F_+(x)$ is not positive, and let us attach to every such point an interval $(x, x+h)$ of length less than d and such that

$$\frac{F(x+h)-F(x)}{h} < k.$$

Then the sum of the positive increments of $F(x)$ over any non-overlapping set of these last intervals is less than $k(b-a)$, if (a, b) is the whole segment in which we are working.

We have now attached to each point of (a, b) an interval on the right of it of length less than d. Hence, by the lemma, we can choose out a finite number of these intervals, nowhere overlapping, and such that the complementary intervals are respectively less than different terms of any convenient monotone descending sequence $q_1, q_2, \ldots$, with zero as limit.

Let us choose the q-sequence so that $q_1 < d$, and in any interval $(y, y+k)$ of length less than q_n, $(n = 1, 2, \ldots)$,

$$F(y+k)-F(y) < 2^{-n}e,$$

which is possible, since $F(x)$ is upper semi-continuous on the right. Then

the sum of the positive increments of $F(x)$ over these complementary intervals is less than e.

Also, by what has already been pointed out, the sum of the positive increments of $F(x)$ over the chosen intervals is less than

$$e+\sum_{i=1}2^{-i}e+k(b-a) < 2e+k(b-a).$$

Now, since $F(x)$ is upper semi-continuous on the right and lower semi-continuous on the left, we can choose d so small that the positive increment of $F(x)$ over the finite set of intervals consisting of the chosen intervals and the complementary intervals has the positive variation P, finite or infinite, of $F(x)$ for limit as $d \to 0$.* Hence $P \leqslant 2e+k(b-a)$.

Thus, since k and e are as small as we please, P must be zero. This proves that $F(x)$ is a monotone non-increasing function of x, and hence that, under the circumstances assumed, (ii) is true. The alternative circumstances are given in (i). Thus the theorem is true.

5. The corresponding theorem with regard to the derivate of a function which is lower semi-continuous on the right and upper semi-continuous on the left follows, of course, by a mere change of sign. It may be well also to point out, though this is scarcely less obvious, that the theorems are equally true if we replace the particular derivate referred to in the enunciation by any of the other derivates, upper, lower, or intermediate, provided it is always on the right hand or always on the left hand. In other words, if we regard the right-hand derivates, for example, as defining a many-valued function, defined for the whole interval, we may take any of the one-valued functions which may be formed by taking at each point of the interval, by any law whatever, some one of the values of the many-valued function in question, the statements of this and the preceding article are still true.

Similar remarks apply to the other theorems about to be proved. In fact, in all cases the case considered in our enunciations is the most unfavourable case; or, if we prefer it, instead of deducing the remaining cases from the unfavourable case as an immediate consequence, we may content ourselves with remarking that the proof is, word for word, the same in such remaining cases.

6. If we prefer to use instead of the lemma of § 2 the theorems of Lebesgue, as extended by myself,† which state (α) that we can find a chain of

* "Functions of Bounded Variation," *Quarterly Journal of Mathematics*, Vol. XLII, § 5, p. 59.
† *Ibid.*, § 19 and § 18, pp. 72–74.

the given intervals placed so that each interval has for left-hand end-point a right-hand end-point of another interval or a limiting point of intervals lying on its left, every point of (a, b) being either an internal point or a left-hand end-point of the intervals of the chain, and (β) that the positive variation may be calculated by means of such chains, instead of only by means of divisions of (a, b) into a finite number of parts, we can give an alternative ending to the proof of § 4, beginning after the sentence: "We have now attached to each point of (a, b) an interval on the right of it of length less than d." We proceed as follows.

By Lebesgue's lemma we can choose out a chain of these intervals stretching from a to b. The sum of the positive increments of $F(x)$ over the intervals of the chain is, by what has been pointed out, less than

$$e+\sum_{i=1} 2^{-i}e+k(b-a) < 2e+k(b-a).$$

Now, since $F(x)$ is upper semi-continuous on the right and lower semi-continuous on the left, the positive increment of $F(x)$ over the intervals of the chain has, as d decreases indefinitely the positive variation P of $F(x)$ as limit, since all the intervals of the chain are of length less than d. Hence

$$P \leqslant 2e+k(b-a),$$

and therefore since e and k are as small as we please,

$$P = 0.$$

This proves that $F(x)$ is a monotone non-increasing function of x, and that, under the circumstances assumed, (ii) is true. The alternative circumstances are given in (i). Thus this proves the theorem.

7. From the theorems given in §§ 4 and 5 the generalisations of Lebesgue's theorem stated in the introduction at once follow. For definiteness, take that which refers to a function which is upper semi-continuous on the right and lower semi-continuous on the left, and denote this function by $f(x)$. Let f_1 denote the integral of, say, the lower right-hand derivate over the set of points where that derivate is positive. Then f_1 possesses a differential coefficient except at a set of content zero. Moreover, its derivates are everywhere $\geqslant 0$. Hence the lower right-hand derivate of $f-f_1$ is nowhere greater than the corresponding derivate of f. Hence it assumes the value $+\infty$ at a countable set of points at most. Again, since f_1 is an integral, its differential coefficient agrees with the integrand, except at a set of content zero; it therefore agrees with the

lower right-hand derivate of f at points where the latter is $\geqslant 0$, with the exception at most of a set of content zero. Hence the lower right-hand derivate of $f-f_1$ is $\leqslant 0$, except at a set of content zero. Moreover, it assumes the value $+\infty$ at a countable set of points at most. Hence, by the theorem just proved, $f-f_1$ is a monotone nowhere increasing function, and therefore, since f_1 is an integral, f has the form stated in the enunciation. Finally, the positive variation of f is clearly the positive variation of f_1; that is, f_1 itself.

Thus our theorem is proved. Similarly the corresponding theorem is proved.

PROOFS OF CERTAIN GENERAL THEOREMS RELATING TO ORDERS OF COINCIDENCE*

By J. C. FIELDS.

[Received December 3rd, 1912.—Read January 9th, 1913.]

IN a paper† read before Section I of the Fifth International Congress of Mathematicians, the writer gave a proof of the Complementary Theorem which dispensed with the preliminary determination of certain numbers elsewhere utilized in proving the theorem. The method of proof employed, however, furnishes at the same time the means of determining these numbers, and the principal object of the present paper is to effect their determination. The expressions obtained for the numbers [see formula (24)] are thereafter used to prove *the independence of the conditions of adjointness* relative to the general algebraic equation, the method of proof being in principle the same as that already given in the writer's book‡ on the algebraic functions in the case where the fundamental algebraic equation is integral. That is to say, the total number of the independent conditions imposed on a rational function of sufficiently general form by adjointness relative to all values of the independent variable z, is equal to or less than the sum of the numbers of the conditions imposed on the function by adjointness relative to the individual values of z. It cannot, however, be less than the sum in question, for this would entail a result at variance with the complementary theorem. The conditions of adjointness must therefore be independent. The independence of the conditions of adjointness is, in fact, a direct corollary of the complementary theorem, so soon as we have a general expression for

* The substance of the present paper is in part that of a paper presented to Section A at the Dundee Meeting of the British Association for the Advancement of Science, September, 1912.

† "Direct Derivation of the Complementary Theorem from Elementary Properties of Rational Functions," *Proceedings of the Fifth International Congress of Mathematicians*, Cambridge, 1912.

‡ *Theory of the Algebraic Functions of a Complex Variable*, pp. 150–155, Mayer and Müller, Berlin, 1906.

the number of the conditions imposed on a sufficiently general rational function by adjointness relative to an arbitrary individual value of z.

Let

$$f(z, u) = u^n + f_{n-1}u^{n-1} + \ldots + f_0 = 0 \tag{1}$$

be any algebraic equation, reducible or irreducible, offering, however, no repeated factor. The orders of coincidence of the function $f'_u(z, u)$ with the branches of the r_∞ cycles corresponding to the value $z = \infty$, we indicate by the symbols $\mu_1^{(\infty)}, \ldots, \mu_{r_\infty}^{(\infty)}$. Suppose $(\tau)^{(\infty)}$ and $(\bar{\tau})^{\infty}$ to be partial bases whose orders of coincidence satisfy the conditions

$$\tau_s^{(\infty)} + \bar{\tau}_s^{(\infty)} = \mu_s^{(\infty)} + 1 + \frac{1}{\nu_s^{(\infty)}} \quad (s = 1, 2, \ldots, r_\infty). \tag{2}$$

The general rational function of (z, u) conditioned by the partial basis $(\tau)^{(\infty)}$ can be represented in the form

$$R\left(\frac{1}{z}, u\right) = \sum_{t=1}^{n} \sum_{r} \beta_{-r, n-t} z^{-r} u^{n-t}. \tag{3}$$

We have proved elsewhere* that the necessary and sufficient conditions in order that a rational function $\psi(z, u)$ may have, for the value $z = \infty$, orders of coincidence which are complementary adjoint to the order 2 to the orders of coincidence furnished by the partial basis $(\tau)^{(\infty)}$ are obtained on equating to 0 the principal residue relative to the value $z = \infty$ in the product

$$R\left(\frac{1}{z}, u\right)\psi(z, u). \tag{4}$$

In the representation of the function $R\left(\frac{1}{z}, u\right)$ given in (3), the earlier coefficients $\beta_{-r, n-t}$ only are conditioned by the partial basis $(\tau)^{(\infty)}$. We shall assume that the coefficients $\beta_{-r, n-t}$ are arbitrary for $r > j+1$. Consider a product of the form

$$\sum_{t=1}^{n} \sum_{r=-i}^{i+1} \beta_{-r, n-t} z^{-r} u^{n-t} \sum_{t=1}^{n} \sum_{r=i+1}^{-i} \alpha_{r-1, t-1} z^{r-1} u^{t-1}, \tag{5}$$

in which the first factor is conditioned by the partial basis $(\tau)^{(\infty)}$. We shall here take i not less than the degree in z of either of the general

* "On the Foundations of the Theory of Algebraic Functions of One Variable," § 2, *Phil. Trans. Roy. Soc.*, 1912.

rational functions conditioned by the partial bases $(\tau)^{(\infty)}$ and $(\bar{\tau})^{(\infty)}$ respectively, and at the same time we shall take it $\geqslant j+1$. Of the $2n(i+1)$ coefficients $\beta_{-r,\,n-t}$ in the first factor a certain number $2n(i+1)-\lambda$ are arbitrary, the remaining λ coefficients being linearly expressible in terms of these $2n(i+1)-\lambda$ arbitrary constants. Writing for brevity

$$d = 2n(i+1)-\lambda,$$

we shall represent the d arbitrary constants in question by the notation $\beta_1, \ldots, \beta_d$. The first factor of the product (5) will then have the form

$$\sum_{s=1}^{d} \beta_s \phi_s \left(\frac{1}{z},\ u\right), \tag{6}$$

where the d functions $\phi_s\left(\frac{1}{z},\ u\right)$ are linearly independent of one another. The coefficients $a_{r-1,\,t-1}$ in the second factor we shall first suppose to be arbitrary. Afterwards we shall impose on them certain conditions obtained on equating to 0 the principal residue relative to the value $z = \infty$ in the product (5).

Suppose $\phi\left(\frac{1}{z},\ u\right)$ to be a specific function included under the first factor of (5), and suppose furthermore that $u^{n-\tau}$ is the highest power of u which appears in the function, and that a term $\bar{\beta}_{-\rho,\,n-\tau} z^{-\rho} u^{n-\tau}$ actually presents itself in which $\bar{\beta}_{-\rho,\,n-\tau} \neq 0$.

In the product

$$\phi\left(\frac{1}{z},\ u\right) \sum_{t=1}^{n} \sum_{r=i+1}^{-i} a_{r-1,\,t-1}\, z^{r-1} u^{t-1}, \tag{7}$$

the principal residue relative to the value $z = \infty$ is an expression linear in the coefficients $a_{r-1,\,t-1}$ of the second factor, the multiplier of the coefficient $a_{\rho-1,\,\tau-1}$ evidently being the constant $\bar{\beta}_{-\rho,\,n-\tau}$ which is different from 0. The principal residue* in the product (7) is therefore not identically 0. In the d products

$$\phi_s\left(\frac{1}{z},\ u\right) \sum_{t=1}^{n} \sum_{r=i+1}^{-i} a_{r-1,\,t-1}\, z^{r-1} u^{t-1} \quad (s = 1, 2, \ldots, d), \tag{8}$$

then the principal residues are not identically 0. Furthermore, regarded as linear expressions in the coefficients $a_{r-1,\,t-1}$, these residues are linearly independent of one another. For, if the principal residues in the d pro-

* The abbreviation "principal residue" is employed to designate "principal residue relative to the value $z = \infty$."

ducts (8) were connected by a linear relation with the constant multipliers $\bar{\beta}_1, \ldots, \bar{\beta}_d$, we could construct a function

$$\phi\left(\frac{1}{z}, u\right) = \sum_{s=1}^{d} \bar{\beta}_s \phi_s\left(\frac{1}{z}, u\right),$$

such that the principal residue in the product (7) would be identically 0. This, however, we have seen to be impossible. The principal residues in the d products (8) are consequently linearly independent of one another. On equating to 0 the principal residue in the product (5), we therefore impose on the coefficients $\alpha_{r-1,\,t-1}$ in the second factor just d conditions, where d is the number of the coefficients in the first factor which are arbitrary. This then leaves λ of the coefficients $\alpha_{r-1,\,t-1}$ arbitrary.

Among the conditions imposed on the coefficients $\alpha_{r-1,\,t-1}$, in equating to 0 the principal residue in the product (5), are included the conditions obtained on equating to 0 the principal residues in the $n(i-j)$ products

$$z^{-\rho} u^{n-\tau} \sum_{t=1}^{n} \sum_{r=i+1}^{-i} \alpha_{r-1,\,t-1}\, z^{r-1} u^{t-1} \tag{9}$$

$$(\rho = j+2, \ldots, i+1\,;\ \tau = 1, 2, \ldots, n),$$

since for the values of ρ and τ here in question the coefficients $\beta_{-\rho,\,n-\tau}$ are arbitrary. On equating to 0 the principal residues in the $i-j$ products (9) corresponding to the value $\tau = n$, we obtain the conditions

$$\alpha_{\rho-1,\,n-1} = 0 \quad (\rho = j+2, \ldots, i+1). \tag{10}$$

On equating to 0 the principal residue in one of the products (9), we obtain $\alpha_{\rho-1,\,\tau-1}$ in terms of coefficients $\alpha_{r-1,\,t-1}$, in which the second suffix is $> \tau-1$. Equating to 0 the principal residues in all the products (9), we then obtain for each coefficient $\alpha_{\rho-1,\,\tau-1}$ in which the first suffix is $\geqslant j+1$, a linear expression in terms of coefficients $\alpha_{r-1,\,t-1}$ with increased second suffix. Each of these latter coefficients, in which the first suffix is $\geqslant j+1$, we can in like manner express in terms of coefficients with increased second suffix. By repetition of the process we see that we ultimately express $\alpha_{\rho-1,\,\tau-1}$ in terms of coefficients $\alpha_{r-1,\,t-1}$ in which the first suffix is $\leqslant j$, and of coefficients in which the second suffix has the value $n-1$. The $i-j$ coefficients with $n-1$ as second suffix have, however, been equated to 0 in (10). It follows that on equating to 0 the principal residues in the $n(i-j)$ products (9), we obtain for each of the $n(i-j)$ coefficients

$$\alpha_{\rho-1,\,\tau-1} \quad (\rho = j+2, \ldots, i+1\,;\ \tau = 1, 2, \ldots, n), \tag{11}$$

a linear expression in terms of the coefficients $\alpha_{r-1,\,t-1}$ in which $r \leqslant j+1$.

This shews the independence of the conditions imposed on the coefficients $a_{r-1,\,t-1}$, in which the first suffix is $\geqslant j+1$, when we equate to 0 the principal residues in the $n(i-j)$ products (9).

On equating to 0 the principal residues in the products (9), and deriving therefrom the expression for a coefficient $a_{\rho-1,\,\tau-1}$ in terms of the coefficients $a_{r-1,\,t-1}$ in which $r \leqslant j+1$, it may be that we simply obtain 0 for the value of the coefficient in question. This has been the case for the coefficients with the second suffix $n-1$ in (10). When we take i and ρ sufficiently large it can furthermore be proved that the expression obtained for $a_{\rho-1,\,\tau-1}$ must be 0. To prove this we note first that the reduced forms of the powers u^n, u^{n+1}, ..., u^{2n-2}, given by the equation (1), are of limited degree in z. We shall assume then that none of these reduced forms is of degree in z greater than a given positive integer m. Going back now to the study of the products (9) and their principal residues, let us first consider the principal residue relative to the value $z = \infty$ in the reduced form of the element

$$a_{r-1,\,t-1}\, z^{r-\rho-1}\, u^{n-\tau+t-1}.$$

Here the degree in z of the reduced form is $\leqslant m+r-\rho-1$, and this is <-1, when $r<\rho-m$. This element then contributes nothing to the principal residue in a product of the form (9) if $r<\rho-m$. The principal residue in the product (9) is therefore the sum of $a_{\rho-1,\,\tau-1}$ and a linear expression in coefficients $a_{r-1,\,t-1}$ in which $r \geqslant \rho-m$. On equating this principal residue to 0, we therefore express $a_{\rho-1,\,\tau-1}$ linearly in terms of coefficients $a_{r-1,\,t-1}$ in which $r \geqslant \rho-m$, $t>\tau$.

Equating to 0 the principal residue in each of the $n(i-j)$ products (9), we may then write

$$a_{\rho-1,\,\tau-1} = A_{\rho-1,\,\tau-1} \quad (\rho = j+2, \ldots, i+1;\ \tau = 1, 2, \ldots, n), \tag{12}$$

where, by $A_{\rho-1,\,\tau-1}$, we designate an expression linear in coefficients $a_{\rho'-1,\,\tau'-1}$ whose first suffixes are $\geqslant \rho-m-1$, and whose second suffixes are greater than $\tau-1$. If $\rho-m \geqslant j+2$, each of these coefficients $a_{\rho'-1,\,\tau'-1}$ can in turn be represented linearly in terms of coefficients $a_{\rho''-1,\,\tau''-1}$ in which the first suffixes are $\geqslant \rho-2m-1$, while the second suffixes are $>\tau$. In this case then we can express the coefficient $a_{\rho-1,\,\tau-1}$ linearly in terms of coefficients $a_{r-1,\,t-1}$ in which the first suffixes are $\geqslant \rho-2m-1$, while the second suffixes are $>\tau$. If furthermore we have $\rho-(n-\tau)\,m \geqslant j+2$, we can evidently, by successive applications of the above process, express $a_{\rho-1,\,\tau-1}$ linearly in terms of coefficients $a_{r-1,\,t-1}$ in which the first suffixes are $\geqslant \rho-(n-\tau)\,m-1 \geqslant j+1$, while the second suffixes are $\geqslant \tau-1+(n-\tau)$ and must therefore be $=n-1$.

From (10), however, all such coefficients $\alpha_{r-1,\,t-1}$ have the value 0. On taking i large enough and equating to 0 the principal residue in each of the $n(i-j)$ products (9), we then have, among other conditions,

$$\alpha_{\rho-1,\,\tau-1} = 0 \quad \text{for} \quad \rho-(n-\tau)\,m \geqslant j+2.$$

For $\rho \geqslant (n-1)\,m+j+2$, therefore, we shall certainly have $\alpha_{\rho-1,\,\tau-1} = 0$. Writing, for brevity, $j' = (n-1)\,m+j$, and equating to 0 the principal residues in the $n(i-j)$ products (9), we have among other conditions

$$\alpha_{\rho-1,\,\tau-1} = 0 \quad (\rho = j'+2,\ \ldots,\ i+1\,;\ \tau = 1,\ 2,\ \ldots,\ n). \tag{13}$$

On taking i sufficiently large and equating to 0 the principal residue in the product (5), independently of the values of the arbitrary coefficients $\beta_{-r,\,n-t}$ involved in the first factor, we impose on the coefficients $\alpha_{r-1,\,t-1}$ of the second factor, among other conditions, the conditions (13). The general function represented by the second factor, conditioned by equating to 0 the principal residue in the product (5), is then evidently the same function as that represented by the second factor in the product

$$\sum_{t=1}^{n}\ \sum_{r=-i}^{i+1} \beta_{-r,\,n-t}\, z^{-r} u^{n-t} \sum_{t=1}^{n}\ \sum_{r=j'+1}^{-i} \alpha_{r-1,\,t-1}\, z^{r-1} u^{t-1}, \tag{14}$$

and conditioned by equating to 0 the principal residue in this product. Now, if i has been taken sufficiently large, it is evident that for $\rho > i+1$, the principal residue in the product

$$z^{-\rho} u^{n-\tau} \sum_{t=1}^{n}\ \sum_{r=j'+1}^{-i} \alpha_{r-1,\,t-1}\, z^{r-1} u^{t-1}$$

is 0, whatever the values of the coefficients $\alpha_{r-1,\,t-1}$ may be. It follows that, on equating to 0 the principal residue in the product (14), we condition the second factor in precisely the same manner as when we equate to 0 the principal residue in the product

$$R\left(\frac{1}{z},\ u\right) \sum_{t=1}^{n}\ \sum_{r=j'+1}^{-i} \alpha_{r-1,\,t-1}\, z^{r-1} u^{t-1}, \tag{15}$$

where $R\left(\frac{1}{z},\ u\right)$ is the general rational function of $(z,\ u)$ conditioned by the partial basis $(\tau)^{(\infty)}$. The necessary and sufficient conditions in order that the second factor in (15) may have orders of coincidence, which are complementary adjoint to the order 2 to the orders of coincidence furnished by the partial basis $(\tau)^{(\infty)}$, are obtained on equating to 0 the principal residue in (15). These necessary and sufficient conditions are therefore also obtained on equating to 0 the principal residue in the product (14). Now the conditions obtained on equating to 0 the principal

residue in the product (5) are evidently necessary in order that the second factor may have the orders of coincidence furnished by the partial basis $(\bar{\tau})^{(\infty)}$. These conditions, however, reduce the second factor in (5) to the form of the second factor in (14), and require the principal residue in (14) to be 0. The conditions obtained on equating to 0 the principal residue in the product (5) are therefore sufficient as well as necessary in order that the second factor may have orders of coincidence which are complementary adjoint to the order 2 to the orders of coincidence furnished by the partial basis $(\tau)^{(\infty)}$.

The conditions imposed on the coefficients of the second factor in the product

$$R\left(\frac{1}{z},\ u\right) \sum_{t=1}^{n} \sum_{r=i+1}^{-i} a_{r-1,\,t-1}\, z^{r-1} u^{t-1}, \tag{16}$$

on equating to 0 the principal residue in the product, are then the same as the conditions obtained on equating to 0 the principal residue in the product (5), if only the integer i is chosen sufficiently large. The number of these conditions, however, we have seen to be

$$d = 2n(i+1)-\lambda.$$

The number of the arbitrary constants involved in the general function of the form of the second factor in (5), and conditioned by the partial basis $(\bar{\tau})^{(\infty)}$ is therefore λ, while the number of the arbitrary constants involved in the first factor of the product, supposed to be conditioned by the partial basis $(\tau)^{(\infty)}$, is $2n(i+1)-\lambda$. Now, as a matter of fact, the first factor of the product (5), as well as the second, is included under the general form

$$\sum_{t=1}^{n} \sum_{r=i+1}^{-i} a_{r-1,\,t-1}\, z^{r-1} u^{t-1}. \tag{17}$$

The general function of this form when conditioned by the partial basis $(\tau)^{(\infty)}$ then involves $2n(i+1)-\lambda$ arbitrary constants, whereas it involves λ arbitrary constants when conditioned by the partial basis $(\bar{\tau})^{(\infty)}$. Otherwise expressed, we may say that two general functions of the form (17) conditioned, one by the partial basis $(\tau)^{(\infty)}$, and the other by the partial basis $(\bar{\tau})^{(\infty)}$, involve between them just $2n(i+1)$ arbitrary constants—it being understood of course that the integer i has been taken large enough.

Let us now select a partial basis $(t)^{(\infty)}$ which is lower than either of the partial bases $(\tau)^{(\infty)}$ or $(\bar{\tau})^{(\infty)}$. No order of coincidence furnished by the partial basis $(t)^{(\infty)}$ then exceeds the corresponding order of coincidence furnished by either of the partial bases $(\tau)^{(\infty)}$ or $(\bar{\tau})^{(\infty)}$. Imposing on the

general rational function of (z, u) conditioned by the partial basis $(t)^{(\infty)}$, the orders of coincidence required by the partial basis $(\tau)^{(\infty)}$, we subject the coefficients of the function to $\sum_{s=1}^{r_\infty} (\tau_s^{(\infty)} - \bar{\tau}_s^{(\infty)})\, \nu_s^{(\infty)}$ conditions. Imposing on the general rational function of (z, u) conditioned by the partial basis $(t)^{(\infty)}$, the orders of coincidence required by the partial basis $(\bar{\tau})^{(\infty)}$, we subject the coefficients of the function to $\sum_{s=1}^{r_\infty} (\bar{\tau}_s^{(\infty)} - t_s^{(\infty)})\, \nu_s^{(\infty)}$ conditions. The difference in the numbers of the conditions imposed by the partial bases $(\tau)^{(\infty)}$ and $(\bar{\tau})^{(\infty)}$ respectively, on the coefficients of the general rational function possessing the orders of coincidence required by the partial basis $(t)^{(\infty)}$, is then given by the expression

$$\sum_{s=1}^{r_\infty} (\tau_s^{(\infty)} - \bar{\tau}_s^{(\infty)})\, \nu_s^{(\infty)} . \tag{18}$$

Also the conditions here in question do not affect terms involving z to a power lower than z^{-i-1}. It evidently follows that the difference between the number λ of the arbitrary constants involved in the general function of the form (17) conditioned by the partial basis $(\bar{\tau})^{(\infty)}$, and the number $2n(i+1)-\lambda$ of the arbitrary constants involved in the general function of the form (17) conditioned by the partial basis $(\tau)^{(\infty)}$, must be equal to the difference between the number of the conditions imposed by the partial basis $(\tau)^{(\infty)}$ on the coefficients of the general rational function conditioned by the partial basis $(t)^{(\infty)}$ and the number of the conditions imposed on these coefficients by the partial basis $(\bar{\tau})^{(\infty)}$. We therefore have

$$2\lambda - 2n(i+1) = \sum_{s=1}^{r_\infty} (\tau_s^{(\infty)} - \bar{\tau}_s^{(\infty)})\, \nu_s^{(\infty)}. \tag{19}$$

From formula (2) we have, however, also

$$\sum_{s=1}^{r_\infty} \left(\mu_s^{(\infty)} + 1 + \frac{1}{\nu_s^{(\infty)}}\right) \nu_s^{(\infty)} = \sum_{s=1}^{r_\infty} (\tau_s^{(\infty)} + \bar{\tau}_s^{(\infty)})\, \nu_s^{(\infty)}. \tag{20}$$

The two last formulæ give us

$$\lambda = ni + \sum_{s=1}^{r_\infty} \tau_s^{(\infty)} \nu_s^{(\infty)} - \tfrac{1}{2} \sum_{s=1}^{r_\infty} \left(\mu_s^{(\infty)} - 1 + \frac{1}{\nu_s^{(\infty)}}\right) \nu_s^{(\infty)}. \tag{21}$$

This then is an expression for the number of the conditions imposed by the partial basis $(\tau)^{(\infty)}$ on the coefficients of the general function of the form (17), and therefore for the number of the conditions imposed by the partial basis in question on the coefficients of the general rational function

of (z, u) of degree i in z, the integer i being supposed of course to be chosen sufficiently large.

It immediately appears that it suffices to choose for i any integer so large that the general rational function of (z, u) conditioned by the partial basis $(\tau)^{(\infty)}$ involves no power of z higher than z^i. For assuming the formula to hold for a larger number i', and writing

$$\lambda' = ni' + \sum_{s=1}^{r_\infty} \tau_s^{(\infty)} \nu_s^{(\infty)} - \tfrac{1}{2} \sum_{s=1}^{r_\infty} \left(\mu_s^{(\infty)} - 1 + \frac{1}{\nu_s^{(\infty)}} \right) \nu_s^{(\infty)},$$

we see that among the λ' conditions, imposed by the partial basis $(\tau)^{(\infty)}$ on the coefficients of the general rational function of (z, u) of degree i' in z, are included the $n(i'-i)$ conditions obtained on equating to 0 the coefficients of terms involving z to a power higher than z^i. On subtracting $n(i'-i)$ from λ', we then obtain the expression (21) for the number of the conditions imposed by the partial basis $(\tau)^{(\infty)}$ on the coefficients of the general rational function of (z, u) of degree i in z.

If, by the notation $(\bar{\tau})^{(\infty)}$, we designate the partial basis which is complementary adjoint to the partial basis $(\tau)^{(\infty)}$, we readily see on replacing the product (5) by the product

$$(22) \qquad \sum_{t=1}^{n} \sum_{r=-i}^{i-1} \beta_{-r,\, n-t}\, z^{-r} u^{n-t} \sum_{t=1}^{n} \sum_{r=i-1}^{-i} a_{r+1,\, t-1}\, z^{r+1} u^{t-1},$$

and taking i sufficiently large, that we can also deduce the formula (21) from considerations relating to the constant coefficient of the element zu^{n-1} in the reduced form of the latter product where the first factor is conditioned by the partial basis $(\tau)^{(\infty)}$. It has been proved elsewhere* that the necessary and sufficient conditions, in order that the second factor in (22) should have orders of coincidence which are complementary adjoint to the orders of coincidence furnished by the partial basis $(\tau)^{(\infty)}$, are obtained on equating to 0 the coefficient of the element zu^{n-1} in the reduced form of the product

$$R\left(\frac{1}{z}, u\right) \sum_{t=1}^{n} \sum_{r=i-1}^{-i} a_{r+1,\, t-1}\, z^{r+1} u^{t-1},$$

where $R\left(\frac{1}{z}, u\right)$ is the general rational function conditioned by the partial basis $\tau^{(\infty)}$. These conditions, however, are also obtained on

* *Loc. cit.*, "On the Foundations," &c., § 2.

equating to 0 the coefficient of the element zu^{n-1} in the reduced form of the product (22), as might readily be shewn by an argument analogous to that by which we proved that the conditions to which we subject the coefficients of the second factor in the product (5), on equating to 0 the principal residue in the product, are the same as those obtained on equating to 0 the principal residue in the product (16). The necessary and sufficient conditions in order that the second factor in the product (22) should have orders of coincidence which are complementary adjoint to those furnished by the partial basis $(\tau)^{(\infty)}$, are then obtained on equating to 0 the coefficient of the element zu^{n-1} in the product.

Denoting by $2ni-\lambda$ the number of the arbitrary coefficients in the first factor of the product (22), this also will be the number of the conditions imposed on the coefficients of the second factor, when we equate to 0 the coefficient of the element zu^{n-1} in the reduced form of the product. The number of the arbitrary coefficients in the second factor, supposed to be conditioned by the partial basis $(\bar{\tau})^{(\infty)}$, will then be λ. Now, in passing from the general rational function of (z, u) conditioned by a lower partial basis $(t)^{(\infty)}$, to the general rational function conditioned by the partial basis $(\tau)^{(\infty)}$, we impose on the coefficients of the former function just $\sum_{s=1}^{r_\infty} (\tau_s^{(\infty)}-\bar{\tau}_s^{(\infty)})\,\nu_s^{(\infty)}$ conditions, and in passing from the general rational function conditioned by the lower partial basis $(t)^{(\infty)}$, to the general rational function conditioned by the partial basis $(\bar{\tau})^{(\infty)}$, we impose on the coefficients of the former function just $\sum_{s=1}^{r_\infty} (\bar{\tau}_s^{(\infty)}-t_s^{(\infty)})\,\nu_s^{(\infty)}$ conditions. The difference between these numbers is, however, evidently equal to the difference in the numbers of the conditions imposed by the partial bases $(\tau)^{(\infty)}$ and $(\bar{\tau})^{(\infty)}$ respectively on the coefficients of the general function of the form

$$\sum_{t=1}^{n} \sum_{r=i-1}^{-i} a_{r+1,\,t-1}\, z^{r+1} u^{t-1},$$

the integer i being assumed to be taken sufficiently large. Under this general form are included both the factors of the product (22), and the difference in question is therefore the difference between λ and $2ni-\lambda$. We consequently have

$$2\lambda-2ni = \sum_{s=1}^{r_\infty} (\tau_s^{(\infty)}-\bar{\tau}_s^{(\infty)})\,\nu_s^{(\infty)}. \tag{23}$$

Furthermore, we have

$$\sum_{s=1}^{r_\infty} \left(\mu_s^{(\infty)}-1+\frac{1}{\nu_s^{(\infty)}}\right) \nu_s^{(\infty)} = \sum_{s=1}^{r_\infty} (\tau_s^{(\infty)}+\bar{\tau}_s^{(\infty)})\,\nu_s^{(\infty)}.$$

From these two formulæ we immediately derive formula (21).

In all that precedes we can evidently replace the element z^{-1} corresponding to the value $z = \infty$ by the element $z - a_\kappa$ corresponding to any finite value $z = a_\kappa$. In analogy with formula (21), we then have the expression

$$\lambda_\kappa = n i_\kappa + \sum_{s=1}^{r_\kappa} \tau_s^{(\kappa)} \nu_s^{(\kappa)} - \tfrac{1}{2} \sum_{s=1}^{r_\kappa} \left(\mu_s^{(\kappa)} - 1 + \frac{1}{\nu_s^{(\kappa)}} \right) \nu_s^{(\kappa)}, \tag{24}$$

for the number of the conditions imposed by the orders of coincidence $\tau_1^{(\kappa)}, \ldots, \tau_{r_\kappa}^{(\kappa)}$ on the constants in the general rational function of (z, u), whose coefficients expanded in powers of $z - a_\kappa$ involve no power lower than $(z - a_\kappa)^{-i_\kappa}$, the integer i_κ being taken at least so great, that the presence of a lower power would be incompatible with the set of orders of coincidence in question. On introducing the symbol ∞, as index and suffix, into formula (21) we may suppose this formula, obtained for $z = \infty$, to be included in (24). We shall then assume that formula (24) has reference to the value $z = \infty$, as well as to the finite values $z = a_\kappa$.

We are now in a position to prove the *independence of the conditions of adjointness*. Consider any basis (τ) and construct the rational function

$$\sum_\kappa \frac{\phi^{(i_\kappa)}(z, u)}{(z - a_\kappa)^{i_\kappa}} + P(z, u), \tag{25}$$

where $\phi^{(i_\kappa)}(z, u)$ designates the general polynomial of degree $i_\kappa - 1$ in z, and $n - 1$ in u, and where $P(z, u)$ denotes the general polynomial of degree i_∞ in z and $n - 1$ in u. The summation with regard to κ is supposed to extend to every finite value $z = a_\kappa$, which is such that the general rational function of (z, u) conditioned by the orders of coincidence $\tau_1^{(\kappa)}, \ldots, \tau_{r_\kappa}^{(\kappa)}$ presents a negative power of $z - a_\kappa$, the integer i_κ furthermore being chosen at least so large that the general rational function in question cannot involve a power of $z - a_\kappa$ lower than $(z - a_\kappa)^{-i_\kappa}$. The degree i_∞ in z of the polynomial $P(z, u)$ is at least as great as the degree in z of the general rational function of (z, u) conditioned by the orders of coincidence $\tau_1^{(\infty)}, \ldots, \tau_{r_\infty}^{(\infty)}$. The expression (25) evidently involves

$$\sum_\kappa n i_\kappa + n \tag{26}$$

arbitrary constants, where it is now to be understood that the summation with regard to κ involves among others the element $n i_\infty$.

Let us now condition the general function of the form (25) by the basis (τ). The number of the conditions thus imposed on the arbitrary constants involved in the general form (25) is certainly not greater than the sum of

the numbers λ_κ given in (24). For the number of the independent conditions imposed on the constant coefficients of the function (25) by the basis (τ) we may then evidently write

$$(27) \qquad \sum_\kappa n i_\kappa + \sum_\kappa \sum_{s=1}^{r_\kappa} \tau_s^{(\kappa)} \nu_s^{(\kappa)} - \tfrac{1}{2} \sum_\kappa \sum_{s=1}^{r_\kappa} \left(\mu_s^{(\kappa)} - 1 + \frac{1}{\nu_s^{(\kappa)}}\right) \nu_s^{(\kappa)} - \epsilon,$$

where ϵ is 0 or a positive integer and where the summations with regard to κ are supposed to extend to the value $z = \infty$ as well as to the finite values $z = a_\kappa$. If ϵ is 0, the conditions imposed on the general form (25) by the orders of coincidence furnished for the individual values of z by the basis (τ) are independent of one another. In this case also the conditions imposed, for example, by the orders of coincidence $\tau_1^{(\kappa)}, \ldots, \tau_{r_\kappa}^{(\kappa)}$ on the general function of the form (25) are the same in number as the conditions imposed by these orders of coincidence on the general rational function of (z, u), whose principal part corresponding to the value $z = a_\kappa$ is

$$\phi^{(i_\kappa)}(z, u)(z - a_\kappa)^{-i_\kappa}.$$

The number of the arbitrary constants involved in the general function of the form (25) conditioned by the basis (τ) is obtained on subtracting the expression (27) from the expression (26). The number of the arbitrary constants in question is then given by the expression

$$(28) \qquad n - \sum_\kappa \sum_{s=1}^{r_\kappa} \tau_s^{(\kappa)} \nu_s^{(\kappa)} + \tfrac{1}{2} \sum_\kappa \sum_{s=1}^{r_\kappa} \left(\mu_s^{(\kappa)} - 1 + \frac{1}{\nu_s^{(\kappa)}}\right) \nu_s^{(\kappa)} + \epsilon.$$

Now the general function of the form (25) conditioned by the basis (τ) is evidently the general rational function of (z, u) built on the basis (τ). The expression (28) then represents the number of the arbitrary constants involved in the general rational function of (z, u) built on the basis (τ). By the complementary theorem, however, we have * for this number the expression

$$(29) \qquad N_\tau = N_{\bar\tau} + n - \sum_\kappa \sum_{s=1}^{r_\kappa} \tau_s^{(\kappa)} \nu_s^{(\kappa)} + \tfrac{1}{2} \sum_\kappa \sum_{s=1}^{r_\kappa} \left(\mu_s^{(\kappa)} - 1 + \frac{1}{\nu_s^{(\kappa)}}\right) \nu_s^{(\kappa)},$$

where $\bar\tau$ is the basis complementary adjoint to the basis (τ). On identifying this expression with the expression given in (28), we derive $\epsilon = N_{\bar\tau}$. We consequently have $\epsilon = 0$ when $N_{\bar\tau}$ has the value 0. When N_τ has the value 0, it follows that the conditions imposed on the general form

* *Loc. cit.*, "On the Foundations," &c. (Formula 81).

(25) by the orders of coincidence furnished for the individual values of z by the basis (τ) are independent of one another, and that their total number is given by the expression

$$(30) \qquad \sum_{\kappa} ni_{\kappa} + \sum_{\kappa} \sum_{s=1}^{r_{\kappa}} \tau_s^{(\kappa)} \nu_s^{(\kappa)} - \tfrac{1}{2} \sum_{\kappa} \sum_{s=1}^{r_{\kappa}} \left(\mu_s^{(\kappa)} - 1 + \frac{1}{\nu_s^{(\kappa)}} \right) \nu_s^{(\kappa)},$$

derived from (27) on putting 0 for ϵ.

The general ϕ-function is the general rational function built on the basis which furnishes the orders of coincidence requisite to adjointness for finite values of the variable z, while for the value $z = \infty$ it furnishes orders of coincidence which severally exceed by 2 the orders of coincidence requisite to adjointness for that value of the variable. We shall call this basis the ϕ-basis. If now (τ) is a lower basis than the ϕ-basis, and if furthermore it falls short of the ϕ-basis in the orders of coincidence which it furnishes for the branches of one cycle at least corresponding to each of the irreducible equations included in the fundamental equation, we readily see that we must have $N_{\bar{\tau}} = 0$. For the relations

$$\tau_s^{(\kappa)} + \bar{\tau}_s^{(\kappa)} = \mu_s^{(\kappa)} - 1 + \frac{1}{\nu_s^{(\kappa)}} \quad (s = 1, \ldots, r_{\kappa}),$$

$$\tau_s^{(\infty)} + \bar{\tau}_s^{(\infty)} = \mu_s^{(\infty)} + 1 + \frac{1}{\nu_s^{(\infty)}} \quad (s = 1, \ldots, r_{\infty}),$$

shew, in the case in question, that the basis $(\bar{\tau})$ furnishes no negative orders of coincidence, and that it furnishes at least one positive order of coincidence corresponding to each of the irreducible equations included in the equation (1). It follows that the general rational function built on the basis $(\bar{\tau})$ is 0, and consequently that we have $N_{\bar{\tau}} = 0$. In the case of a basis (τ) then such as we have just described, the conditions imposed on the general form (25) by the orders of coincidence furnished for the individual values of z are independent of one another, and their total number is given by the expression (30).

If in particular the basis in question is the adjoint basis, that is, the basis whose orders of coincidence define adjointness for all values of z, the conditions imposed on the general form (25) by adjointness relative to the individual values of z are independent of one another. The number of the conditions imposed on the form (25) by adjointness for all values of the variable z is then given by the expression

$$(31) \qquad \sum_{\kappa} ni_{\kappa} + \tfrac{1}{2} \sum_{\kappa} \sum_{s=1}^{r_{\kappa}} \left(\mu_s^{(\kappa)} - 1 + \frac{1}{\nu_s^{(\kappa)}} \right) \nu_s^{(\kappa)},$$

obtained on substituting in (30) the values $\mu_s^{(\kappa)}-1+\frac{1}{\nu_s^{(\kappa)}}$ for the symbols $\tau_s^{(\kappa)}$. In like manner, from (28), we obtain the expression

$$(32) \qquad n-\tfrac{1}{2}\sum_{\kappa}\sum_{s=1}^{r_\kappa}\left(\mu_s^{(\kappa)}-1+\frac{1}{\nu_s^{(\kappa)}}\right)\nu_s^{(\kappa)}$$

for the number of the arbitrary constants involved in the general adjoint function.

Where N is the degree in (z, u) of the fundamental equation (1), it has been proved* that the aggregate degree of a rational function of (z, u), which is adjoint relatively to the value $z=\infty$, must be $\leqslant N-1$. Choosing for (τ) the adjoint basis and constructing the form (25) with reference to this basis, we are evidently at liberty to take $N-1$ for i_∞, the degree in z of the polynomial $P(z, u)$. The number of the conditions imposed on the general rational function of (z, u) of degree $N-1$ in z, by the requirement of adjointness relative to the value $z=\infty$, is then

$$(33) \qquad n(N-1)+\tfrac{1}{2}\sum_{s=1}^{r_\infty}\left(\mu_s^{(\infty)}-1+\frac{1}{\nu_s^{(\infty)}}\right)\nu_s^{(\infty)},$$

obtained from (24) by substituting $N-1$ for i_∞ and $\mu_s^{(\infty)}-1+\frac{1}{\nu_s^{(\infty)}}$ for $\tau_s^{(\infty)}$. Among the conditions here in question are those which reduce the rational function of degree $N-1$ in z to the form in which its aggregate degree in (z, u) is $N-1$. The number of these latter conditions is evidently $\frac{1}{2}n(n-1)$. Subtracting this number from the number given in (33) we obtain

$$n(N-1)-\tfrac{1}{2}n(n-1)+\tfrac{1}{2}\sum_{s=1}^{r_\infty}\left(\mu_s^{(\infty)}-1+\frac{1}{\nu_s^{(\infty)}}\right)\nu_s^{(\infty)}$$

for the number of the conditions imposed on the general rational function of (z, u) of degree $N-1$, by the requirement of adjointness for the value $z=\infty$. At the same time supposing $P(z, u)$ in the form (25) to represent the general polynomial of aggregate degree $N-1$ in (z, u), we obtain, instead of (31), the expression

$$(34) \qquad n(N-1)-\tfrac{1}{2}n(n-1)+\Sigma'ni_\kappa+\tfrac{1}{2}\sum_{\kappa}\sum_{s=1}^{r_\kappa}\left(\mu_s^{(\kappa)}-1+\frac{1}{\nu_s^{(\kappa)}}\right)\nu_s^{(\kappa)}$$

for the number of the conditions imposed on the coefficients of the form by

---

* *Loc. cit.*, "On the Foundations," &c., § 1.

adjointness for all values of the variable z, the accent over the sign of summation with regard to κ here signifying that the summation is not extended to the value $z = \infty$.

In the case where the fundamental equation (1) is an integral algebraic equation, it can be shewn * that a rational function of (z, u) which is adjoint for a finite value $z = a_\kappa$ must be integral with regard to the element $z - a_\kappa$. If for (τ), then, we choose the adjoint basis and construct the form (25) with reference to this basis, selecting for $P(z, u)$ the general polynomial of aggregate degree $N-1$ in (z, u), we see that we can take 0 for the value of each of the integers i_κ in the summation. The form (25) then reduces to the general polynomial in (z, u) of aggregate degree $N-1$, and for the number of the conditions imposed on this polynomial by the requirement that it be adjoint for all values of z we obtain from (34) the expression

$$(35) \qquad n(N-1) - \tfrac{1}{2}n(n-1) + \tfrac{1}{2} \sum_{\kappa}{}' \sum_{s=1}^{r_\kappa} \left(\mu_s^{(\kappa)} - 1 + \frac{1}{\nu_s^{(\kappa)}}\right) \nu_s^{(\kappa)},$$

a result already given in the writer's book † on the algebraic functions. At the same time we obtain, of course, the expression

$$n(N-1) + \tfrac{1}{2} \sum_{\kappa}{}' \sum_{s=1}^{r_\kappa} \left(\mu_s^{(\kappa)} - 1 + \frac{1}{\nu_s^{(\kappa)}}\right) \nu_s^{(\kappa)}$$

for the number of the conditions imposed on the general polynomial in (z, u) of degree $N-1$ in z by the requirement that it be adjoint for all values of z, and more generally the expression

$$(36) \qquad n i_\infty + \tfrac{1}{2} \sum_{\kappa}{}' \sum_{s=1}^{r_\kappa} \left(\mu_s^{(\kappa)} - 1 + \frac{1}{\nu_s^{(\kappa)}}\right) \nu_s^{(\kappa)}$$

for the number of the conditions imposed by adjointness on the general polynomial in (z, u) of degree i_∞ in z, where $i_\infty \geqslant N-1$.

The last result has reference to the case in which the fundamental equation is integral. It is evidently included as special case under the general statement, that the expression (31) represents the number of the conditions imposed by adjointness relative to any algebraic equation on the form (25) constructed with reference to such equation. Con-

---

* *Loc. cit.*, "On the Foundations," &c., § 3.

† Chap. XIII, p. 154.

versely, however, it may be noted that the general result is already implied in the special, and that, from the independence of the conditions of adjointness in the case of an integral algebraic equation, may be deduced the independence of the conditions of adjointness for any algebraic equation.

Whatever the form of the fundamental equation (1) may be we can transform to an integral equation $F(z, v) = 0$ on writing

$$v = \{(z-a_1) \dots (z-a_\rho)\}^\sigma u, \tag{37}$$

where the integer σ is chosen sufficiently large, the values $a_1, \dots, a_\rho$ being those values of z for which equation (1) is not of integral character. Employing as before the notation $\mu_s^{(\kappa)} - 1 + \frac{1}{\nu_s^{(\kappa)}}$ to designate the orders of coincidence requisite to adjointness relative to the equation (1), the same notation will designate the orders of coincidence requisite to adjointness relative to the equation $F(z, v) = 0$ for all values of z except the ρ finite values, $a_1, \dots, a_\rho$ and the value $z = \infty$. For these values of z the orders of coincidence requisite to adjointness will evidently be given by the expressions $(n-1)\sigma + \mu_s^{(\kappa)} - 1 + \frac{1}{\nu_s^{(\kappa)}}$ and $-(n-1)\rho\sigma + \mu_s^{(\infty)} - 1 + \frac{1}{\nu_s^{(\infty)}}$ respectively. Substituting these expressions for the numbers $\mu_s^{(\kappa)} - 1 + \frac{1}{\nu_s^{(\kappa)}}$ corresponding to the ρ finite values of z here in question, and for the number $\mu_s^{(\infty)} - 1 + \frac{1}{\nu_s^{(\infty)}}$ corresponding to the value $z = \infty$ in the double summation in (36), this expression remains unaltered. The expression (36) as it stands then represents the total number of the conditions imposed on the general polynomial in (z, v) of degree i_∞ in z, by adjointness relative to the equation $F(z, v) = 0$ for all values of z.

The general polynomial in (z, v) of degree i_∞ in z involves $n(i_\infty + 1)$ arbitrary constants. Transformed to terms of (z, u) it is evidently a polynomial of degree $i_\infty + (n-1)\rho\sigma$ in z and derived from the general polynomial in (z, u) of degree $i_\infty + (n-1)\rho\sigma$ in z on subjecting the coefficients of the latter to $n\{i_\infty + 1 + (n-1)\rho\sigma\} - n(i_\infty + 1)$ conditions. Adding this number to the expression (36) we obtain the expression

$$n\{i_\infty + (n-1)\rho\sigma\} + \tfrac{1}{2} \sum_\kappa \sum_{s=1}^{r_\kappa} \left(\mu_s^{(\kappa)} - 1 + \frac{1}{\nu_s^{(\kappa)}}\right) \nu_s^{(\kappa)} \tag{38}$$

for the number of the conditions imposed on the general polynomial in (z, u) of degree $i_\infty + (n-1)\rho\sigma$ in z by the orders of coincidence which

define adjointness relative to the equation $F(z, v) = 0$. Dividing the general polynomial in (z, u) of degree $i_\infty + (n-1)\rho\sigma$ in z by $\{(z-a_1), \ldots, (z-a_\rho)\}^{(n-1)\sigma}$ we then see that the expression (38) represents the number of the conditions imposed on the quotient by the orders of coincidence which define adjointness relative to the equation (1). The general quotient here in question we can evidently represent in the form

$$\sum_\kappa \frac{\phi_\kappa(z, u)}{(z-a_\kappa)^{(n-1)\sigma}} + P(z, u), \tag{39}$$

where the summation with regard to κ is extended only to the ρ values $z = a_1, \ldots, a_\rho$, the numerators $\phi_\kappa(z, u)$ representing general polynomials in (z, u) of degree $(n-1)\sigma - 1$ in z, while $P(z, u)$ is the general polynomial in (z, u) of degree i_∞ in z. Also it is evident, from the manner in which we arrived at the form (39), that the general rational function of (z, u) conditioned by adjointness relative to the equation (1) for all values of z must be of this form.

Supposing now that $i_\kappa \leqslant (n-1)\sigma$ is an integer such that a rational function of (z, u) which is adjoint for the value $z = a_\kappa$ cannot involve a power of $z-a_\kappa$ lower than $(z-a_\kappa)^{-i_\kappa}$, we see that the general rational function of (z, u) conditioned by adjointness for all values of z must be of the form (25). The form (25), however, is here included under the form (39) and obtained from it by imposing on its coefficients $\sum_{\kappa=1}^{\rho} n\{(n-1)\sigma - i_\kappa\}$ conditions. Subtracting this number from the number given in (38) we obtain the expression (31) for the number of the conditions imposed on the general rational function of the form (25) by adjointness relative to all values of the variable z, where, however, it understood for the moment that the summation with regard to κ in (25) involves elements corresponding to the ρ values $a_1, \ldots, a_\rho$ only. We have obtained this result under the limitation $i_\kappa \leqslant (n-1)\sigma$. Having arrived at the result under this restriction, however, it is immediately apparent that we can drop the restriction and say that the number of the conditions imposed on the form (25) by adjointness for all values of the variable z is given by the expression (31), the integers i_κ being subject only to the limitation that a power of $z-a_\kappa$ lower than $(z-a_\kappa)^{-i_\kappa}$ cannot be present in a rational function of (z, u) which is adjoint for the value $z = a_\kappa$. It is now evident that the statement holds also when we include in the summation in (25) elements corresponding to other values $z = a_\kappa$ besides the ρ values $a_1, \ldots, a_\rho$. Since the expression (31) represents the sum of the numbers of the conditions imposed on the general rational functions of the several forms $(z-a_\kappa)^{-i_\kappa}((z-a_\kappa, u))$ by adjointness relative to the corresponding indi-

vidual values of z, we have established the principle of the independence of the conditions of adjointness for the general equation (1). Furthermore, we have here deduced this principle of the independence of the conditions of adjointness for the general equation (1) as a consequence of the independence of the conditions of adjointness in the case of an integral algebraic equation. In making our deduction we have assumed for the number of the conditions imposed on a general rational function of the form $(z-a_\kappa)^{-i_\kappa}(z-a_\kappa, u))$ by adjointness relative to any algebraic equation for the individual value $z = a_\kappa$ the expression furnished by formula (24). It is readily seen, however, that this expression also in the case of any algebraic equation can be deduced from the corresponding expression in the case of an integral algebraic equation.

ON A CONNEXION BETWEEN THE FUNCTIONS OF HERMITE* AND THE FUNCTIONS OF LEGENDRE

H. E. J. CURZON.

[Received November 5th, 1912.—Read December 12th, 1912.]

Introductory.

For positive integral values of n, the Hermite function is a polynomial and is defined by the relation

$$U_n(x) = e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

Thus, if n is a positive integer,

$$U_n(x) = \frac{n!}{2\pi i} \int_D \frac{e^{x^2-t^2}}{(t-x)^{n+1}} dt,$$

where D is a simple contour containing x.

Hence
$$U_n(x) = \frac{n!}{2\pi i} \int_C e^{-2ux-u^2} u^{-n-1} du,$$

where C is a simple contour enclosing the origin.

It can easily be proved that $U_n(x)$ satisfies the differential equation

$$\frac{d^2u}{dx^2} - 2x \frac{du}{dx} + 2nu = 0. \qquad \text{(A)}$$

Whittaker (see footnote) then shows that for *all* values of n,

$$\frac{\Gamma(n+1)}{2\pi i} \int_{\Gamma_u} e^{-2ux-u^2} u^{-n-1} du$$

---

* Hermite, *Comptes Rendus*, t. LVIII, pp. 266-273. Whittaker, "Functions associated with the Parabolic Cylinder," *Proc. London Math. Soc.*, 1903, pp. 420 *et seq.* Watson, *Proc. London Math. Soc.*, Ser. 2, Vol. 8, pp. 393 *et seq.* Myller-Lebedoff, *Math. Ann.*, Vol. 64, pp. 388 *et seq.*

satisfies the differential equation (A), Γ_u being a contour in the plane of u, beginning at the positive end of the real axis, circulating in a counter-clockwise direction round the origin and returning again to positive infinity on the real axis. In the case when n is a positive integer this contour may clearly be deformed into a simple contour enclosing the origin without altering the value of the integral.

In general then $U_n(x)$ is defined by the relation

$$U_n(x) = \frac{\Gamma(n+1)}{2\pi i} \int_{\Gamma_u} \frac{e^{-2ux-u^2}}{u^{n+1}}\, du. \tag{I}$$

In Whittaker's paper it is then shewn that a second independent solution of (A) is given by

$$V_n(x) = e^{x^2} U_{-n-1}(-ix). \tag{II}$$

The object of this paper is to establish a connexion between these Hermite functions of the first and second kind and the Legendre functions of the first and second kind by means of relations involving integrals analogous to those connecting the Bessel and Legendre functions.

1. Suppose n is an even positive integer, then

$$U_n(x) = \frac{n!}{2\pi i} \int_D \frac{e^{-2ux-u^2}}{u^{n+1}}\, du,$$

where D is a simple contour enclosing the origin.

Hence, on expanding e^{-2ux} in ascending powers of ux,

$$U_n(x) = \sum_{r=0}^{\infty} \frac{(-2x)^r n!}{r!\, 2\pi i} \int_D e^{-u^2} u^{-n-1+r}\, du$$

$$= \sum_{s=0}^{s=\frac{1}{2}n} \frac{(-2x)^{2s} n!}{(2s)!\, 2\pi i} \int_D e^{-u^2} u^{-n-1+2s}\, du$$

$$= \sum_{s=0}^{s=\frac{1}{2}n} \frac{(2x)^{2s} n!}{(2s)!} \frac{(-1)^{\frac{1}{2}n-s}}{\left(\frac{n}{2}-s\right)!}.$$

Now $\int_{-\infty}^{\infty} e^{-t^2} t^{2s}\, dt = \Gamma(s+\tfrac{1}{2}) = \dfrac{\Gamma(s+\frac{1}{2})\,\Gamma(s+1)}{\Gamma(s+1)} = \dfrac{\pi^{\frac{1}{2}}\,\Gamma(2s+1)}{2^{2s}\,\Gamma(s+1)_{\frac{1}{2}}}$;

therefore

$$\int_{-\infty}^{\infty} e^{-t^2} U_n(xt)\,dt = \sum_{s=0}^{s=\frac{1}{2}n} \frac{(2x)^{2s}\, n!\, (-1)^{\frac{1}{2}n-s}}{(2s)!\left(\frac{n}{2}-s\right)!} \frac{\pi^{\frac{1}{2}}\,(2s)!}{2^{2s}\, s!}$$

$$= \frac{n!\,\pi^{\frac{1}{2}}}{\left(\frac{n}{2}\right)!} \sum_{s=0}^{\frac{1}{2}n} \frac{\left(\frac{n}{2}\right)!\,(-1)^{\frac{1}{2}n-s}}{s!\left(\frac{n}{2}-s\right)!}\, x^{2s}$$

$$= \frac{n!\,\pi^{\frac{1}{2}}}{\left(\frac{n}{2}\right)!}\,(x^2-1)^{\frac{1}{2}n}.$$

Now, from the definition of $U_n(x)$, viz.,

$$U_n(x) = \frac{\Gamma(n+1)}{2\pi i}\int_{\Gamma_u} \frac{e^{-2ux-u^2}}{u^{n+1}}\,du,$$

clearly, $$\frac{dU_n(x)}{dx} = -2\,\frac{\Gamma(n+1)}{2\pi i}\int_{\Gamma_u} \frac{e^{-2ux-u^2}}{u^n}\,du = -2nU_{n-1}(x), \qquad \text{(III)}$$

and we have proved that

$$\int_{-\infty}^{\infty} e^{-t^2} U_n(xt)\,dt = \frac{n!\,\pi^{\frac{1}{2}}}{\left(\frac{n}{2}\right)!}\,(x^2-1)^{\frac{1}{2}n}.$$

On differentiating each side of this equation $\frac{1}{2}n$ times with respect to x, we therefore obtain

$$(-2)^{\frac{1}{2}n}\frac{n!}{\left(\frac{n}{2}\right)!}\int_{-\infty}^{\infty} e^{-t^2} t^{\frac{1}{2}n} U_{\frac{1}{2}n}(xt)\,dt = \frac{n!}{\left(\frac{n}{2}\right)!}\,\pi^{\frac{1}{2}}\,\frac{d^{\frac{1}{2}n}}{dx^{\frac{1}{2}n}}\,(x^2-1)^{\frac{1}{2}n}$$

$$= \frac{n!}{\left(\frac{n}{2}\right)!}\,\pi^{\frac{1}{2}}\,2^{\frac{1}{2}n}\left(\frac{n}{2}\right)!\,P_{\frac{1}{2}n}(x).$$

Hence for all positive integral values of m, replacing $\frac{1}{2}n$ by m,

$$P_m(x) = \frac{(-1)^m}{\pi^{\frac{1}{2}}\, m!}\int_{-\infty}^{\infty} e^{-t^2} t^m U_m(xt)\,dt.$$

2. Expansion of $U_n(x)$ in ascending powers of x.

For all values of p it is well known that

$$\int_{\Gamma_u} e^{-u^2} u^{p-1}\,du = ie^{p\pi i}\sin p\pi\,\Gamma\left(\frac{p}{2}\right). \qquad \text{(IV)}$$

Now $$U_n(x) = \frac{\Gamma(n+1)}{2\pi i}\int_{\Gamma_u} e^{-2ux-u^2}u^{-n-1}\,du\ldots,$$

so that, if it is permissible to expand e^{-2ux} in ascending powers of x, and integrate term by term, the required expansion is at once obtained on applying the above formula.

Consider $$\int_{\Gamma_u}\sum_{r=s}^{\infty}\frac{(-2x)^r u^r}{r!}u^{-n-1}e^{-u^2}du.$$

Suppose first of all that s is chosen large enough to secure that the real part of $s-n$ is positive.

Then this integral is equal to

$$(1-e^{-2n\pi i})\int_{\infty}^{0}\sum_{r=s}^{\infty}\frac{(-2x)^r u^r}{r!}u^{-n-1}e^{-u^2}du.$$

Now let $|-2x|$ be denoted by ξ, and let the real part of n be represented by N, and then examine

$$\int_0^{\infty}\sum_{r=s}^{\infty}\frac{\xi^r u^r}{r!}u^{-N-1}e^{-u^2}du.$$

Choose a positive quantity λ, so that

$$\frac{s}{2\xi} < \lambda < \frac{s+1}{2\xi},$$

and rewrite the integral in the form

$$\int_0^{\lambda}\sum_{r=s}^{\infty}\frac{\xi^r u^r}{r!}u^{-N-1}e^{-u^2}du+\int_{\lambda}^{\infty}\sum_{r=s}^{\infty}\frac{\xi^r u^r}{r!}u^{-N-1}e^{-u^2}du,$$

which is less than

$$\int_0^{\lambda}\frac{\xi^s u^s}{s!}\,\frac{u^{-N-1}e^{-u^2}}{1-\dfrac{\xi u}{s+1}}du+\int_{\lambda}^{\infty}e^{-u^2+\xi u}u^{-N-1}\,du,$$

which is less than

$$\frac{2\xi^s}{s!}\int_0^{\lambda}u^{s-N-1}e^{-u^2}du+\int_{\lambda}^{\infty}e^{-u^2+\xi u}u^{-N-1}du.$$

Now let λ tend to infinity as the result of s tending to infinity.

Then since $\int^{\infty}e^{-u^2+\xi u}u^{-N-1}du$ is a convergent integral for all finite values of ξ and all finite values of N (a being a real positive quantity), it

follows that

$$\int_{\lambda}^{\infty} e^{-u^2+\zeta u} u^{-N-1}\, du$$

tends to zero. While

$$\frac{2\zeta^s}{s!}\int_0^{\lambda} u^{s-N-1} e^{-u^2}\, du < \frac{2\zeta^s}{s!}\,\tfrac{1}{2}\Gamma\left(\frac{s-N}{2}\right),$$

and this tends to zero as s tends to infinity, for all finite values of ζ, and finite values of N.

Hence for all finite values of x, and all finite values of n, it is permissible to expand e^{-2ux} under the integral sign and integrate term by term. Thus

$$U_n(x)$$

$$= \frac{\Gamma(n+1)}{2\pi i}\sum_{r=0}^{\infty}\frac{(-2x)^r}{r!}\int_{\Gamma_u} e^{-u^2} u^{-n-1+r}\, du$$

$$= \frac{\Gamma(n+1)}{2\pi i}\sum_{r=0}^{\infty}\frac{(-2x)^r}{r!}\, ie^{-(n-r)\pi i}\sin[-(n-r)]\,\pi.\Gamma\left(-\frac{n-r}{2}\right)$$

$$= \frac{e^{-n\pi i}\Gamma(n+1)}{2\pi}\sum_{r=0}^{\infty}\frac{(2x)^r}{r!}\, 2\sin\left(-\frac{n-r}{2}\pi\right)\cos\left(-\frac{n-r}{2}\pi\right)\Gamma\left(-\frac{n-r}{2}\right)$$

$$= e^{-n\pi i}\Gamma(n+1)\sum_{r=0}^{\infty}\frac{(2x)^r}{r!}\cos\frac{n-r}{2}\pi\,\frac{1}{\Gamma\left(\frac{n-r}{2}+1\right)}$$

$$= e^{-n\pi i}\Gamma(n+1)\sum_{s=0}^{\infty}\frac{(2x)^{2s}}{(2s)!}\cos\frac{n\pi}{2}\,\frac{(-1)^s}{\Gamma\left(\frac{n-2s}{2}+1\right)}$$

$$+e^{-n\pi i}\Gamma(n+1)\sum_{s=0}^{\infty}\frac{(2x)^{2s+1}}{(2s+1)!}\sin\frac{n\pi}{2}\,\frac{(-1)^s}{\Gamma\left(\frac{n-2s-1}{2}+1\right)}$$

$$= \frac{e^{-n\pi i}\Gamma(n+1)\cos\frac{n\pi}{2}}{\Gamma\left(\frac{n}{2}+1\right)}\sum_{s=0}\frac{(-1)^s\frac{n}{2}\left(\frac{n}{2}-1\right)\ldots\left(\frac{n}{2}-s+1\right)}{(2s)!}x^{2s}$$

$$+\frac{e^{-n\pi i}\Gamma(n+1)\sin\frac{n\pi}{2}}{\Gamma\left(\frac{n+1}{2}\right)}\sum_{s=0}^{\infty}\frac{(-1)^s\frac{n-1}{2}\left(\frac{n-1}{2}-1\right)\ldots\left(\frac{n-1}{2}-s+1\right)}{(2s+1)!}x^{2s+1}. \qquad (V)$$

If n is a positive integer one of the series vanishes and the other terminates.

If n is a negative integer, write $n = -m$, and then the factor multiplying each term of the first series is written

$$\frac{e^{m\pi i}\,\Gamma(-m+1)\cos\frac{m\pi}{2}}{\Gamma\left(-\frac{m}{2}+1\right)} = e^{m\pi i}\,\frac{\pi}{\Gamma(m)\sin m\pi}\,\frac{\sin\frac{m\pi}{2}\,\Gamma\left(\frac{m}{2}\right)}{\pi}\cos\frac{m\pi}{2}$$

$$= \frac{e^{m\pi i}\,\Gamma\left(\frac{m}{2}\right)}{2\Gamma(m)}, \qquad \text{(VI)}$$

while the factor multiplying each term in the second series becomes

$$-\frac{e^{m\pi i}\,\Gamma(-m+1)\sin\frac{m\pi}{2}}{\Gamma\left(\frac{-m+1}{2}\right)} = -\frac{e^{m\pi i}\,\Gamma\left(\frac{m+1}{2}\right)}{2\Gamma(m)}.$$

Thus, if n is a negative integer, neither of the series vanishes, and neither of the series terminates.

3. Investigation of an upper limit for $|U_n(z)|$.

For brevity, let

$$A = \frac{\Gamma(n+1)\cos\frac{n\pi}{2}}{e^{n\pi i}\,\Gamma\left(\frac{n}{2}+1\right)}, \qquad B = \frac{\Gamma(n+1)\sin\frac{n\pi}{2}}{e^{n\pi i}\,\Gamma\left(\frac{n+1}{2}\right)}.$$

Then

$$U_n(x) = A\left[1-\frac{n}{2!}\,2x^2+\frac{n(n-2)}{4!}\,(2x^2)^2-\ldots\right.$$
$$\left.+(-1)^{\kappa-1}\,\frac{n(n-2)\ldots n+2\kappa+4}{(2\kappa-2)!}\,(2x^2)^{\kappa-1}\right]$$

$$+2Bx\left[1-\frac{n-1}{3!}\,2x^2+\frac{(n-1)(n-3)}{5!}\,(2x^2)^2-\ldots\right.$$
$$\left.+(-1)^{k-1}\,\frac{(n-1)(n-3)\ldots(n-2\kappa+3)}{(2\kappa-1)!}\,(2x^2)^{\kappa-1}\right]$$

$$+A\phi_1(x)+2Bx\phi_2(x),$$

where $$\phi_1(x) = \sum_{p=\kappa}^{p=\infty} (-1)^p \frac{n(n-2)\dots(n-2p+2)}{(2p)!} (2x^2)^p,$$

and $$\phi_2(x) = \sum_{p=\kappa}^{p=\infty} (-1)^p \frac{(n-1)(n-3)\dots(n-2p+1)}{(2p+1)!} (2x^2)^p.$$

Now, let $\nu = |n|$ and $\xi = |x|$. Then

$$|\phi_1(x)| \leqslant \sum_{p=\kappa}^{\infty} \frac{\nu(\nu+2)\dots(\nu+2p-2)}{(2p)!} 2^p \xi^{2p}$$

$$\leqslant \sum_{p=\kappa}^{\infty} \frac{\nu(\nu+2)\dots(\nu+2p-2)}{1.3\dots(2p-1)} \frac{\xi^{2p}}{p!}.$$

Now, let κ be the smallest integer greater than (or equal to) $\frac{\nu}{2}$.

Then $|\phi_1(x)|$

$$\leqslant \sum_{p=\kappa}^{\infty} \frac{\nu(\nu+2)\dots(\nu+2p-2\kappa-2)}{(2\kappa+1)(2\kappa+3)\dots(2p-1)} \frac{(\nu+2p-2\kappa)\dots(\nu+2p-2)}{1.3\dots(2\kappa-1)} \frac{\xi^{2p}}{p!}$$

$$\leqslant \sum_{p=\kappa}^{\infty} \frac{(\nu+2p-2\kappa)(\nu+2p-2\kappa+2)\dots(\nu+2p-2)}{1.3\dots(2\kappa-1)} \frac{\xi^{2p}}{p!}$$

$$\leqslant \frac{1}{1.3\dots(2\kappa-1)} \sum_{p=\kappa}^{\infty} \frac{(\nu+2p-2\kappa)(\nu+2p-2\kappa+2)\dots(\nu+2p-2)}{(p-\kappa+1)(p-\kappa+2)\dots p} \frac{\xi^{2p}}{(p-\kappa)!}$$

$$\leqslant \frac{2^\kappa . \xi^{2\kappa}}{1.3\dots(2\kappa-1)} \sum_{p=\kappa}^{\infty} \frac{\left(\frac{\nu}{2}+p-\kappa\right)\left(\frac{\nu}{2}+p-\kappa+1\right)\dots\left(\frac{\nu}{2}+p-1\right)}{(p-\kappa+1)(p-\kappa+2)\dots p} \frac{\xi^{2(p-\kappa)}}{(p-\kappa)!}$$

$$\leqslant \frac{2^\kappa . \xi^{2\kappa}}{1.3\dots(2\kappa-1)} \sum_{r=0}^{\infty} \left(1+\frac{\nu-2}{2(r+1)}\right)\left(1+\frac{\nu-2}{2(r+2)}\right)\dots\left(1+\frac{\nu-2}{2(r+\kappa)}\right) \frac{\xi^{2r}}{r!}.$$

Hence, if ν is greater than 2,

$$|\phi_1(x)| \leqslant \frac{2^\kappa . \xi^{2\kappa}}{1.3\dots(2\kappa-1)} \sum_{r=0}^{\infty} \left(1+\frac{\nu-2}{2(r+1)}\right)^\kappa \frac{\xi^{2r}}{r!}$$

$$\leqslant \frac{2^\kappa . \xi^{2\kappa}}{1.3\dots(2\kappa-1)} \sum_{r=0}^{\infty} \left(1+\frac{\nu-2}{2}\right)^\kappa \frac{\xi^{2r}}{r!}$$

$$\leqslant \frac{\nu^\kappa . \xi^{2\kappa}}{1.3\dots(2\kappa-1)} e^{\xi^2},$$

while, if ν is not greater than 2, it is essentially positive, and so

$$|\phi_1(x)| \leqslant \frac{2^\kappa . \xi^{2\kappa}}{1.3\ldots(2\kappa-1)} \sum_{r=0}^{\infty} \frac{\xi^{2r}}{r!}$$

$$\leqslant \frac{2^\kappa . \xi^{2\kappa}}{1.3\ldots(2\kappa-1)} e^{\xi^2} \leqslant 2\xi^2 . e^{\xi^2},$$

since $\kappa = 1$ in this case.

Similarly, we can prove that, if ν is greater than 1, then

$$|\phi_2(x)| \leqslant \frac{(\nu+1)^\kappa . \xi^{2\kappa}}{1.3\ldots(2\kappa+1)} e^{\xi^2},$$

while, if ν is not greater than 1,

$$|\phi_2(x)| \leqslant \frac{2\xi^2}{3} e^{\xi^2}.$$

4. If $y_n(v)$ is any solution of

$$\frac{d^2u}{dv^2} - 2v\frac{du}{dv} + 2nu = 0,$$

then, under certain conditions governing x, it will be shewn that

$$I = \int_{\Gamma_u} e^{-u^2} u^n y_n(xu)\, du$$

satisfies Legendre's equation.

For $$(1-x^2)\frac{d^2I}{dx^2} - 2x\frac{dI}{dx} + n(n+1)I$$

$$= \int_{\Gamma_t} e^{-t^2} t^n [t^2(1-x^2) y_n''(xt) - 2xt y_n'(xt) + n(n+1) y_n(xt)]\, dt.$$

Now $$\frac{d}{dt}[e^{-t^2} t^{n+1} \{n y_n(xt) - xt y_n'(xt)\}]$$

$$= e^{-t^2} t^n [n(n+1) y_n(xt) - 2xt y_n'(xt) + t^2(1-x^2) y_n''(xt)$$

$$- t^2 \{y_n''(xt) - 2xt y_n'(xt) + 2n y_n(xt)\}]$$

$$= e^{-t^2} t^n [n(n+1) y_n(xt) - 2xt y_n'(xt) + t^2(1-x^2) y_n''(xt)].$$

Hence $$(1-x^2)\frac{d^2I}{dx^2}-2x\frac{dI}{dx}+n(n+1)I$$

$$=\int_{\Gamma_t}\frac{d}{dt}\left[e^{-t^2}t^{n+1}\left\{ny_n(xt)-xty'_n(xt)\right\}\right]dt,$$

and this is equal to zero if

$$\operatorname*{Lt}_{t\to\infty}\ e^{-t^2}t^{n+1}\left\{ny_n(xt)-xty'_n(xt)\right\}=0.$$

This condition may be expressed in a simpler form.

Since $$\int_{\Gamma_u}\frac{d}{du}(e^{-2ux-u^2}u^{-n})\,du=0\,;$$

therefore

$$-2x\int_{\Gamma_u}e^{-2ux-u^2}u^{-n}\,du-2\int_{\Gamma_u}e^{-2ux-u^2}u^{-n+1}\,du-n\int_{\Gamma_u}e^{-2ux-u^2}u^{-n-1}\,du=0,$$

i.e., $$2xU_{n-1}(x)+U_n(x)+2(n-1)\,U_{n-2}(x)=0. \qquad \text{(IX)}$$

Hence $$nU_n(v)-vU'_n(v)=nU_n(v)+2nvU_{n-1}(v)\quad[\text{by (III)}]$$

$$=-2n(n-1)\,U_{n-2}(v)\quad[\text{by (IX)}].$$

Further $$V_n(v)=e^{v^2}U_{-n-1}(-iv)\quad[\text{by (II)}]\,;$$

therefore

$$nV_n(v)-vV'_n(v)$$

$$=e^{v^2}[nU_{-n-1}(-iv)-2v^2U_{-n-1}(-iv)+2(n+1)ivU_{-n-2}(-iv)]\ [\text{by (III)}]$$

$$=e^{v^2}[nU_{-n-1}(-iv)+iv\left\{2ivU_{-n-1}(-iv)+2(n+1)\,U_{-n-2}(-iv)\right\}]$$

$$=e^{v^2}[nU_{-n-1}(-iv)+ivU_{-n}(-iv)]\quad[\text{by (IX)}]$$

$$=\tfrac{1}{2}e^{v^2}U_{-n+1}(-iv)\quad[\text{by (IX)}].$$

Hence $ny_n(v)-vy'_n(v)$ must have the form

$$c_1U_{n-2}(v)+c_2e^{v^2}U_{-n+1}(-iv),$$

since $y_n(v)$, being a solution of Hermite's differential equation, has the form

$$\gamma_1U_n(v)+\gamma_2V_n(v).$$

Hence we have to examine

$$\operatorname*{Lt}_{t\to\infty} e^{-t^2} t^{n+1} U_{n-2}(xt) \quad \text{and} \quad \operatorname*{Lt}_{t\to\infty} e^{-t^2+x^2t^2} t^{n+1} U_{-n+1}(-ixt).$$

On applying the results of § 3, assuming n is finite, the first of these expressions tends to zero if $|x|<1$, and the second tends to zero if the real part of $\{-1+x^2+|x|^2\}$ is negative.

In applying the results of paragraph 3, it is to be remembered that κ is merely the smallest integer greater than $\frac{|n|}{2}$.

The condition that the real part of $\{-1+x^2+|x|^2\}$ is negative may be more briefly expressed by saying that the real part of x is numerically less than $\frac{1}{\sqrt{2}}$. For, on putting $x=re^{\phi i}$,

$$\begin{aligned} -1+x^2+|x|^2 &= -1+r^2\cos 2\phi+r^2+ir^2\sin 2\phi \\ &= -1+2r^2\cos^2\phi+ir^2\sin 2\phi. \end{aligned}$$

Both conditions are satisfied if $|x|<\frac{1}{\sqrt{2}}$.

The following results, however, are obviously true.

If $|x|<1$, then $\int_{\Gamma_t} e^{-t^2} t^n U_n(xt)\,dt$ satisfies Legendre's equation.

If |the real part of $x|<\frac{1}{\sqrt{2}}$; then $\int_{\Gamma_t} e^{-t^2} t^n V_n(xt)\,dt$ is a solution.

5. Let $$y=\frac{1}{i.\pi^{\frac{1}{2}}.\Gamma(n+1)\sin n\pi}\int_\Gamma e^{-t^2} t^n U_n(xt)\,dt.$$

This is a solution of Legendre's equation if $|x|<1$. It will be identified with $P_n(x)$ by shewing that when x tends to zero y tends to $P_n(0)$, and dy/dx tends towards $P_n'(0)$.

$P_n(x)=F\left(-n,\ n+1,\ 1,\ \frac{1-x}{2}\right)$ if $|1-x|<2$ (see, for instance, Whittaker, *Mod. Anal.*, § 118).

Therefore $$P_n(0)=F(-n,\ n+1,\ 1,\ \tfrac{1}{2}).$$

But, by one of Kummer's formulæ,

$$F(2a,\ 2\beta,\ a+\beta+\tfrac{1}{2},\ x) = F[a,\ \beta,\ a+\beta+\tfrac{1}{2},\ 4x(1-x)]$$

if $|x|<1$, and if $|4x(1-x)|\leqslant 1$.

Hence

$$P_n(0) = F\left(-\frac{n}{2},\ \frac{n+1}{2},\ 1,\ 1\right) = \frac{\Gamma(1)\,\Gamma\left(1+\frac{n}{2}-\frac{n+1}{2}\right)}{\Gamma\left(1+\frac{n}{2}\right)\Gamma\left(1-\frac{n+1}{2}\right)}$$

$$= \frac{\Gamma(\tfrac{1}{2})\,\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(1+\frac{n}{2}\right)\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(1-\frac{n+1}{2}\right)} = \frac{\Gamma\left(\frac{n+1}{2}\right)\cos\frac{n\pi}{2}}{\pi^{\frac{1}{2}}.\Gamma\left(\frac{n}{2}+1\right)}. \quad \text{(XII)}$$

Further $$(x^2-1)\frac{dP_n(x)}{dx} = nx\,P_n(x) - n\,P_{n-1}(x);$$

therefore $$P_n'(0) = nP_{n-1}(0) = \frac{2.\Gamma\left(\frac{n+2}{2}\right)\sin\frac{n\pi}{2}}{\pi^{\frac{1}{2}}.\Gamma\left(\frac{n+1}{2}\right)}. \quad \text{(XIII)}$$

On turning back to the series for $U_n(x)$,

$$U_n(0) = \frac{\Gamma(n+1)\cos\frac{n\pi}{2}}{e^{n\pi i}.\Gamma\left(\frac{n}{2}+1\right)}, \qquad U_n'(0) = \frac{2.\Gamma(n+1)\sin\frac{n\pi}{2}}{e^{n\pi i}.\Gamma\left(\frac{n+1}{2}\right)}.$$

Hence

$$[y]_{x=0} = \frac{1}{\pi^{\frac{1}{2}}.i.\Gamma(n+1)\sin n\pi}\,\frac{\Gamma(n+1)\cos\frac{n\pi}{2}}{e^{n\pi i}.\Gamma\left(\frac{n}{2}+1\right)}$$

$$\times ie^{(n+1)\pi i}\sin(n+1)\pi.\Gamma\left(\frac{n+1}{2}\right) \quad \text{[using (IV)]}$$

$$= \frac{\cos\frac{n\pi}{2}\,\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{1}{2}}.\Gamma\left(\frac{n}{2}+1\right)} = P_n(0),$$

and

$$\left[\frac{dy}{dx}\right]_{x=0} = \frac{1}{\pi^{\frac{1}{2}}.i.\Gamma(n+1)\sin n\pi}\int_{\Gamma_t} e^{-t^2}t^{n+1}\left[-2nU_{n-1}(xt)\right]dt \quad \text{[by (III)]}$$

$$= \frac{-2n}{\pi^{\frac{1}{2}}.i.\Gamma(n+1)\sin n\pi}\,\frac{\Gamma(n)\cos\frac{n-1}{2}\pi}{e^{(n-1)\pi i}.\Gamma\left(\frac{n+1}{2}\right)}$$

$$\times ie^{(n+2)\pi i}\sin(n+2)\pi.\Gamma\left(\frac{n+2}{2}\right)$$

$$= \frac{2\Gamma\left(\frac{n}{2}+1\right)\sin\frac{n\pi}{2}}{\pi^{\frac{1}{2}}.\Gamma\left(\frac{n+1}{2}\right)} = P_n'(0).$$

Hence throughout the domain $|x|<1$, which contains no singularities of $P_n(x)$,

$$P_n(x) = \frac{1}{i.\pi^{\frac{1}{2}}.\Gamma(n+1)\sin n\pi}\int_{\Gamma_t} e^{-t^2}t^n\,U_n(xt)\,dt.$$

If the real part of $(n+1)$ is positive, this result may be put into the form

$$P_n(x) = \frac{e^{2n\pi i}-1}{i.\pi^{\frac{1}{2}}.\Gamma(n+1)\sin n\pi}\int_0^\infty e^{-t^2}t^n\,U_n(xt)\,dt$$

$$= \frac{2e^{n\pi i}}{\pi^{\frac{1}{2}}.\Gamma(n+1)}\int_0^\infty e^{-t^2}t^n\,U_n(xt)\,dt.$$

If n is a positive integer, then $t^nU_n(xt)$ is an even function of t, and in that case

$$P_n(x) = \frac{(-1)^n}{\pi^{\frac{1}{2}}.n!}\int_{-\infty}^\infty e^{-t^2}t^n\,U_n(xt)\,dt,$$

which is the result obtained in § 1.

6. Investigation of the corresponding relation between V_n and Q_n.

We shall follow the same procedure as in the preceding paragraph, but in determining $Q_n(0)$ a difficulty is encountered, for in defining $Q_n(x)$ a slit is made in the plane of x from $+1$ to -1 along the real axis.

Now we have shown that $\int_{\Gamma_t} e^{-t^2}t^n\,V_n(xt)\,dt$ exists, at any rate, inside the domain $|\text{real part of } x| < \frac{1}{\sqrt{2}}$.

The cut between -1 and $+1$ separates this strip into two half-strips, so we shall discuss the relation between $Q_n(x)$ and this integral, first, for a point x in the upper half-strip, and secondly for a point x in the lower half-strip.

For a point x actually on the real axis between $+\frac{1}{\sqrt{2}}$ and $-\frac{1}{\sqrt{2}}$ we shall use the usual definition

$$Q_n(x) = \operatorname*{Lt}_{\epsilon \to 0} \frac{Q_n(x+\epsilon i)+Q_n(x-\epsilon i)}{2}.$$

Let us first discuss the value of

$$\operatorname*{Lt}_{x \to 0} Q_n(x),$$

where x approaches 0 from above the real axis, which, for the sake of brevity, we will denote by

$$\operatorname*{Lt}_{x \to 0} Q_n(x).$$

Now it is well known that, if the real part of $(n+1)$ is positive, and if x is not a real number less than -1, then

$$Q_n(x) = \int_0^\infty \{x+\sqrt{x^2-1}\cosh\theta\}^{-n-1}\, d\theta,$$

where that branch of $\{x+\sqrt{x^2-1}\cosh\theta\}^{-n-1}$ is to be taken which, on continuation along a straight line from x to 1, takes the value 1 at $x=1$, ... (see, *e.g.*, Whittaker, *Mod. Anal.*, § 125).

In order to pass from a real value of x greater than unity to a point in the upper half-plane near the origin we have to avoid $x=1$ by means of a semicircle in the upper half of the plane. Thus $(x-1)$ has its argument *increased* by π on avoiding $x=1$, so that for real values of x less than 1,

$$(x-1) = (1-x)\, e^{\pi i},$$

if x approaches the real axis from above. Thus

$$\operatorname*{Lt}_{x \to 0} Q_n(x) = \operatorname*{Lt}_{x \to 0} \int_0^\infty \{x+\sqrt{(1-x^2)\, e^{\pi i}}\cosh\theta\}^{-n-1}\, d\theta \quad [\text{if } R(n+1)>0]$$

$$= e^{-\frac{1}{2}(n+1)\pi i} \int_0^\infty \cosh^{-n-1}\theta\, d\theta \quad [\text{if } R(n+1)>0].$$

Now $$\int_0^\infty \cosh^{-n-1}\theta\, d\theta = \int_0^1 \xi^{n+1}\xi^{-1}(1-\xi^2)^{-\frac{1}{2}}\, d\xi,$$

on putting $$\cosh\theta = \frac{1}{\xi};$$

therefore

$$\operatorname*{Lt}_{x\to 0} Q_n(x) = \tfrac{1}{2}\frac{\Gamma\left(\frac{n+1}{2}\right)\pi^{\frac{1}{2}}}{\Gamma\left(\frac{n+2}{2}\right)} e^{-\frac{1}{2}(n+1)\pi i} \quad [\text{if } R(n+1) > 0]. \qquad \text{(XIV}a\text{)}$$

Now for all values of n,

$$nQ_n(x) - (2n-1)\,x\,Q_{n-1}(x) + (n-1)\,Q_{n-2}(x) = 0$$

(see, *e.g.*, Whittaker, *Mod. Anal.*, § 124); therefore

$$\operatorname*{Lt}_{x\to 0} Q_{n-2}(x) = -\frac{n}{n-1}\operatorname*{Lt}_{x\to 0} Q_n(x),$$

and by means of this relation it is clear that the condition $R(n+1) > 0$, of (XIVa), may be removed.

Again, since $(x^2-1)\dfrac{dQ_n(x)}{dx} = nxQ_n(x) - nQ_{n-1}(x)$;

therefore $\operatorname*{Lt}_{x\to} \dfrac{dQ_n(x)}{dx} = n \operatorname*{Lt}_{x\to 0} Q_{n-1}(x)$

$$= \frac{n}{2}\frac{\Gamma\left(\frac{n}{2}\right)\pi^{\frac{1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} e^{-(\frac{1}{2}n\pi i)} = \frac{\Gamma\left(\frac{n+2}{2}\right)\pi^{\frac{1}{2}}.e^{-(\frac{1}{2}n\pi i)}}{\Gamma\left(\frac{n+1}{2}\right)}. \qquad \text{(XV}a\text{)}$$

Similarly, $$\operatorname*{Lt}_{x\to 0} Q_n(x) = \tfrac{1}{2}\frac{\Gamma\left(\frac{n+1}{2}\right)\pi^{\frac{1}{2}}}{\Gamma\left(\frac{n+2}{2}\right)} e^{+\frac{1}{2}(n+1)\pi i}, \qquad \text{(XIV}b\text{)}$$

and $$\operatorname*{Lt}_{x\to 0} \frac{dQ_n(x)}{dx} = \frac{\Gamma\left(\frac{n+2}{2}\right)\pi^{\frac{1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} e^{+(\frac{1}{2}n\pi i)}. \qquad \text{(XV}b\text{)}$$

Now consider $$y = \frac{2^n}{e^{\frac{1}{2}n\pi i}\sin n\pi}\int_{\Gamma_t} e^{-t^2} t^n V_n(xt)\,dt,$$

where x is restricted to the upper half of the strip $|R(x)| < \dfrac{1}{\sqrt{2}}$.

Now $$V_n(v) = e^{v^2} U_{-n-1}(-iv);$$

therefore $$V_n(0) = U_{-n-1}(0) = \frac{e^{(n+1)\pi i}\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma(n+1)} \quad \text{[by (VI)]}, \qquad \text{(XVI)}$$

and $$V_n'(0) = -iU'_{-n-1}(0) = -2i(n+1)\,U_{-n-2}(0) \quad \text{[by (III)]}$$

$$= \frac{-2i(n+1)\,e^{(n+2)\pi i}\Gamma\left(\frac{n+2}{2}\right)}{2\Gamma(n+2)} = \frac{ie^{(n+1)\pi i}\Gamma\left(\frac{n+2}{2}\right)}{\Gamma(n+1)}; \qquad \text{(XVII)}$$

therefore

$$[y]_{x=0} = \frac{2^n}{e^{\frac{1}{2}n\pi i}\sin n\pi}\,\frac{e^{(n+1)\pi i}\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma(n+1)}\,ie^{(n+1)\pi i}\sin(n+1)\,\pi\Gamma\left(\frac{n+1}{2}\right)$$

[by (XVI) and (IV)]

$$= \frac{-2^n.i.\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2e^{\frac{1}{2}n\pi i}\Gamma(n+1)}\,\frac{\Gamma\left(\frac{n+2}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}$$

$$= \frac{-2^n.i.\Gamma\left(\frac{n+1}{2}\right)\Gamma(n+1)\,\pi^{\frac{1}{2}}.2^{-n}}{2e^{\frac{1}{2}n\pi i}\Gamma(n+1)\,\Gamma\left(\frac{n+2}{2}\right)} \quad \text{[using formula } (\alpha)\text{]}$$

$$= \frac{e^{-\frac{1}{2}(n+1)\pi i}\Gamma\left(\frac{n+1}{2}\right)\pi^{\frac{1}{2}}}{2\Gamma\left(\frac{n+2}{2}\right)} = \operatorname*{Lt}_{x\to 0} Q_n(x) \quad \text{[by (XIV}\alpha\text{)]}.$$

While

$$\left[\frac{dy}{dx}\right]_{x=0} = \frac{2^n}{e^{\frac{1}{2}n\pi i}\sin n\pi}\int_{\Gamma_t} e^{-t^2}t^{n+1}\,V_n'(0)\,dt$$

$$= \frac{2^n}{e^{\frac{1}{2}n\pi i}\sin n\pi}\,\frac{ie^{(n+1)\pi i}\Gamma\left(\frac{n+2}{2}\right)}{\Gamma(n+1)}\,ie^{(n+2)\pi i}\sin(n+2)\,\pi.\Gamma\left(\frac{n+2}{2}\right)$$

[by (XVII) and (IV)]

$$= \frac{2^n e^{-(\frac{1}{2}n\pi i)}\Gamma\left(\frac{n+2}{2}\right)\pi^{\frac{1}{2}}.\Gamma(n+1)\,2^{-n}}{\Gamma(n+1)\,\Gamma\left(\frac{n+1}{2}\right)}$$

$$= \frac{e^{-(\frac{1}{2}n\pi i)}\Gamma\left(\frac{n+2}{2}\right)\pi^{\frac{1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} = \operatorname*{Lt}_{x\to 0}\frac{dQ_n(x)}{dx} \quad \text{(by XV}\alpha\text{)}.$$

Hence we have proved that, throughout the upper half of the domain $|R(x)| < \frac{1}{\sqrt{2}}$, y satisfies Legendre's equation, and that when x is restricted to this domain and approaches the origin, then y and dy/dx respectively take the values that $Q_n(x)$ and $\frac{dQ_n(x)}{dx}$ take on approaching the origin in the same way.

Hence throughout this domain

$$Q_n(x) = y = \frac{2^n}{e^{\frac{1}{2}n\pi i}\sin n\pi}\int_\Gamma e^{-t^2}t^n V_n(xt)\,dt.$$

On turning our attention to the lower half of the strip it is to be noticed that

$$\frac{\underset{x\to 0}{\mathrm{Lt}}\ \frac{dQ_n(x)}{dx}}{\underset{x\to 0}{\mathrm{Lt}}\ Q_n(x)} = -\frac{\underset{x\to 0}{\mathrm{Lt}}\ \frac{dQ_n(x)}{dx}}{\underset{x\to 0}{\mathrm{Lt}}\ Q_n(x)}.$$

Hence in seeking for a relation holding good in the lower half-strip it is not sufficient to alter the coefficient of the integral, and so we are led to consider

$$y_1 = -\frac{2^n}{e^{\frac{1}{2}n\pi i}\sin n\pi}\int_{\Gamma_t} e^{-t^2}t^n V_n(-xt)\,dt.$$

Clearly $[y_1]_{x=0} = e^{(n+1)\pi i}[y]_{x=0}$

$$= \frac{e^{(n+1)\pi i}e^{-\frac{1}{2}(n+1)\pi i}\Gamma\left(\frac{n+1}{2}\right)\pi^{\frac{1}{2}}}{2\Gamma\left(\frac{n+2}{2}\right)} = \underset{x\to 0}{\mathrm{Lt}}\ Q_n(x) \quad \text{[by (XIV}b\text{)]},$$

while $\left[\frac{dy_1}{dx}\right]_{x=0} = e^{(n+1)\pi i}\frac{2^n}{e^{\frac{1}{2}n\pi i}\sin n\pi}(-1)\int_{\Gamma_t} e^{-t^2}t^{n+1}V_n'(0)\,dt$

$$= e^{n\pi i}\left[\frac{dy}{dx}\right]_{x=0} = \frac{e^{\frac{1}{2}n\pi i}\Gamma\left(\frac{n+2}{2}\right)\pi^{\frac{1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} = \underset{x\to 0}{\mathrm{Lt}}\left(\frac{dQ_n(x)}{dx}\right)$$

[by (XVb)]

Hence throughout the lower strip

$$Q_n(x) = y_1 = \frac{-2^n}{e^{\frac{1}{2}n\pi i}\sin n\pi}\int_{\Gamma_t} e^{-t^2}t^n V_n(-xt)\,dt.$$

Thus, throughout the domain $|R(x)| < \frac{1}{\sqrt{2}}$,

$$Q_n(x) = \frac{2^n . i . e^{\mp \frac{1}{2}(n+1)\pi i}}{e^{2n\pi i} \sin n\pi} \int_{\Gamma_t} e^{-t^2} t^n V_n(\pm xt)\, dt,$$

where the upper or lower sign is to be taken according as x lies above or below the real axis.

If x lies on the real axis between $-\frac{1}{\sqrt{2}}$ and $+\frac{1}{\sqrt{2}}$,

$$Q_n(x) = \frac{2^n . i}{e^{2n\pi i} \sin n\pi} \int_{\Gamma_t} e^{-t^2} t^n \frac{e^{-\frac{1}{2}(n+1)\pi i} V_n(xt) + e^{\frac{1}{2}(n+1)} V_n(-xt)}{2}\, dt.$$

For a value of x lying on the real axis between $+1$ and -1, it is known that

$$\operatorname*{Lt}_{\epsilon \to 0} [Q_n(x+\epsilon i) - Q_n(x-\epsilon i)] = -i\pi P_n(x) \ldots$$

(Heine, *Kugelfunctionen*, Vol. 1, p. 130).

It is of interest to verify that the preceding results are consistent with this formula.

Using the results established in this paragraph,

$$\operatorname*{Lt}_{\epsilon \to 0} [Q_n(x+\epsilon i) - Q_n(x-\epsilon i)]$$
$$= \frac{2^n . i}{e^{2n\pi i} \sin n\pi} \int_{\Gamma_t} e^{-t^2} t^n \left[e^{-\frac{1}{2}(n+1)\pi i} V_n(xt) - e^{\frac{1}{2}(n+1)} V_n(-xt)\right] dt,$$

if x lies on the real axis between $+\frac{1}{\sqrt{2}}$ and $-\frac{1}{\sqrt{2}}$.

Now, since $U_n(x)$, $V_n(x)$, and $V_n(-x)$ are three solutions of Hermite's equation, there must exist a relation of the form

$$U_n(x) = A V_n(x) + B V_n(-x),$$

where A and B are constants. We have

$$U_n(0) = (A+B) V_n(0) \quad \text{and} \quad U_n'(0) = (A-B) V_n'(0);$$

therefore $A+B = \frac{U_n(0)}{V_n(0)} = \frac{e^{-n\pi i} \Gamma(n+1) \cos \frac{n\pi}{2}}{\Gamma\left(\frac{n}{2}+1\right)} \frac{2\Gamma(n+1)}{e^{(n+1)\pi i} \Gamma\left(\frac{n+1}{2}\right)}$

[by (V) and (XVI)];

therefore $A+B=\dfrac{2e^{-(2n+1)\pi i}\,\Gamma(n+1)\cos\dfrac{n\pi}{2}}{\pi^{\frac{1}{2}}.2^{-n}}$,

and $$A-B=\frac{U_n'(0)}{V_n'(0)}=\frac{2.e^{-n\pi i}\,\Gamma(n+1)\sin\dfrac{n\pi}{2}}{\Gamma\left(\dfrac{n+1}{2}\right)}\,\frac{\Gamma(n+1)}{i.e^{(n+1)\pi i}\,\Gamma\left(\dfrac{n+2}{2}\right)}$$

[by (V) and (XVII)];

therefore $A-B=\dfrac{-2.i.e^{-(2n+1)\pi i}\,\Gamma(n+1)\sin\dfrac{n\pi}{2}}{\pi^{\frac{1}{2}}.2^{-n}}$.

Hence $$A=\frac{2^n.e^{-(2n+1)\pi i}\,\Gamma(n+1)}{\pi^{\frac{1}{2}}}e^{-(\frac{1}{2}n\pi i)},$$

$$B=\frac{2^n.e^{-(2n+1)\pi i}\,\Gamma(n+1)}{\pi^{\frac{1}{2}}}e^{+(\frac{1}{2}n\pi i)};$$

therefore

$$U_n(x)=\frac{2^n.e^{-(2n+1)\pi i}\,\Gamma(n+1)}{\pi^{\frac{1}{2}}}\left[e^{-(\frac{1}{2}n\pi i)}\,V_n(x)+e^{\frac{1}{2}n\pi i}\,V_n(-x)\right];$$

therefore

$$U_n(xt)=\frac{2^n.e^{-(2n+1)\pi i}\,\Gamma(n+1)i}{\pi^{\frac{1}{2}}}\left[e^{-\frac{1}{2}(n+1)\pi i}\,V_n(xt)-e^{\frac{1}{2}(n+1)\pi i}\,V_n(-xt)\right].$$

Hence $$\operatorname*{Lt}_{\epsilon\to 0}\left[Q_n(x+\epsilon i)-Q_n(x-\epsilon i)\right]$$

$$=\frac{2^n}{e^{2n\pi i}\sin n\pi}\,\frac{\pi^{\frac{1}{2}}}{2^n.e^{-(2n+1)\pi i}\,\Gamma(n+1)}\int_{\Gamma_t}e^{-t^2}t^n\,U_n(xt)\,dt$$

$$=-\frac{\pi^{\frac{1}{2}}}{\sin n\pi.\,\Gamma(n+1)}\,\frac{\pi^{\frac{1}{2}}.i.\,\Gamma(n+1)\sin n\pi}{1}\,P_n(x)\quad\text{(by § 5)}$$

$$=-i\pi.P_n(x).\qquad\text{Q.E.D.}$$

7a. Let $y_n(v)$ satisfy the equation

$$(1-v^2)\frac{d^2y}{dv^2}-2v\frac{dy}{dv}+n(n+1)\,y=0.$$

Secondly, let L be a contour in the plane of t, beginning at a point where the real part of t is infinitely great and positive and greater in absolute

magnitude than the imaginary part of t, finishing at a point where the real part of t is infinitely great and negative and greater in absolute magnitude than the imaginary part of t, and avoiding the origin and the points $\pm ix$. Then it will be shown that

$$z = \int_L e^{-t^2} t^{-n} y_n\left(\frac{ix}{t}\right) dt$$

is a solution of Hermite's differential equation. For

$$\frac{d^2z}{dx^2} - 2x\frac{dz}{dx} + 2nz$$

$$= \int_L e^{-t^2} t^{-n}\left[-\frac{1}{t^2}\, y_n''\left(\frac{ix}{t}\right) - \frac{2ix}{t}\, y_n'\left(\frac{ix}{t}\right) + 2n y_n\left(\frac{ix}{t}\right)\right] dt.$$

Now

$$\frac{d}{dt}\left[e^{-t^2} t^{-n-2} ix y_n'\left(\frac{ix}{t}\right) - e^{-t^2} t^{-n-1} n y_n\left(\frac{ix}{t}\right)\right]$$

$$= e^{-t^2} t^{-n}\left[-2\frac{ix}{t}\, y_n'\left(\frac{ix}{t}\right) + 2n y_n\left(\frac{ix}{t}\right) - (n+2)\frac{ix}{t^3}\, y_n'\left(\frac{ix}{t}\right)\right.$$

$$\left. + n(n+1)\frac{1}{t^2}\, y_n\left(\frac{ix}{t}\right) + \frac{x^2}{t^4}\, y_n''\left(\frac{ix}{t}\right) + n\frac{ix}{t^3}\, y_n'\left(\frac{ix}{t}\right)\right]$$

$$= e^{-t^2} t^{-n}\left[-\frac{1}{t^2}\, y_n''\left(\frac{ix}{t}\right) - \frac{2ix}{t}\, y_n'\left(\frac{ix}{t}\right) + 2n y_n\left(\frac{ix}{t}\right)\right.$$

$$\left. + \frac{1}{t^2}\left\{\left(1+\frac{x^2}{t^2}\right) y_n''\left(\frac{ix}{t}\right) - \frac{2ix}{t}\, y_n'\left(\frac{ix}{t}\right) + n(n+1)\, y_n\left(\frac{ix}{t}\right)\right\}\right]$$

$$= e^{-t^2} t^{-n}\left[-\frac{1}{t^2}\, y_n''\left(\frac{ix}{t}\right) - \frac{2ix}{t}\, y_n'\left(\frac{ix}{t}\right) + 2n y_n\left(\frac{ix}{t}\right)\right].$$

Hence
$$\frac{d^2z}{dx^2} - 2z\frac{dz}{dx} + 2nz = 0,$$

if, at the extremities of L, $e^{-t^2} t^{-n-1}\left[\frac{ix}{t}\, y_n'\left(\frac{ix}{t}\right) - n y_n\left(\frac{ix}{t}\right)\right]$ is zero.

Now, for infinite values of t, $y_n'\left(\frac{ix}{t}\right)$ and $y_n\left(\frac{ix}{t}\right)$ are finite quantities, since $y_n(v) = AP_n(v) + BQ_n(v)$, and the values of $P_n(0)$, $P_n'(0)$, $Q_n(0)$, $Q_n'(0)$ are finite [see (XII), (XIII), (XIV), and (XV)].

And, from the nature of the end-points of L, $e^{-t^2}t^{-n-1}$ becomes zero at each end-point.

Hence the proposition is proved.

7*b*. It will be proved that

$$U_n(x) = \int_{+\infty}^{-\infty} \frac{\Gamma(n+1)}{e^{\frac{1}{2}(n+2)\pi i}\,\pi^{\frac{1}{2}}}\, e^{-t^2}t^{-n}P_n\left(\frac{ix}{t}\right)dt,$$

where the origin is avoided by an anti-clockwise semicircle [and where the point $(-ix)$ is avoided by a similar semicircle if x is purely imaginary].

In proving this we shall require the value of

$$I_n = \int_{\infty}^{-\infty} e^{-t^2}t^{-n}dt,$$

where the origin is avoided in the manner mentioned.

It is easily seen that

$$I_n = ie^{-\frac{1}{2}(n+1)\pi i}\sin\left(-\frac{n+1}{2}\pi\right)\Gamma\left(-\frac{n-1}{2}\right)$$

$$= \frac{e^{-(\frac{1}{2}n\pi i)}\pi}{\Gamma\left(\frac{n+1}{2}\right)} \text{ for all values of } n. \qquad \text{(XVIII)}$$

Now, let $$z = \frac{\Gamma(n+1)}{e^{\frac{1}{2}(n+2)\pi i}\,\pi^{\frac{1}{2}}}\int_{+\infty}^{-\infty} e^{-t^2}t^{-n}P_n\left(\frac{ix}{t}\right)dt,$$

where singularities of the integrand are to be avoided in the prescribed manner. Then

$$[z]_{x=0} = \frac{\Gamma(n+1)}{e^{\frac{1}{2}(n+2)\pi i}\,\pi^{\frac{1}{2}}}\,\frac{(-1)\,e^{-(\frac{1}{2}n)\pi i}\,\pi}{\Gamma\left(\frac{n+1}{2}\right)}\,\frac{\Gamma\left(\frac{n+1}{2}\right)\cos\frac{n\pi}{2}}{\pi^{\frac{1}{2}}.\Gamma\left(\frac{n+2}{2}\right)} \quad \text{[by (XII)]}$$

$$= \frac{\Gamma(n+1)\cos\frac{n\pi}{2}}{e^{n\pi i}.\Gamma\left(\frac{n+2}{2}\right)} = U_n(0) \quad \text{[by (V)]},$$

while

$$\left[\frac{dz}{dx}\right]_{x=0} = \frac{\Gamma(n+1)}{e^{\frac{1}{2}(n+2)\pi i}\,\pi^{\frac{1}{2}}}\, iP'_n(0)\, I_{n+1}$$

$$= \frac{\Gamma(n+1)}{e^{\frac{1}{2}(n+2)\pi i}\,\pi^{\frac{1}{2}}}\, i\, \frac{2\Gamma\left(\frac{n+2}{2}\right)\sin\frac{n\pi}{2}}{\pi^{\frac{1}{2}}.\Gamma\left(\frac{n+1}{2}\right)}\, \frac{(-1)\,e^{-\frac{1}{2}(n+1)\pi i}\,\pi}{\Gamma\left(\frac{n+2}{2}\right)}$$

[by (XIII) and (XVIII)]

$$= \frac{2.\Gamma(n+1)\sin\frac{n\pi}{2}}{e^{n\pi i}.\Gamma\left(\frac{n+1}{2}\right)} = U'_n(0) \quad [\text{by (V)}],$$

and z is a solution of Hermite's equation.

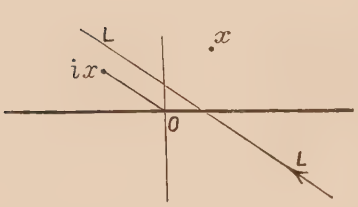
Therefore $z = U_n(x)$, and this is the formula to be established.

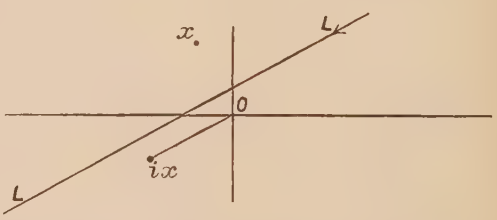
7c. In searching for a similar relation between V_n and Q_n the path of integration along which t travels must be such that ix/t does not cross over the cut between -1 and $+1$ used in defining $Q_n(z)$. To secure this we choose L to be a straight line parallel to the straight line joining 0 to ix, and proceeding to infinity in each direction.

In order that L shall satisfy the requirements of § 7a, it is clear that the argument of x must lie between $\frac{1}{4}\pi$ and $\frac{3}{4}\pi$, or between $-\frac{1}{4}\pi$ and $-\frac{3}{4}\pi$.

Let us further define L so that the argument of ix/t always lies between 0 and π, and let t travel along L so that the angular motion of t about the origin is anti-clockwise. Then the two cases occur illustrated by the following diagrams.

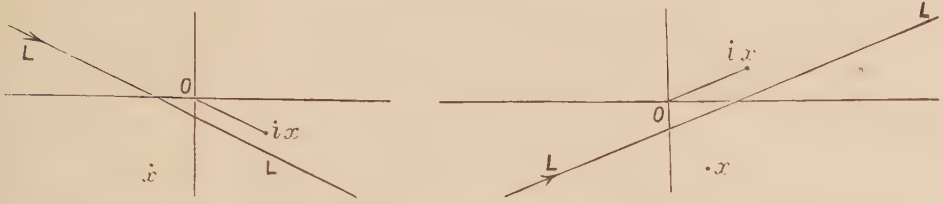
CASE I.





Here the argument of t at the beginning of the path lies between $-\frac{1}{4}\pi$ and $+\frac{1}{4}\pi$, the argument of x lying between $\frac{1}{4}\pi$ and $\frac{3}{4}\pi$.

Case II.



Here the argument of t at the beginning of the path lies between $-\frac{5}{4}\pi$ and $-\frac{3}{4}\pi$, the argument of x lying between $-\frac{3}{4}\pi$ and $-\frac{1}{4}\pi$.

Now in each of the two cases let us examine the integral

$$M = \int_L e^{-t^2} t^{-n}\, dt,$$

where that branch of t^{-n} is to be taken which, on the continuation of t along any path till its argument is zero and its modulus 1, takes the value 1.

In Case I, L may clearly be deformed into a contour beginning at positive infinity on the real axis, avoiding the origin by an anti-clockwise semi-circle, and ending at negative infinity on the real axis, that is to say, in Case I, $M = I_n$.

In Case II, the argument of t at the beginning of the path is between $-\frac{5}{4}\pi$ and $-\frac{3}{4}\pi$; so on applying the substitution $t = e^{-\pi i}.u$, it is clear that in this case $M = e^{(n+1)\pi i}.I_n = K_n$, say.

First suppose that the argument of x lies between $\frac{1}{4}\pi$ and $\frac{3}{4}\pi$ and examine

$$y = \frac{e^{(2n+1)\pi i}}{2^n.\pi i} \int_L e^{-t^2} t^{-n} Q_n\left(\frac{ix}{t}\right) dt.$$

This satisfies Hermite's differential equation, and when L is defined and x is restricted as above, then $Q_n(ix/t)$ is a one-valued continuous function of (ix/t) for all values of t along L. Now suppose x travels along a straight line to the origin, then the one-valued character of $Q_n(ix/t)$ will be maintained for all values of t along L, and ix/t will approach the origin from above the real axis, and when x arrives at the origin,

$$Q_n\left(\frac{ix}{t}\right) \text{ takes the value } \tfrac{1}{2}\frac{\Gamma\left(\frac{n+1}{2}\right)\pi^{\frac{1}{2}}.e^{-\frac{1}{2}(n+1)\pi i}}{\Gamma\left(\frac{n+2}{2}\right)} \quad [\text{by (XIV}a)],$$

$$Q_n'\left(\frac{ix}{t}\right) \text{ takes the value } \frac{\Gamma\left(\frac{n+2}{2}\right)\pi^{\frac{1}{2}}.e^{-(\frac{1}{2}n\pi i)}}{\Gamma\left(\frac{n+1}{2}\right)} \quad [\text{by (XV}a)];$$

therefore

$$[y]_{x=0} = \frac{e^{(2n+1)\pi i}}{2^n.\pi i}\,\tfrac{1}{2}\,\frac{\Gamma\left(\frac{n+1}{2}\right)\pi^{\frac{1}{2}}.e^{-\frac{1}{2}(n+1)\pi i}}{\Gamma\left(\frac{n+2}{2}\right)}\,I_n$$

$$= \frac{e^{(2n+1)\pi i}}{2^n.\pi i}\,\tfrac{1}{2}\,\frac{\Gamma\left(\frac{n+1}{2}\right)\pi^{\frac{1}{2}}.e^{-\frac{1}{2}(n+1)\pi i}}{\Gamma\left(\frac{n+2}{2}\right)}\,\frac{(-1)\,e^{-(\frac{1}{2}n\pi i)}\,\pi}{\Gamma\left(\frac{n+1}{2}\right)} \quad \text{[by (XVIII)]}$$

$$= \frac{e^{(n-1)\pi i}.\pi^{\frac{1}{2}}}{2^{n+1}.\Gamma\left(\frac{n+2}{2}\right)} = \frac{e^{(n-1)\pi i}.\pi^{\frac{1}{2}}.\Gamma\left(\frac{n+1}{2}\right)}{2^{n+1}.\Gamma\left(\frac{n+2}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}$$

$$= \frac{e^{(n-1)\pi i}.\Gamma\left(\frac{n+1}{2}\right)}{2.\Gamma(n+1)} = [V_n(x)]_{x=0} \quad \text{[by (XVI)]},$$

while

$$\left[\frac{dy}{dx}\right]_{x=0} = \frac{e^{(2n+1)\pi i}}{2^n.\pi i}\,i.I_{n+1}\,\frac{\Gamma\left(\frac{n+2}{2}\right)\pi^{\frac{1}{2}}.e^{-(\frac{1}{2}n\pi i)}}{\Gamma\left(\frac{n+1}{2}\right)}$$

$$= \frac{e^{(2n+1)\pi i}}{2^n.\pi}\,\frac{(-1)\,e^{-\frac{1}{2}(n+1)\pi i}\,\pi}{\Gamma\left(\frac{n+2}{2}\right)}\,\frac{\Gamma\left(\frac{n+2}{2}\right)\pi^{\frac{1}{2}}.e^{-(\frac{1}{2}n\pi i)}}{\Gamma\left(\frac{n+1}{2}\right)}$$

$$= \frac{i.e^{(n-1)\pi i}\,\pi^{\frac{1}{2}}}{2^n.\Gamma\left(\frac{n+1}{2}\right)} = \frac{i.e^{(n-1)\pi i}\,\Gamma\left(\frac{n+2}{2}\right)}{\Gamma(n+1)} = \left[\frac{d}{dx}\,V_n(x)\right]_{x=0}$$

[by (XVII)];

therefore $$y = V_n(x).$$

Thus, if the argument of x lies between $\frac{1}{4}\pi$ and $\frac{3}{4}\pi$, and if L is an infinite straight line parallel to the straight line joining ix to the origin, and drawn on that side of this straight line which makes the argument of ix/t lie always between 0 and π as t describes the straight line L in a direction anti-clockwise about the origin, then

$$V_n(x) = \frac{e^{(2n+1)\pi i}}{2^n.\pi i}\int_L e^{-t^2}t^{-n}\,Q_n\left(\frac{ix}{t}\right)dt,$$

where that branch of t^{-n} is to be taken which, on the continuation of t along any path till its argument is zero and its modulus 1, gives the value 1.

Now suppose the argument of x lies between $-\frac{1}{4}\pi$ and $-\frac{3}{4}\pi$, and examine

$$y_1 = \frac{e^{n\pi i}}{2^n . \pi i} \int_L e^{-t^2} t^{-n} Q_n\left(\frac{ix}{t}\right) dt.$$

Clearly $[y_1]_{x=0} = e^{-(n+1)\pi i}[y]_{x=0} \frac{K_n}{I_n} = [y]_{x=0} = [V_n(-x)]_{x=0}$,

while $\left[\frac{dy_1}{dx}\right]_{x=0} = e^{-(n+1)\pi i}\left[\frac{dy}{dx}\right]_{x=0} \frac{K_{n+1}}{I_{n+1}} = -\left[\frac{dy}{dx}\right]_{x=0} = \left[\frac{d}{dx} V_n(-x)\right]_{x=0}$,

since $$\frac{K_n}{I_n} = e^{(n+1)\pi i} \quad \text{(see p. 257).}$$

Hence, if the argument of x lies between $-\frac{1}{4}\pi$ and $-\frac{3}{4}\pi$, then the formula of the immediately preceding proposition must be replaced by

$$V_n(-x) = \frac{e^{n\pi i}}{2^n . \pi i} \int_L e^{-t^2} t^{-n} Q_n\left(\frac{ix}{t}\right) dt.$$

ON FUNCTIONS AND THEIR ASSOCIATED SETS OF POINTS

By W. H. YOUNG.

[Received December 8th, 1912.—Read December 12th, 1912.]

1. Starting with continuous functions, or even with simple upper and lower semi-continuous functions, constant in a finite number of closed and open stretches into which the fundamental region of integration has been divided, we may, by the method of monotone sequences, obtain all known or mathematically knowable functions of one or any number of variables. That the functions so obtained are identical in their ensemble with the functions of Baire's classes follows from the fact that the unique limit of a sequence of functions of a certain type can be obtained as the limit of a monotone sequence of functions each of which is the limit of a monotone sequence of functions of the same type as the given functions. It is only, however, by the use of what may be called transfinite processes that all such functions can be obtained. After any number (finite or transfinite of the second class) of monotone sequences, by the use of another monotone sequence, we can get a new function not belonging to any preceding type. Baire's process and the method of monotone sequences agree, not only in giving us the same functions in their entirety, but in requiring the same transfinite numbers for the same types of functions. On the other hand, Baire's process classes together functions of different types; thus, it makes no distinction between l and u-functions, and none even between these functions and those which are both lu and ul-functions. It would seem then that Baire's original classification has decided disadvantages compared with that which I have advanced, which arises, in fact, out of the very nature of things.

The superiority of the newer classification is also evident when we consider the properties of the different types of functions furnished successively by the monotone process. I propose in the present communication to illustrate this fact. With the exception of a single indication I confine myself, however, to functions which can be characterized by any finite number of alternate l's and u's. To go beyond this would quite

unnecessarily complicate the notation to be employed and would introduce considerations which must be regarded as of only theoretical interest—at least in the present state of science. In two or three steps we pass from the simple functions to functions of a generality of type rarely presenting themselves. Thus, the upper and lower derivates of a continuous function are ul's and lu's respectively, while in the theory of the integration of even unbounded functions we do not require to go beyond lul's and ulu's. We can in fact, always find functions of one of these, or of lower, types which have the same integral as any assigned function, and, in the case of bounded functions, we can find a lu and an ul having the same integral, the former not greater and the latter not less than the given function.

It is evident then that, next to upper and lower semi-continuous functions (l and u-functions), lu's and ul's are worthy of careful consideration. It is found, however, that properties possessed by them, when once proved, are generalisable as they stand for any function of the types in question. The step from l and u-functions to lu's and ul's is not immediate, and, indeed, even the determination of the nature of the set of points at which an l or u-function assumes a particular value has not hitherto been effected. The reason is not far to seek; it is that even at the present moment a systematic account of the theory of open sets does not exist. In stating some of the properties of inner and outer limiting sets in a paper published in the *Proceedings* of this Society some years ago* I took the first step in a direction which the present investigation shows to be necessary. I now take up the matter *de novo* and consider the question of open sets in general. There is precise parallelism between the Theory of Sets and the Theory of Functions, more especially between the Theory of Sets and the Theory of Two-valued Functions. Whereas, however, the general process of proceeding to the limit has been conceived as the natural one by which to extend the notion of function, no one, so far as I know, has attempted to apply it as a generating process to sets of points. A linear set might clearly be defined as a limiting set of a sequence of sets in the following manner: Think of the set G_n as making correspond to each point of the continuum either a white point or a black point, and this for all values of n; then one of two things may happen: either to each point of the continuum there corresponds, after a certain value of n, different in general for the points of the interval, either a white point or a black point, or this is not the case; if this is the case we suppose such a white point or black point attached finally to the

* "Open Sets and the Theory of Content," 1904, *Proc. London Math. Soc.*, Ser. 2, Vol. 2, pp. 16–51.

interval. Considering then only the white points in the set G so obtained and in the generating sets, we may with propriety say that the set G is the limit of the sets $G_1, G_2, \ldots, G_n, \ldots$. It is evident that the set G so obtained will in general be neither the outer nor the inner limiting set of the sets $G_1, G_2, \ldots, G_n, \ldots$. It might appear, therefore, that we had devised a more general generating process than that which corresponds to the use of outer and inner limiting processes. It might, moreover, happen, and would do so in the general case, that the white points and the black points did not definitely assert themselves at each point; in this case, instead of a limiting set G, we should have an upper and lower limiting set.

The method of monotone sequences leads, however, naturally to all the sets obtained by such processes. This follows from the fact that our process, when applied to two-valued functions, as to more general functions, leads to all the two-valued functions of Baire's classes. There is, therefore, evidently no reason whatever, when we are occupied with the classification of functions or of sets, to employ any but monotone processes.

In other words, we may say that all known or mathematically knowable sets of points can be obtained from a finite number of open or closed stretches by means of monotone processes alone. Here, again, it is only those obtained in a finite number of steps with which we need concern ourselves. It is unnecessary to recapitulate; it is sufficient to repeat that there is a precise analogy in this matter between two-valued functions and sets of points.

2. In analogy with the classification of functions which I have already introduced, I denote an ordinary inner and outer limiting set respectively as an i-set and an o-set, and indicate the succession of monotone processes of forming from these successively inner and outer limiting sets by i's and o's. The known theorem that an inner (outer) limiting set of ordinary inner (outer) limiting sets is a set of the same type thus gives us the symbolic statement that an ii is an i and an oo is an o. Thus we get at the second stage two new types of sets, and two only, an io-set and an oi-set, respectively the inner limiting set of ordinary outer and the outer limiting limiting set of ordinary inner limiting sets. The difference of two ordinary inner or outer limiting sets is shown to belong to both of these new types. The fundamental theorems which reproduce themselves at each stage may, as an instance, be stated here for the two new types of sets which appear at the third stage:

I. *The common points of two ioi's, or oio's, is a set of the same type; the points belonging to one or other of two ioi's or oio's is a set of the same type.*

II. *An ioi is both an ioio and an oioi, and the same is true of an oio.*

III. *The points common to an ioi and an oio form both an ioio and an oioi.*

The last of these theorems, in conjunction with the obvious fact that the complementary set of one of these types is a set of the twin type, gives us the theorem that *the difference of two ioi's, or two oio's, one of which is a subset of the other, is both an ioio and an oioi.*

The analogy between the sets thus introduced, and systematically arranged, and functions assuming two values only, and belonging to the system of functions obtained by monotone processes from simple l and u-functions, is at once evident if we take corresponding to each set the two-valued function whose value at each point of the set is unity and zero elsewhere. We then have the theorem that *the function corresponding to an o-set is a lu-function, to an io-set is a ulu-function, to an oio-set is a lulu-function, and so on, the general law being that to a final o in the name of the set corresponds a final lu in the name of the function, and elsewhere for an u we write i and for an l we write o; for the sets whose names end in i the rule only differs in that the final i we write ul.*

From this correspondence the whole theory of the content of open sets follows immediately from the theory of generalised integration as developed in my "New Method in the Theory of Integration" and elaborated in my recent paper on "The New Theory of Integration." Moreover, the dual view of content as the upper bound of the content of closed subsets contained in the set and the lower bound of the content of sets of completely open intervals containing the set, which I advanced in my earliest work on the subject, and from which I have never departed, gains a new justification. It arises inevitably and opens at once all the far-reaching vistas leading from this concept as centre.

Whether by definition, or as a deduction from the geometrical definition, the content of a set is the integral of the corresponding two-valued function.

The connection between functions and sets is still further brought out by the theorems concerning the nature of the sets of points at which functions assume a certain value k or a certain range of values. They show us that it is by their behaviour with respect to certain sets of points that functions are characterised. We have, in fact, the two following main theorems :—

(A) *The name of the set of points at which a function assumes a value k begins with an i and contains as many letters as the name of the*

function. Thus, the points at which an l or a u is $= k$ *form an i-set: the points at which a lu or an ul is* $= k$ *form an io-set, and so on.*

(B) *The necessary and sufficient condition that a function should belong to a certain type is the following: the name of the set of points at which a function whose name begins with* $l \leqslant k$, *or with* $u \geqslant k$, *begins with an i, and contains one less letter than the name of the set. Thus, the points at which a* $lu \leqslant k$, *or an* $ul \geqslant k$, *form an i-set; the points at which a* $lul \leqslant k$, *or a* $ulu \geqslant k$, *form an io-set, and so on.*

It then follows at once from the fundamental theorems that *the points at which a function whose name begins with an* $l < k$, *or with a* $u > k$, *form a set of the twin type to that of the set of points where the function assumes the value of k.* Thus, for instance, *the points at which a* $lu < k$ *form an oi-set.*

Hence, in particular, we have the final form of a theorem which I had already partly obtained: *the points at which the upper right-hand (left-hand) derivate of a continuous function is* $+\infty$, *or is* $\geqslant k$, *form an ordinary inner limiting set; the points at which the lower right-hand (left-hand) derivate is* $-\infty$, *or* $\leqslant k$, *form an ordinary inner limiting set; the points at which any one of the four derivates assumes the value k form an io-set.*

These results enable us to determine the type of the function obtained from a given function by changing the value of the latter to zero wherever it is greater than a certain value k, or wherever it is less than k. In particular, *if the given function is one whose name begins with an l, we may put it zero wherever it is* $\leqslant k$ *without changing its type, and the same is true if its name begins with a u, and we change it to zero wherever it is* $\geqslant k$.

3. We start with a pair of types of sets, complementary to one another, a closed set and a set of completely open intervals.

By Cantor's Theorem * the inner limiting set of a succession of closed sets, each containing the next after it, is a closed set. Reciprocally the outer limiting set of a succession of sets of intervals, each set containing the one preceding it, is a set of intervals. We have two new types of sets, an ordinary outer limiting set, or o-set, consisting of all the points belonging to a succession of closed sets, each containing the next before it, and an ordinary inner limiting set, or i-set, consisting of all the points common to a succession of sets of intervals, each contained in the next before it.

* See Young's *Theory of Sets of Points*, § 20, p. 38.

We have now the following fundamental theorems :—

I. *The common points of two closed sets, or of two sets of intervals, form a set of the same type (closed set or set of intervals, as the case may be). The points belonging to one or other of two closed sets or to two sets of intervals form a set of the same type.**

II. *A closed set is both an i-set and an o-set. A set of intervals is both an i-set and an o-set.*

Surround each point of the closed set G by an interval of length e_n, where $e_1, e_2, \ldots$, is any monotone descending sequence with zero as limit

These intervals have an equivalent set of non-overlapping completely open intervals with the same internal points, say D_n. The set D_n evidently contains D_{n+1}, and the sets therefore have in common an ordinary inner limiting set, or i-set, say S. Each set D_n contains the closed set G; therefore S contains G. But, if P is any point not belonging to the closed set G, it will have a positive minimum distance from G, and will therefore for sufficiently large n certainly be external to all the intervals which generated the set D_n, and will therefore be external to D_n. Thus it will not be a point of S. Hence S is precisely the closed set G, which is accordingly an i-set.

That a closed set is an o-set is evident, since the points of a closed set inside any closed interval form a closed set, and we have only to take a succession of intervals, one inside the other, gradually sweeping out the whole fundamental region, to obtain our closed set as the outer limiting set of closed sets.

III. *The set of points common to a closed set and a set of intervals is both an i-set and an o-set.*

Let G be the closed set and H the set of intervals. Express G as an i-set, and let G_n be the typical set of intervals of the succession having G as inner limiting set. Then, symbolically,

$$G_1 \geqslant G_2 \geqslant \ldots \to G\,;$$

therefore
$$(H, G_1) \geqslant (H, G_2) \geqslant \ldots \to (H, G),$$

that is to say, in words, since G_n contains G_{n+1}, the set of intervals which is common to G_n and H contains that common to G_{n+1} and H. Also, since the inner limiting set of the sets of intervals G_n is G, the inner

* *Ibid.*, p. 84.

limiting set of the sets of intervals (H, G_n) consists of the points common to H and G. Thus the common points of H and G form an i-set.

Again express H as an outer limiting set,

$$H_1 \leqslant H_2 \leqslant \ldots \rightarrow H;$$

therefore $$(G, H_1) \leqslant (G, H_2) \leqslant \ldots \rightarrow (G, H).$$

Thus, since (G, H_n) is a closed set, (G, H) is an o-set. This proves the theorem.

4. These theorems repeat themselves now exactly for i-set and o-sets.

We have first the theorem that an inner limiting set of ordinary inner limiting sets is an ordinary inner limiting set, and an outer limiting set of ordinary outer limiting sets is an ordinary outer limiting set, or, symbolically

an ii-set is an i-set,

an oo-set is an o-set,

or, still more concisely,

an ii is an i,

an oo is an o.

These theorems are proved on p. 72 of *The Theory of Sets of Points.*

We have two new types of sets, an inner limiting set of ordinary outer limiting sets, or *io*-set, or *io*, and an outer limiting set of ordinary inner limiting sets, or *oi*-set, or *oi*. It is then at once evident that an *oi* is complementary to an *io*, and *vice versa*. The fundamental theorems with respect to i and o-sets are, then, the following :—

I. *The common points of two i-sets or of two o-sets is a set of the same type. The points belonging to one or other of two i-sets or to two o-sets is a set of the same type.*

Let G and H be two i-sets ; then we may write, symbolically,

$$G_1 \geqslant G_2 \geqslant \ldots \rightarrow G,$$

$$H_1 \geqslant H_2 \geqslant \ldots \rightarrow H,$$

and therefore, considering the sets of points common to corresponding generating sets G_n and H_n, and, remembering that this set is of the same type as G_n and H_n themselves,

$$(G_1, H_1) \geqslant (G_2, H_2) \geqslant \ldots \rightarrow (G, H).$$

This shows that the set (G, H) of points common to G and H is an i-set.*

* This result was given recently by A. Rosenthal, "Ueber die Singularitäten der reellen ebenen Kurven," 1912, *Habilitationsschrift*, Munich, p. 23.

Similarly, by considering the generating sequences, we prove that if G and H are both o-sets, so is (G, H).

This proves the first part of the theorem, and the second part follows in like manner.

II. *An i-set and an o-set are both io and oi-sets.*

This follows at once from the corresponding theorem for the pair of functions at the preceeding stage, viz., closed sets and sets of intervals. Thus, if G is an i-set, and

$$G_1 \geqslant G_2 \geqslant \ldots \to G,$$

since each G_n is a special kind of o-set, G is an oi-set.

On the other hand, by the preceding theorem, the sub-set of G in any interval is an i-set. Thus, letting the interval expand till it generates the whole fundamental region, we have G expressed as an oi-set.

This proves that an i-set is both an io and an oi; similarly an o-set is both an io and an oi.

III. *The points common to an i-set and an o-set form both an io and an oi-set.*

The proof is formally the same as that of the corresponding theorem in § 3, the auxiliary theorems there used having all been proved.

5. It is clear now that we can progress any number of stages. The argument is the same at each stage. To take the next stage we have the theorem that

an iio is an io,

an ooi is an oi,

and we have a pair of new kinds of sets, viz., *ioi*-sets, or *ioi*'s, and *oio*-sets, or *oio*'s. Such sets are again evidently complementary to one another. The fundamental theorems reproduce themselves:

I. *The common points of two io's, or oi's, is a set of the same type. The points belonging to one another of two io's, or oi's is a set of the same type.*

II. *An io is both an ioi and an oio, and the same is true of an oi.*

III. *The points common to an io and an oi form both an ioi and an oio.*

It is clear that the generalisation is immediate. The notation employed will, of course, need modification if we are to include all the sets

to be employed obtained by the processes of which we here have a vista. Thus, the outer limiting set of a succession of sets of which the first is an *o*, the second an *io*, the third an *oio*, and so forth, will not belong to a type coming under this nomenclature. All Cantor's transfinite numbers of the second class can be employed without exhausting all possible varieties of these sets. The correlatives, however, of the fundamental theorems I, II, and III will evidently hold for all such varieties of sets, as is shown by the mode of proof given above.

6. It is an immediate corollary from the fundamental theorem III that *the difference of two sets of the same type, one of which is a sub-set of the other, is a set of both the next pair of types.*

Thus, for instance, if G is an *oi*-set, and H is another *oi*-set, contained in G, the complementary set K of H is an *io*-set, and therefore the difference $G-H$ is the set (G, K) of the points common to the *oi*-set G and the *io*-set H, and is therefore, by III, both an *ioi* and an *oio*.

7. We now consider the functions obtained by assigning at each point of a set, such as any we have obtained by monotone processes, the value unity, and zero at all the remaining points of the continuum. I shall, for brevity, speak of such a function as the function corresponding to the set in question.

We have first to notice that these functions are special cases of the functions to some of whose properties I have called attention,* and that they fall naturally into the pairs of classes which I have characterized by the letters l and u alternately.

It is evident, moreover, that, if the values assumed by a function are two only in number, we can, by means of the fundamental processes of addition, subtraction, multiplication, and division, reduce the function to one assuming only the values 0 and 1.

Suppose, then, we try to build up, in the most general manner possible, a function having the values 0 and 1 only. All such functions which are capable of mathematical expression will be obtainable by means of repeated application of the method of monotone sequences. The simple functions with which we start will belong to one of two types: either the value will be zero at the points of a finite number of completely open intervals and unity elsewhere or *vice versa*. In the former case the function

* "On the New Theory of Integration," 1912, *Roy. Soc. Proc.*, A, Vol. 88, pp. 170–178. "On a New Method in the Theory of Integration," 1910, *Proc. London Math. Soc.*, Ser. 2, Vol. 9, pp. 15–50.

will be a simple u-function (upper semi-continuous function), and in the latter case it will be a simple l-function (lower semi-continuous function). We may express this by saying that the functions corresponding to a finite number of completely open intervals and to a finite number of closed intervals are respectively simple l-functions and simple u-functions.

The function corresponding to an infinite set of open intervals is, then, clearly the limit of a monotone ascending sequence of simple l-functions; in other words, it is an l-function. Again, the function corresponding to a closed set of points is—by, for example, the use of the extended Heine-Borel Theorem—seen to be the limit of a descending sequence of simple u-functions; that is, it is a u-function.

Corresponding to an i-set we have the function obtained by a monotone descending succession of l-functions; that is, an ul-function. And corresponding to an o-set we have a function obtained by a monotone ascending sequence of u-functions; that is a lu-function.

The general law at once becomes apparent, and we may at once, without proof, enunciate the following theorem:—

THEOREM.—*If at each point of a set we construct a function whose value is unity at the points of a set and zero elsewhere, then the function corresponding to an o-set is a lu-function, to an i-set is an ul-function, to an io-set is a ulu-function, to an oi-set is a lul-function, and so on, the general rule being for a final o in the notation used to characterise the set write lu, for a final i write ul, and elsewhere write u for i and l for o.*

9. It follows, then, as a particular case of the general theory, that all such two-valued functions possess integrals, and that these integrals may be obtained by term-by-term integration of monotone sequences alone, the first sequence of the series consisting of functions corresponding to a finite number of closed and a finite number of open intervals respectively.

Now, in the case of these latter functions, the integral, in whatever sense the term may be used, is the sum of the lengths of the intervals. We are hence led to introduce the idea of length, or, as it is more convenient to say, "content," in relation to the sets of points to which our two-valued functions correspond. We may, in fact, indifferently define the content of such a set of points as the integral of the corresponding two-valued function, or as the unique limit obtained by monotone sequences, the first sequence of the series consisting of the lengths of a finite number of closed or open intervals. That the latter process is allowable depends on the uniqueness in question, and has already been proved in the paper cited in the more general case.

It has, however, been shown in the same paper, by means of monotone sequences alone, that the most general function obtainable by such processes is such that an *ul*-function not less than it and a *lu*-function not greater than it can be found whose integrals have the same value. This is therefore true of the two-valued functions, and the associated *ul* and *lu*-functions will then obviously be also two-valued functions.

Interpreting this fact in the language of sets, we have then the following result :—

THEOREM.—*Given any set of points obtained by repeated application of the method of monotone sequences from a finite number of closed or open intervals, an i-set which contains it and an o-set which is contained in it can be found having the same content.*

In other words, given any such set, if we find the upper bound of the contents of all closed sets contained in the given set and the lower bound of the contents of all infinite sets of intervals containing the given set, the upper bound of the former is equal to the lower bound of the latter.

The common value is then what we call the content of the set and is the integral of the two-valued function which is unity at the points of the set and zero elsewhere.

10. We now pass to the consideration of the general function. By general function I mean the most general function obtained by monotone processes from simple l and u-functions ; these functions are, however, in point of fact, the most general functions which are capable of mathematical expression.

We wish now to investigate the nature of the sets of points at which a general function of the various types obtainable has a particular value or has values belonging to an assigned closed or open range of values. In the case of an l-function we know that the set of points at which it is not greater than k form a closed set, and the set of points at which a u-function is not less than k form a closed set. In the case of a continuous function we know, further, that the set of points at which the function has the value k is a closed set. We wish to generalise these results.

We have first to investigate the nature of the sets at which an upper or a lower semi-continuous function (l-function or u-function) assumes the value k.

As the method employed is, however, perfectly general, and depends on a simple theorem, I shall first explain this result.

THEOREM.—*If the points at which f has values belonging to a closed or*

open range of values always form a set of a certain type, the points at which f assumes any particular value is an inner limiting set of sets of this type.

The importance of this theorem consists in the fact that it is sufficient to know the type in question for a closed, or a partly open, or a completely open range of values, for us to be able to deduce at once the nature of the set at which a particular value is assumed.

To prove this theorem let us suppose the variable range of values concerned to be that corresponding to the interval $(k-e_i, k+e_i)$, where $e_1, e_2, \ldots$, is a monotone descending sequence with zero as limit. Let G_i denote the set of points at which $f(x)$ has a value in this interval, open or closed, or half-open, as the case may be. Then at every stage the points at which $f(x) = k$ are contained in the set G_i. Also the set G_i contains the set at the next stage G_{i+1}, so that the points common to all the sets G_i form an inner limiting set which contains all the points at which $f(x)$ has the value k. Further, it contains no points at which $f(x)$ has a different value; for, if $f(x)$ differs from k, we can find i so that $f(x)$ differs from k by more than e_i, so that the point x is certainly not a point of the set G_i, and therefore it is not a point of the inner limiting set. Thus the inner limiting set consists precisely of the points at which $f(x)$ has the value k. This proves the result in question.

11. Now let $f(x)$ denote an l-function, then the points at which $f(x) > k$ form the internal points of a set of completely open intervals, and the set of points at which $f(x) \leqslant k$ form a closed set. Thus the points at which

$$k-e_i < f(x) \leqslant k+e_i$$

form the difference of two closed sets. Now the difference of two closed sets is an i-set, and the inner limiting set of i-sets is an i-set. Hence *the points at which the l-function $f(x)$ has the value k form an i-set.*

Next let $f(x)$ denote an u-function; then the points at which $f(x) \geqslant k$ form a closed set, and therefore the points at which

$$k-e_i \leqslant f(x) < k+e_i$$

form the difference of two closed sets. Hence, as before, *the points at which the u-function $f(x)$ has the value k form an i-set.*

12. We now pass to the consideration of ul and lu-functions. We

require another new principle, which is again perfectly general in its application. It is embodied in the following statement:—

THEOREM.—*If the points at which the functions of a monotone ascending succession are greater than k form for each function a set of a certain type, the points at which the limiting function is greater than k form an outer limiting set of sets of that type.*

If the points at which the functions of a monotone descending succession are less than k form for each function a set of a certain type, the points at which the limiting function is less than k form an outer limiting set of sets of that type.

Take the first case for definiteness, and let $f_1 < f_2 < \ldots$ be the succession and f the limiting function.

Let G_n denote the set of points at which $f_n > k$, and G the set at which $f > k$. Then a point at which $f_n > k$ is, since the succession is monotone ascending, certainly a point at which $f_{n+1} > k$. Hence the set G_n is contained in the set G_{n+1}. Moreover, at a point at which $f_n > k, f > k$; hence the set G contains the outer limiting set traced out by G_n.

But, as at each point the value of f is the limit of the value of f_n, as $n \to \infty$, a point at which $f > k$, must also be a point at which, for some value of n, and all subsequent values, $f_n > k$. Thus every point of G belongs to G_n for some value of n and all subsequent values. Thus G is identical with the outer limiting set traced out by G_n. This proves the theorem.

It should be noticed that the sets G_n need not all be of the same type.

13. In the case of an u-function the points at which $f > k$ form a set which is the difference of two sets, of which the more embracing is a closed set and the other is an i-set. Thus *the points at which the u-function $f > k$ form an o-set.*

In a similar way *the points at which an l-function $f < k$ form an o-set.*

Hence *the points at which a lu-function $> k$, or an ul-function $< k$, form an o-set*, since the former is the limit of a monotone descending succession of u's and the latter of a descending succession of l's.

14. Now let $f(x)$ denote a lu-function; then the points at which

$$k - e_i < f(x) \leqslant k + e_i$$

form the difference of two o-sets, which is therefore an io-set. Hence *the points at which the lu-function $f(x)$ has the value k form an io-set.*

Next let $f(x)$ denote an *ul*-function; then the points at which

$$k-e \leqslant f(x) < k+e,$$

form the difference of two *o*-sets. Hence, as before, *the points at which the ul-function $f(x)$ has the value k form an io-set.*

15. Hence, if $f(x)$ denote a *lu*-function, the points at which $f < k$ form a set which is the difference of two sets, of which the more embracing is the set complementary to an *o*-set, which is an *i*-set, and the other is an *io*-set. In other words, *the points at which a lu-function $f < k$ form an oi-set.*

Now take a monotone descending succession of *lu*-functions, having an *ulu*-function for limiting function. Then it follows, by the argument of § 12, that *the points at which an ulu-function $f(x) < k$ form an oi-set.*

In precisely similar way *the points at which a lul-function $f(x) > k$ form an oi-set.*

16. It at once follows, by an application of § 10, that *the points at which either a lul-function or an ulu-function assumes the value k form an ioi-set.* It is unnecessary to continue the discussion at length, as we have merely to repeat the argument. The final result is already beyond a doubt, namely,

THEOREM.—*To obtain the nature of the set at which one of our functions, which can be characterised in the ordinary way by alternate l's and u's, has the value k, we have merely to begin with an i and write alternately o's and i's till the number of letters is the same as those used in the denomination of the function.*

To obtain the nature of the set of the points at which such a function is greater or less than an assigned value k, if the first letter in the name of the function is u, the theorem tells us the nature of the set of points at which the function is less than k, and if the first letter in the name of the function is l, the theorem tells us the nature of the set of points at which the function is greater than k. We have merely to begin with an o and write alternately i's and o's till the number of letters is one less than those used in the denomination of the function.

Evidently, therefore, if we want the nature of the set of points at which the former function is greater and the latter function less than k, it will be the opposite to that of the set of points at which the function has the value k; that is, it will be got by interchanging the letters i and o.

17. We now pass to the consideration of the following problem :—

What is the nature of the function obtained from a given function by changing the value of the function to zero wherever it is greater than a certain positive quantity k or wherever it is not less than k ?

Since the function which has the value k everywhere belongs to the same type as f itself, the result of attributing the value k wherever the given function was greater than k will not alter the type of function.* Except, however, in the trivial case in which k is zero the process we are considering does, as will be seen, alter the type of the function in general.

For definiteness consider a *lul*-function, say $f(x)$. The points at which $f(x) = k$ form an *ioi*-set, and the points at which it is $> k$ form an *oi*-set. Hence the function which has the value k wherever $f > k$ and is zero elsewhere is a *lul*-function (§ 8). Hence, subtracting this from the given function f, we have a function which is both an *ulul* and a *lulu*-function. This is then the type of the function obtained when we make the original function zero wherever it was $> k$. If, however, we make it zero wherever it was $\geqslant k$ we shall have to subtract, for the same reason, an *ulu*-function, and therefore get a function which is a *lulu*-function only in general.

Now suppose k' less than k, and let us make $f(x)$ zero wherever $f \leqslant k'$. This is clearly equivalent to subtracting a function of the opposite type to that which we subtracted when we made the original function zero wherever it was $> k$; that, is to the subtraction of an *ulu*-function. Thus, if we perform this process in addition to either of the processes described we shall not alter the type of function obtained.

On the other hand, it will be remarked that, if we only perform the process last described, the nature of the function is not altered. In other words, *if, wherever, a lul-function is* $\leqslant k$, *we assign to it the value zero, this will not alter the type of function.*

More generally, for any of our functions whose name begins with an l, the same statement is true. For those of our functions whose names begin with a u, the same will be true if we substitute $\geqslant$ *for* $\leqslant$. These results are generalizations of known results for *l* and *u*-functions.

18. Now, a countable set of points is an ordinary outer limiting set whose generating sets consist of a finite number of points only. Hence *a function which is never negative and is zero except at a countable set*, being the limit of monotone ascending sequence of functions which are

---

* "A New Method in the Theory of Integration," *loc. cit.*, § 7.

zero except at a finite number of points where they are positive *is a lu-function.*

Similarly *the function which is zero except at a countable set where it is negative is an ul-function.*

Thus, *a function which is zero except at a countable set of points is in general both an ulu and a lul-function.*

The familiar function which is zero at all the irrational points and unity at the rational points is a *lu*-function.

To change the values of a function at a countable set of points is equivalent to adding a function which is zero except at a countable set. This shows why, if the function is of a fairly simple type, such a change has the effect of complicating its type.

19. The results of § 16 enable me to give a final form to a statement made in my paper on "Oscillating Successions of Continuous Functions," * § 4, Cor. 2. I have already pointed out the obvious incorrectness of the result there stated and have amended it, giving to it a form sufficient for the applications that I had in mind.† The corollary in question should now read as follows :—

The points at which the upper function of a succession of lower semi-continuous functions is $+\infty$ *or is* $\geqslant k$ *form an ordinary inner limiting set.*

Similarly,

The points at which the lower function of a succession of upper semi-continuous functions is $-\infty$ *or* $\leqslant k$ *form an ordinary inner limiting set.*

These results follow at once from the fact that these functions, are respectively ul and lu functions, as proved in the theorem to which these results are a corollary.

We may say in addition

The points at which the upper function of a succession of lower semi-continuous functions assumes the value k form an io-set, and the same kind of set is formed by the points at which the lower function of a succession of upper semi-continuous functions assumes the value k.

Hence, in particular, *the points at which the upper right-* (*left-*) *hand*

* *Proc. London Math. Soc.*, Ser. 2, Vol. 6, p. 303, 1908.

† "A Note on Trigonometrical Series," 1908, *Messenger of Mathematics*, p. 45.

derivate of a continuous function is $+\infty$ or is $\geqslant k$ form an ordinary inner limiting set, and the points at which it is equal to k or to $-\infty$ form an io-set. The points at which the lower right- (left-) hand derivate of a continuous function is $-\infty$ or is $\leqslant k$ form an ordinary inner limiting set, and the points at which it is equal to k or to $+\infty$ form an io-set.

The above results may be generalised for the upper and lower functions of successions of our functions of more complicated type. These upper and lower functions are, in fact, each the limit of a monotone sequence of functions, each of which is the limit of monotone sequence of functions of types depending only on the functions of the given succession.

20. We now proceed to modify our *lul*-function, supposed bounded, and, as before, positive, so as to make it a p-valued function. Let K' and K be any two quantities, K' not less than the upper bound, and K not greater than the lower bound, of the function, and divide the interval (K, K') into a number q say, of equal parts.

We thus obtain q intervals in the whole range of values. Wherever f has a value internal to one of these intervals of values, attribute to f the value of the lower end-point of that interval. Denote the function so obtained by g_q, and denote the corresponding function, formed by the same process, but in which the upper end-point takes the place of the lower end-point of the interval of values, by h_q. We wish to find the nature of the functions g_q and h_q.

Let r and s denote the values at the lower and upper ends respectively of one of the intervals of values; then we have seen that the function obtained from f by assigning to f the value zero wherever $f \leqslant r$ and wherever $f > s$ is the excess of a *lul* over an *ulu*-function, that is a *lul*-function, and also that the function whose value is unity at the points where f is greater than r and not greater than s, and zero elsewhere, is the function corresponding to the difference of two *oio*-sets, that is the difference of two *lul*-functions.

Now g_q is obtainable from f by subtracting the first of these two functions and adding r times the second function, and then performing a corresponding process for each of the q intervals. Hence g_q is the difference of two *lul*-functions.

Similarly h_q is obtainable from f by precisely the same process, only that s takes the place of r as a multiplier. Thus, h_q is of the same type as g_q. The effect, therefore, of our process is to increase the complication of our function to a slight extent only. That the new functions g_q and h_q are of higher type than before may be easily tested by taking, instead

of a *lul*, an *l*-function ; it will be found that there is a rise in the scale of types.

If we now make q approach infinity, g_q will trace out a monotone increasing, and h_q a monotone decreasing, sequence. But these sequences of functions will not, in general, have f for limit. In fact, the limit of g_q is a *lulu* and that of h_q is an *ulul*-function.

We see, therefore, that the process by which Lebesgue defines his integral consists in replacing f by two sequences of function above and below f belonging to a higher type than f, though simpler in this sense than f, that each of them has only a finite number of values. This is virtually equivalent to supposing the theory of the integration of two-valued functions of all classes to have been developed before those of the general functions of the simplest types have been disposed of. Moreover, these sequences of Lebesgue's do not converge to f.

21. On logical grounds Lebesgue's working definition appears, from this point of view, open to some objection. Moreover, it is scarcely a definition which appears to a student of the subject to present itself inevitably. Its great advantage is that it enables the mathematician familiar with the Theory of Sets of Points to get at once to work and to prove, for example, the more difficult relations between a derivate and its primitive.

In the theoretical development of the subject, when a sufficient time has elapsed for the value of the concept of generalised integration to be generally accepted, this advantage may scarcely appear to compensate for the obvious disadvantages.

If, on the other hand, the method of monotone sequences for the Theory of Integration, such as I have exposed elsewhere, be adopted, the Theory of Sets of Points is not introduced except and in so far as it is required. Thus, for many of the purposes of generalised integration, this theory does not present itself. When it is required Lebesgue's definition presents itself naturally, not as a definition, but as a property of the integral otherwise defined.

22. We now show how Lebesgue's working definition may be deduced from the one we have employed. We remark first that the integrals of g_q and h_q have, as q increases, necessarily the same limit. In fact, the difference of the integrals of g_q and h_q is clearly less than the q-th part of the product of length of the interval of integration by $K'-K$. Further, the integral of g_q is the sum of the integrals of q functions, each of which is

obtained by multiplying the two-valued function corresponding to a certain set of points by the corresponding quantity r; that is, by multiplying the content of the set of points by r. The integral of h_q is of the same form, with s instead of r as a multiplier. We thus get at once Lebesgue's two summations,

$$r_1E_1+r_2E_2+\ldots+r_qE_q,$$

and

$$s_1E_1+s_2E_2+\ldots+s_qE_q,$$

whose common limit, when $q \to \infty$, is the integral.

23. In "A New Method in the Theory of Integration," § 20, I employed a method of constructing an *ul* and a *lu*-function having the same integral and between which a given bounded function lies. Write

$$k_r = K+re_n, \quad [(r = 0, 1, \ldots, (n-1)],$$

where $ne_n = K'-K$, and K and K' are respectively the upper bound and the lower bound of $f(x)$, and denote by I_r the set of points at which $f(x) \geqslant k_r$. Then the set of points I_r-I_{r+1} consists of the points at which

$$k_r \leqslant f(x) < k_{r+1},$$

and, if the function $f(x)$ is one of the functions we have been considering, we can, inside this set, find a closed set of content as near as we please to the content of the set I_r-I_{r+1} itself. Assuming, therefore, that $f(x)$ is any function having this property (so-called measurable function), we can find a closed sub-set of I_r-I_{r+1}, which may be denoted by $I'_r-I'_{r+1}$, whose content differs from that of I_r-I_{r+1} by less than $n^{-1}.2^{-n}, \ldots$.* We then define two functions a_n and b_n as follows:—

At every point of I'_{n-1}, $\quad a_n = k_{n-1}, \quad b_n = K'$;

at every point of $I'_{n-2}-I'_{n-1}$, $\quad a_n = k_{n-2}, \quad b_n = k_{n-1}$;

and so on.

At every point of I'_1-I_0, $\quad a_n = K, \quad b_n = k_1$,

---

* The description in the paper cited is not quite clear, since it is not explicitly stated that the sets $I'_r-I'_{r+1}$ are to be closed. For the rest it is evident that we must ensure this, or else b_n would not be an l-function. The set I_n is to be regarded as the null set, so that we begin with I'_{n-1} and not $I'_{n-1}-I'_n$.

at all the remaining points, forming a set of content less than 2^{-n},

$$a_n = K, \qquad b_n = K'.$$

Then the points at which a_n is not less than any quantity z form a closed set, namely, if $k_r < z < k_{r+1}$, the set $I'_{n-1}+(I'_{n-2}-I'_{n+1})+\ldots+(I'_{r-1}-I'_r)$, or I'_{r-1}, and if $z = k_r$, it is the set I'_r. Thus the function a_n is a u-function.

Also the functions a_n form a monotone ascending sequence of u-functions, each not greater than f. Therefore the limiting function $a(x)$ of the sequence is a lu-function not greater than $f(x)$.

Similarly the points at which b_n is not greater than z form a closed set, namely, if $k_r < z < k_{r+1}$, the set $(I'_0-I'_1)+(I'_1-I'_2)+\ldots+(I'_{r-1}-I'_r)$, while if $z = k_{r+1}$ we have to add to this closed set the closed set $(I'_r-I'_{r+1})$. Also the functions b_n form a monotone descending sequence of l-functions not less than f. Therefore the limiting function $b(x)$ of the sequence is an ul-function not less than f.

Finally, $b_n(x)-a_n(x) = e_n$, except at a set of content less than 2^{-n}, where it has the value $K'-K$.

Now
$$b_1-a_1 \geqslant b_2-a_2 \geqslant \ldots \geqslant b_n-a_n \geqslant b_{n+1}-a_{n+1} \geqslant \ldots,$$

each of these functions being an l-function. Hence the exceptional set at which $b_n-a_n = K'-K$ is contained in that at which $b_{n-1}-a_{n-1} = K'-K$. These exceptional sets, each of which, by § 11, is an i-set, lie each inside the preceding, and their inner limiting set is an i-set whose content is zero. Except at this i-set we have $b(x) = a(x)$, and therefore $= f(x)$.

Thus we have constructed a lu-function not less than $f(x)$ and an ul-function not greater than $f(x)$, differing only at an i-set of content zero, where their difference is $(K'-K)$.

This, therefore, proves the required result.

24. The method of the preceding article enables us to prove the following theorem :—

THEOREM.—*The necessary and sufficient condition that a function should belong to a certain type is the following : the name of the set of points at which a function whose name begins with* $l \leqslant k$, *or with* $u \geqslant k$, *begins with an i and contains one letter less than the name of the set. Thus, the points at which a lu-function* $\leqslant k$, *or an ul* $\geqslant k$, *form an i-set ; the points at which a lul* $\leqslant k$, *or an ulu* $\geqslant k$, *form an io-set ; and so on.*

That this condition is necessary has been already shown (§ 16). The proof that it is sufficient depends on the formation of an auxiliary function $a(x)$ of the required type and its identification with the given function $f(x)$. The function $a(x)$ is formed on the same plan as in the preceding article, only without using the content, the auxiliary sets employed being those which are naturally suggested by the given nature of the set of points at which the given function has the given range of values. These sets are, moreover, suitably modified, in accordance with the first fundamental theorem, so as to lead by the familiar process of a doubly monotone sequence, monotone in the same sense, uniformly to the set from which we started.

The proof is perfectly general and depends only on the fundamental theorems and on the process of induction. It is given at length for the case when the given set of points at which $f(x) \leqslant k$ is an i-set, and it is required to prove that $f(x)$ is a lu-function. In any other special case we have only to assume the theorem proved for all the lower stages, and apply the proof here given *mutatis mutandis*.

That the theorem is true when the given set is a closed set, the function being then an l-function or a u-function, is well known and may be proved immediately from the definition of these functions.

We assume now that the points at which $f(x) \leqslant k$ form an i-set.

For brevity let us suppose that the lower bound K of the function $f(x)$ is zero and the upper bound is unity, and let us divide this interval of values at successive stages into 2, 4, 8, ..., equal parts, denoting the points of division by the binary notation.

Then, at the first stage, we have the point of division $\cdot 1$. Let $I_{\cdot 1}$ denote the set of points at which $f(x) \leqslant \cdot 1$; then $I_{\cdot 1}$ is by hypothesis an i-set. Let us express it in this way:

$$I^1_{\cdot 1} > I^2_{\cdot 1} > I^3_{\cdot 1} > \to I_{\cdot 1},$$

where, for each integer r, $I^r_{\cdot 1}$ denotes a set of completely open intervals.

At the next stage we have three points of division, $\cdot 01$, $\cdot 1$, and $\cdot 11$ and three corresponding i-sets, viz., $I_{\cdot 01}$, $I_{\cdot 1}$, and $I_{\cdot 11}$. The first and last of these are new at this stage, and we proceed to express them also as the limits of sets of intervals in such a way that, for each fixed value of r,

$$I^r_{\cdot 01} \leqslant I^r_{\cdot 1} \leqslant I^r_{\cdot 11};$$

this is, of course, possible, since

$$I_{\cdot 01} \leqslant I_{\cdot 1} \leqslant I_{\cdot 11},$$

so that if $I^r_{\cdot 01}$ is not contained in $I_{\cdot 1}$ we may replace the former by their common part, and if $I^r_{\cdot 11}$ does not contain $I^r_{\cdot 1}$ we may replace the former by the set composed of both. By the first fundamental theorem these changes do not affect the type of the sets concerned.

Proceeding thus, stage by stage, we get a doubly infinite set of sets of intervals denoted typically by

$$I^r_{\cdot N1},$$

where N denotes any combination of zeros and ones and r is an integer. These sets have the properties denoted symbolically as follows :—

$$\left.\begin{aligned} I^r_{\cdot N1} &\geqslant I^{r+1}_{\cdot N1} \\ I^r_{\cdot N1} &\geqslant I^r_{\cdot M1} \end{aligned}\right\} \quad (\cdot M1 < \cdot N1).$$

Hence if k is any number between 0 and 1, and we take any monotone descending sequence of binary numbers having k as limit, say

$$\cdot N_1 1, \quad \cdot N_2 1, \quad \cdot N_3 1, \quad \ldots,$$

we shall have a corresponding scheme of sets of intervals and i-sets.

$$\begin{array}{ccccccc}
I^1_{\cdot N_1 1}, & \geqslant & I^2_{\cdot N_1 1}, & \geqslant & \ldots\ldots \rightarrow & I_{\cdot N_1 1}, \\
\mathbb{V} & & \mathbb{V} & & & \mathbb{V} \\
I^1_{\cdot N_2 1}, & \geqslant & I^2_{\cdot N_2 1}, & \geqslant & \ldots\ldots \rightarrow & I_{\cdot N_2 1}, \\
\mathbb{V} & & \mathbb{V} & & & \mathbb{V} \\
\mathbb{V} & & \mathbb{V} & & & \mathbb{V} \\
I^1_{\cdot N_r 1}, & \geqslant & I^2_{\cdot N_r 1}, & \geqslant & \ldots\ldots \rightarrow & I_{\cdot N_r 1}, \\
 & & & & & \mathbb{V} \\
 & & & & & \downarrow \\
 & & & & & I_k ;
\end{array}$$

whence it follows that $I^s_{\cdot N_r 1}$ has, as r and s approach infinity in any manner, the unique double limit I_k ; that is to say, if r and s describe monotone increasing successions of integers, the sets of interval $I^s_{\cdot N_r 1}$ lie each inside the preceding, and their inner limiting set is I_k.

Now let us consider the stage of division at which there are n parts into which the segment (K, K') is divided. Let the points of division be denoted by $k_1, k_2, \ldots, k_{n-1}$, and let us define a corresponding function a_n as follows :—

At all the points of the set of intervals $I^n_{k_1}$ let $a_n(x) = K$.

At all the remaining points of the set of intervals $I^n_{k_2}$ let $a_n = k_1$.

At all the remaining points of the set of intervals $I^n_{k_3}$ let $a_n = k_2$; and so on.

At all the remaining points of the set of intervals $I^n_{k_r}$ let $a_n = k_{r-1}$; and so on.

Finally, at all the remaining points of the whole interval, let $a_n = k_{n-1}$.

Then the points at which $a_n(x) < z$ clearly form a set of completely open intervals, namely, $I^n_{k_r}$ if $z = k_r$ or if $k_{r-1} < z < k_r$.

Hence the function $a_n(x)$ is an u-function. Also evidently,

$$a_n(x) \leqslant a_{2n}(x),$$

so that these functions form a monotone ascending sequence whose limiting function $a(x)$ is a lu-function.

Now, if k is as before any number, and we express it, as before, as the limit of a monotone sequence of binary numbers, let us suppose that, at the stage at which we form the function a_n, the new binary number of the sequence introduced is $\cdot N_r 1$, so that the corresponding set of intervals is $I^n_{\cdot N_r 1}$ at every point of which

$$a_n \leqslant k_{r_n}, \tag{1}$$

where

$$k_{r_n} = \cdot N_r 1 - \frac{1}{n}.$$

Then as n, and therefore also r, increase without limit the sets of intervals $I^n_{\cdot N_r 1}$ describe, as we have seen, a sequence having as inner limiting set the set I_k at which $f(x) \leqslant k$. But at every point of this inner limiting set we have, by (1),

$$a(x) \leqslant k.$$

Also, if x is a point at which $a(x)$ has a value greater than k, we can find n so that $a_n(x)$ is greater than k_{r_n}, and therefore x is not a point of $I^n_{\cdot N_r 1}$, and therefore certainly does not belong to the inner limiting set. Thus the inner limiting set consists precisely of the points at which

$$a(x) \leqslant k.$$

Thus the points at which $a(x) \leqslant k$ are identical with the points at which $f(x) \leqslant k$.

Hence it follows at once that the points at which

$$k - e < a(x) \leqslant k + e$$

are identical with the points at which

$$k-e < f(x) \leqslant k+e,$$

and therefore, by the argument of § 11, the points at which

$$a(x) = k$$

are identical with the points at which

$$f(x) = k.$$

Hence $f(x)$ is identical with $a(x)$ and is therefore a lu-function. Thus the theorem is true.

*Appendix on the Connexion of the above Theory with that of Baire's Classes of Functions.**

If $f(x)$ is a function of Baire's first class it is the limit of a sequence of continuous functions, that is, of functions of class 0,

$$f_1(x), \quad f_2(x), \quad \ldots.$$

Denoting by $v_{m,n}(x)$ the function which is at every point the greatest and by $w_{m,n}$ that which is the least of

$$f_m(x), \quad f_{m+1}(x), \quad \ldots, \quad f_{m+n}(x),$$

and by $v_m(x)$ and $w_m(x)$ respectively, the limits of these functions when $n \to \infty$, $v_{m,n}(x)$ and $w_{m,n}(x)$ are functions of the same type as $f_m(x)$, and $v_m(x)$ and $w_m(x)$, being the limits of monotone sequences, respectively ascending and descending, of such functions, are functions of the next two types, viz., $v_m(x)$ is an l and $w_m(x)$ a u. As $m \to \infty$ $v_m(x)$ and $w_m(x)$ describe monotone sequences, the former descending, the latter ascending, having the same limit $f(x)$. Thus $f(x)$ is both an ul and a lu.

Similarly, using this result, a function of Baire's second class is both

* This Appendix has been added, January 1913, in response to a question of a referee. It will be remarked that the two theories are each of them complete in themselves, but that both are necessary for the entire elucidation of the subject. In explaining the connexion I have not scrupled to avail myself of some of the reasoning adopted by Lebesgue in his extremely ingenious paper, "Sur les fonctions représentables analytiquement," 1904, *Jour. de Math., p. et a.* (VI), Vol. I, pp. 139–216, a paper, concerned with Baire's classes, which had previously escaped my notice, and with which my own work has certain points of contact.

an *ulu* and a *lul*, and, by induction, a function of Baire's n-th class is both an *ulu* ... and a *lul* ..., with $(n+1)$ letters.*

It remains to prove the converse of this result in order to establish completely the relation between my classification of functions and that of Baire.

To do this we shall use two properties of our functions. The first is that the limit of a uniformly convergent sequence of functions of any monotone type is a function of the same type. This I have proved elsewhere.† The second is that the limit of a uniformly convergent sequence of functions of Baire's n-th class is a function of the same class. This has been proved by Lebesgue ‡ as follows :—

Let $f_1, f_2, \ldots$, be a uniformly convergent sequence of functions of Baire's n-th class, and f the limiting function. Take any convergent series $\epsilon_1+\epsilon_2+\ldots$, and let R_p denote the remainder after the first p terms. Corresponding to each integer r determine an integer n_r such that

$$|f(x)-f_{n_r}(x)| < \epsilon_r$$

for all points x. We may write

$$f(x) = f_{n_1}(x)+\{f_{n_2}(x)-f_{n_1}(x)\}+\{f_{n_3}(x)-f_{n_2}(x)\}+\ldots,$$

where

$$f_{n_r}(x)-f_{n_{r-1}}(x) = \operatorname*{Lt}_{p\to\infty} \phi_{r,p}(x),$$

$\phi_{r,p}(x)$ being a function of Baire's $(n-1)$-th class, numerically $< 2\epsilon_r$, which is possible since the left hand side of the preceding equation is of that class and numerically $\leqslant 2\epsilon_r$.

Let us write

$$\psi_q(x) = \sum_{r=1}^{r=q} \phi_{r,q}(x) = \sum_{r=1}^{r=p} \phi_{r,q}(x)+\sum_{r=p+1}^{r=q} \phi_{r,q}(x) = \sum_{r=1}^{r=p} \phi_{r,q}(x)+2\theta R_p,$$

where $|\theta| \leqslant 1$. Then $\psi_q(x)$ is also a function of Baire's $(n-1)$-th class. Letting $q\to\infty$, the summation on the right converges to $f_{n_p}(x)$, so that the right-hand side differs from $f(x)$ by less than $2R_{p-1}$. Since R_{p-1} may be made as small as we please by choosing p large enough, this shows that

---

* I have confined myself here to classes of finite type; the reasoning is easily extendible to functions of transfinite type.

† See my paper on "Uniform Oscillation of the First and Second Kind," in these *Proceedings, infra.*

‡ *Loc. cit.*, p. 156.

the function $\psi_q(x)$ converges to $f(x)$ as $q \to \infty$, and that therefore $f(x)$ is of Baire's n-th class. This proves the required theorem.

LEMMA.—*Given an ioi ... set S, with n letters, we can construct a ulu ... function, with n letters, having that set S for the set of its zeros and being elsewhere positive and not greater than unity.*

Let G be the *oio* ... set complementary to S, and let a defining succession of *io* ... sets, with $(n-1)$ letters, be

$$G_1 \leqslant G_2 \leqslant \ldots \to G.$$

The function corresponding to the set G_r, that is the function which is unity at the points of G_r and zero elsewhere, is known to be an *ulu* ... function, with n letters; the same is therefore true if we multiply it by 2^{-r}. Let it then be called $\psi_r(x)$.

Writing

$$f_n(x) = \psi_1(x) + \psi_2(x) + \ldots + \psi_n(x),$$

it follows that $f_n(x)$ is also an *ulu* ... function with n letters.

Now, at the points of G_1, we have

$$f_n(x) = 2^{-1} + 2^{-2} + \ldots + 2^{-n} = 1 - 2^{-n},$$

at the points of $G_r - G_{r-1}$, we have

$$f_n(x) = 2^{-r+1} - 2^{-n} \quad (r = 2, 3, \ldots, n),$$

and at the remaining points $f_n(x) = 0$.

These latter points form the complementary set to G_n and are therefore points of S.

Hence it follows that the functions $f_n(x)$ form, as n increases, a monotone ascending sequence which converges uniformly, since $f_n(x)$ and $f_{n+1}(x)$ only differ by 2^{-n-1}. The limiting function $f(x)$ of the succession is accordingly * a function of the same type as $f_n(x)$, namely, an *ulu* ... function with n letters, and is such that

$$\begin{aligned} f(x) &= 1 \text{ at the points of } G_1 \\ &= 2^{-r+1} \text{ at the points of } G_r - G_{r-1} \quad (r = 2, 3, \ldots, n, n+1, \ldots) \\ &= 0 \text{ at the remaining points, viz., at the points of } S. \end{aligned}$$

Thus $f(x)$ has the properties required. This proves the Lemma.

* "Uniform Oscillation of the First and Second Kind," *loc. cit.*

THEOREM.—*A function which is both a ulu ... and a lul ..., with* $(n+1)$ *letters, is a function of Baire's n-th class.**

Let f be the function. Then, since f is both an *ulu* ... and a *lul* ..., with $(n+1)$ letters, the sets of points at which

$$a \leqslant f$$

and

$$f \leqslant b$$

respectively are both *io* ... sets, with n letters. The same is therefore true of their common part, viz., the set of points at which

$$a \leqslant f \leqslant b.$$

Let A and B be the lower and upper bounds of f, and let us divide the range (A, B) of values of f into $(k+1)$ equal parts, denoting the points of division by $a_0 = A$, a_1, a_2, ..., $a_{k+1} = B$. Then by the lemma we can find an *ulu* ... function ψ_r, with n letters, which is zero wherever

$$a_r \leqslant f \leqslant a_{r+1},$$

and is positive and not greater than unity elsewhere. Let this be done for $r = 0, 1, \ldots, k$, and let us write †

$$u_r = \psi_0 \psi_1 \ldots \psi_{r-1},$$

$$v_r = \psi_{r+1} \psi_{r+2} \ldots \psi_k,$$

$$\phi_r(x) = \frac{u_r}{u_r + v_r}.$$

Then $\phi_r(x)$ is a function which is zero wherever

$$a_0 \leqslant f \leqslant a_r,$$

and positive and less than unity where

$$a_r < f < a_{r+1},$$

while it is unity wherever

$$a_{r+1} \leqslant f \leqslant a_{k+1}.$$

Moreover, since u_r and v_r are both functions of Baire's n-th class, being, in fact, *ulu* ...'s with n letters, and they do not vanish together, $u_r/(u_r+v_r)$, that is $\phi_r(x)$, is a function of the same class. Hence, writing

$$f_k(x) = a_0 + (a_1 - a_0)\phi_0 + (a_2 - a_1)\phi_1 + \ldots + (a_{k+1} - a_k)\phi_k,$$

* Under special circumstances the function might, of course, belong to a lower class.

† Cp. Lebesgue, *loc. cit.*, p. 168.

it follows that $f_k(x)$ is also a function of Baire's n-th class. Also, whatever integer $r \leqslant k$ be taken, at each point where

$$a_r \leqslant f \leqslant a_{r+1},$$

we have

$$\begin{aligned} f_k(x) &= a_0+(a_1-a_0)+(a_2-a_1)+\ldots+(a_r-a_{r-1})+(a_{r+1}-a_r)\,\phi_r \\ &= a_r+(a_{r+1}-a_r)\,\phi_r. \end{aligned}$$

Thus, since $|\phi_r| \leqslant 1$, and $a_{r+1}-a_r = (B-A)/(k+1)$, we have, whatever point x be taken,

$$|f(x)-f_k(x)| \leqslant (B-A)/(k+1).$$

This shows that, as $k \to \infty$, the functions $f_k(x)$ converge uniformly to $f(x)$. Hence $f(x)$ also belongs to Baire's n-th class.

This proves the theorem.

SOME PROPERTIES OF THE EXTENDED ZETA FUNCTION

By G. N. WATSON.

[Received October 9th, 1912.—Read November 14th, 1912.—Revised December 27th, 1912.]

1. The behaviour of the extended ζ-function, defined, when* $\sigma > 1$, by the equation

$$\zeta(s, a) = \sum_{n=0}^{\infty} (a+n)^{-s},$$

for large values of $|s|$ does not appear to have received the attention which it deserves; the known properties† of the function are as follows:

(i) The only singularity of $\zeta(s, a)$ in the finite part of the plane is a simple pole at $s = 1$.

(ii) As $|s| \to \infty$, $e^{-\theta|t|}|\zeta(s, a)| \to 0$, if $\pi > \theta > |\arg a|$, and σ is fixed.

(iii) If $\sigma < 0$, $0 < a \leqslant 1$, then

$$\zeta(s, a) = 2\Gamma(1-s)\left[\sin(\tfrac{1}{2}s\pi)\sum_{n=1}^{\infty}\frac{\cos(2n\pi a)}{(2n\pi)^{1-s}} + \cos(\tfrac{1}{2}s\pi)\sum_{n=1}^{\infty}\frac{\sin(2n\pi a)}{(2n\pi)^{1-s}}\right].$$

In this paper I propose to find inequalities satisfied by $\zeta(s, a)$ as $|t| \to \infty$ when σ is assigned. The results, which are summarised below in formulæ (8a) and (8b), are of interest as being the most precise which have been obtained by contour integration in connection with the theory of zeta functions.

2. Consider first $\zeta(s, a)$, where $R(a) > 1$.

When $\sigma > 1$, it is easy to see that,‡ if N be an arbitrary integer,

$$\zeta(s, a) = \sum_{n=0}^{N} (a+n)^{-s} - \frac{1}{(1-s)(N+a)^{s-1}} - \sum_{n=N}^{\infty} f_n(s), \qquad (1)$$

---

* Following Landau, we write $s = \sigma + it$, where σ, t are real.

† (i) and (ii) are given by Barnes, *Proc. London Math. Soc.*, Ser. 2, Vol. 3, 1905, pp. 253–272; (ii) is a generalisation of a theorem which had previously been incorrectly stated, *Messenger of Mathematics*, Vol. XXIX (1899), p. 114; (iii), which is due to Hurwitz, is a generalisation of Riemann's formula

$$2^{1-s}\Gamma(s)\zeta(s)\cos(\tfrac{1}{2}s\pi) = \pi^s\zeta(1-s);$$

see Lindelöf, *Le Calcul des Résidus* (Borel *Tracts*, 1905), pp. 107 and 58.

‡ Cp. Landau, *Primzahlen*, § 228.

where $$f_n(s) = \frac{1}{1-s}\left\{\frac{1}{(n+1+a)^{s-1}} - \frac{1}{(n+a)^{s-1}}\right\} - \frac{1}{(n+1+a)^s}$$

$$= s\int_n^{n+1} \frac{u-n}{(u+a)^{s+1}}\,du,$$

so that when* $\sigma > 0$,

$$|f_n(s)| < |s| \int_n^{n+1} \frac{1}{|(u+a)^{s+1}|}\,du$$

$$< |s|\,\{1+e^{t\arg a}\}\int_n^{n+1} |(n+a)^{-\sigma-1}|\,du,$$

i.e., $$|f_n(s)| < |s|\,\{1+e^{t\arg a}\}\,|(n+a)^{-\sigma-1}|. \qquad (2)$$

From (2) it follows that the series on the right of (1) is a uniformly convergent series of analytic functions of s when $\sigma > 0$; so that (1) defines the analytic continuation of $\zeta(s, a)$ over the strip $0 < \sigma \leqslant 1$.

Hence, from (1), when $\sigma > 0$,

$$|\zeta(s, a)| \leqslant \sum_{n=0}^{N} |(a+n)^{-s}| + |t^{-1}(N+a)^{1-s}| + \sum_{n=N}^{\infty} |f_n(s)|$$

$$< \{1+e^{t\arg a}\}\left[\sum_{n=0}^{N} |(a+n)|^{-\sigma} + |t^{-1}(a+N)^{-\sigma}| + \sum_{n=N}^{\infty} |s| \int_n^{n+1} |(u+a)|^{-\sigma-1}\,du\right]$$

$$< \{1+e^{t\arg a}\}\left[\sum_{n=0}^{N} |(a+n)|^{-\sigma} + |t^{-1}(a+N)^{-\sigma}| + |s| \int_N^{\infty} (u+1)^{-\sigma-1}\,du\right].$$

Also $$\sum_{n=0}^{N} |(a+n)^{-\sigma}| \leqslant |a|^{-\sigma} + \sum_{n=1}^{N} (n+1)^{-\sigma}$$

$$\leqslant |a|^{-\sigma} + (1-\sigma)^{-1}\{(N+1)^{1-\sigma} - 1\}, \qquad (3)$$

by the Cauchy-Maclaurin sum formula; and

$$\sum_{n=1}^{N} (n+1)^{-\sigma} < \{1+(N+1)^{1-\sigma}\} \sum_{n=1}^{N} (n+1)^{-1}$$

$$< \{1+(N+1)^{1-\sigma}\} \log (N+1). \qquad (3a)$$

---

* $|(u+a)^{-it}| \leqslant 1$ when $t\arg(u+a) \leqslant 0$, and $|(u+a)^{-it}| \leqslant e^{t\arg a}$ when $t\arg(u+a) \geqslant 0$. Hence always $|(u+a)^{-it}| < 1 + e^{t\arg a}$.

Now, let $|t|>1$, $[t]=$ the greatest integer contained in $|t|$, and let $N=|t|$, then, by (3) and (3a),

$$|\zeta(s,a)|<\{1+e^{t\arg a}\}\Big[|a|^{-\sigma}+(1-\sigma)^{-1}\{([t]+1)^{1-\sigma}+1\}$$
$$+|t|^{-1}|([t]+a)^{1-\sigma}|+(|t|+2)\sigma^{-1}([t]+1)^{-\sigma}\Big]$$
$$(0<\sigma\leqslant 2,\ \sigma\neq 1),$$

$$|\zeta(s,a)|<\{1+e^{t\arg a}\}[|a|^{-\sigma}+\{1+([t]+1)^{1-\sigma}\}\log([t]+1)$$
$$+|t|^{-1}.([t]+a)^{1-\sigma}|+(|t|+2)\sigma^{-1}([t]+1)^{-\sigma}]$$
$$(0<\sigma\leqslant 2).$$

An inspection of these inequalities shews that*

$$\zeta(s,a)=O(1)+O(e^{t\arg a}) \qquad (1<\sigma\leqslant 2),$$
$$=O(|t|^{1-\sigma})+O(e^{t\arg a}|t|^{1-\sigma}) \qquad (0<\sigma<1),$$
$$=O(\log|t|)+O(\log|t|\,e^{t\arg a}) \qquad (1\leqslant\sigma\leqslant 2),$$
$$=O(|t|^{1-\sigma}\log|t|)+O(|t|^{1-\sigma}\log|t|\,e^{t\arg a}) \qquad (0<\sigma\leqslant 1).$$

Now let
$$\tau(\sigma)=1-\sigma \qquad (0\leqslant\sigma\leqslant 1),$$
$$\tau(\sigma)=0 \qquad (\sigma\geqslant 1).$$

Also let M be a fixed integer such that $M\geqslant 1$, $R(a+M)>0$; and let $a=a+M+1$, so that

$$\zeta(s,a)=\sum_{n=0}^{M}(a+n)^{-s}+\zeta(s,a),$$

where $R(a)>1$. Hence

$$\zeta(s,a)=\sum_{n=0}^{M}(a+n)^{-s}+O(|t|^{\tau(\sigma)})+O(e^{t\arg a}|t|^{\tau(\sigma)}) \qquad (0<\sigma\leqslant 2,\ \sigma\neq 1)$$
$$=\sum_{n=0}^{M}(a+n)^{-s}+O(|t|^{\tau(\sigma)}\log|t|)+O(e^{t\arg a}|t|^{\tau(\sigma)}\log|t|)$$
$$(0<\sigma\leqslant 2).$$

If a be not real and positive, then if $t\arg a>0$, the first term in

---

* We use the symbols O, o in the sense introduced by Landau, *Primzahlen*, § 12. $f(x)=O[g(x)]$ means that $\operatorname{Lt}_{x\to\infty}\sup\{|f(x)|\div g(x)\}$ is finite; $f(x)=o[g(x)]$ means that
$$\operatorname{Lt}_{x\to\infty}\sup\{|f(x)|\div g(x)\}=0.$$

each summation is $O(e^{t \arg a})$ but not $o(e^{t \arg a})$, while each of the remaining terms is $o(e^{t \arg a})$; so that

$$\left.\begin{aligned} \xi(s, a) &= O(e^{t \arg a}) \\ &\neq o(e^{t \arg a}) \end{aligned}\right\} \quad (0 < \sigma \leqslant 2).$$

If $t \arg a \leqslant 0$, $\left.\begin{aligned} \xi(s, a) &= O(|t|^{\tau(\sigma)}) \\ &= O(|t|^{\tau(\sigma)} \log |t|) \end{aligned}\right\}$ $\begin{aligned} &(0 < \sigma \leqslant 2,\ \sigma \neq 1) \\ &(0 < \sigma < 2). \end{aligned}$

If, however, $\sigma \geqslant 2$,

$$|\xi(\sigma, a)| < (1+e^{t \arg a}) \sum_{n=0}^{\infty} (n+1)^{-2}.$$

This aggregate of results may be written concisely as follows

$$\left.\begin{aligned} \xi(s, a) &= O(e^{t \arg a}) + O(|t|^{\tau(\sigma)}) && (\sigma > 0,\ \sigma \neq 1) && (4) \\ &\neq o(e^{t \arg a}) && (\sigma > 0,\ t \arg a > 0) && (4a) \\ &= O(e^{t \arg a}) + O(|t|^{\tau(\sigma)} \log |t|) && (\sigma > 0) && (4b) \end{aligned}\right\},$$

where

$$\tau(\sigma) = 1-\sigma \quad (0 \leqslant \sigma \leqslant 1),$$

$$\tau(\sigma) = 0 \quad (\sigma > 1).$$

3. We next discuss $\xi(s, a)$, when $\sigma < 1$ and a is real.

Choose a_1 so that $a-a_1$ is an integer and $0 < a_1 \leqslant 1$, then it is easy to see that*

$$\begin{aligned} \xi(s, a)-\xi(s, a_1) &= O(1) && (a > 0) \\ &= O(1) \text{ or } O(e^{\pi |t|}) && (a < 0), \end{aligned}$$

according as t and $\arg a\ (= \pm \pi)$ have opposite or the same signs, so that always

$$\xi(s, a) = \xi(s, a_1) + O(1) + O(e^{t \arg a}).$$

Also, by the formula of Hurwitz, when $\sigma < 0$,

$$\begin{aligned} \xi(s, a_1) &= 2\Gamma(1-s) \left[\sin(\tfrac{1}{2}s\pi) \sum_1^{\infty} \frac{\cos 2n\pi a_1}{(2n\pi)^{1-s}} + \cos(\tfrac{1}{2}s\pi) \sum_1^{\infty} \frac{\sin 2n\pi a_1}{(2n\pi)^{1-s}}\right] \\ &= -i(2\pi)^{s-1} \Gamma(1-s)[\exp(\tfrac{1}{2}s\pi i)\, \xi(s, a_1) - \exp(-\tfrac{1}{2}s\pi i)\, \xi(s, -a_1)], \end{aligned}$$

where

$$\xi(s, a_1) = \sum_1^{\infty} \frac{\exp(2n\pi i a_1)}{n^{1-s}},$$

* The two ζ functions differ by a *finite* series.

so that*

$$\{1-\exp(2\pi i\alpha_1)\}\,\xi(s,\alpha_1) = \exp(2\pi i\alpha_1)+\sum_2^N \exp(2n\pi i\alpha_1)\left\{\frac{1}{n^{1-s}}-\frac{1}{(n-1)^{1-s}}\right\}$$
$$+\sum_{N+1}^{\infty}\int_{n-1}^{n}\exp(2n\pi i\alpha_1)(s-1)u^{s-2}\,du.$$

This gives the analytic continuation of $\xi(s,\alpha_1)$ when $\sigma<1$; by work similar to that of § 2, we may easily shew that

$$\{1-\exp(2\pi i\alpha_1)\}\,\xi_1(s,\alpha_1) = O(1)+O(|t|^{\sigma}) \qquad (\sigma<1,\ \sigma\neq 0)$$
$$= O(\log|t|)+O(|t|^{\sigma}\log|t|) \quad (\sigma<1).$$

But since $\log\Gamma(1-s)\sim(\frac{1}{2}-\sigma)\log|t|-\frac{1}{2}\pi|t|$, we deduce that

$$\left.\begin{aligned}\xi(s,\alpha) &= O(|t|^{\frac{1}{2}-\sigma})+O(|t|^{\frac{1}{3}})+O(e^{t\arg\alpha}) \qquad (\sigma<1,\ \sigma\neq 0) \quad (5a)\\ &= O(|t|^{\frac{1}{2}-\sigma}\log|t|)+O(|t|^{\frac{1}{3}}\log|t|)+O(e^{t\arg\alpha}) \quad (\sigma<1) \quad (5b)\end{aligned}\right\}.$$

4. Finally, we consider $\xi(s,\alpha)$ when $\sigma<1$ and α is not real; choose α as in § 2, so that $R(\alpha)>1$ and $\alpha-a=M$, where M is a positive integer (zero excluded); further, let $\alpha-\frac{1}{2}=\beta=ce^{i\gamma}$, where $c>0$, $0<|\gamma|<\frac{1}{2}\pi$.

By the theory of analytic continuation

$$\xi(s,\alpha) = \sum_{n=0}^{M-1}(\alpha+n)^{-s}+\xi(s,\alpha) = \xi(s,\alpha)+O(1)+O(e^{t\arg\alpha}).$$

There are really two cases to be considered, according as $\arg\alpha \gtrless 0$, *i.e.*, as $\gamma \gtrless 0$. We shall discuss the case $\gamma>0$ in full, indicating the changes to be made when $\gamma<0$.

By Barnes's formula† and Cauchy's theorem,

$$\xi(s,\alpha) = \frac{i\Gamma(1-s)}{2\pi}\int_C \frac{e^{-\alpha y}}{1-e^{-y}}(-y)^{s-1}\,dy$$
$$= \frac{i\Gamma(1-s)}{2\pi}\left[\int_D \frac{e^{-\alpha y}}{1-e^{-y}}(-y)^{s-1}\,dy - 2\pi i\times\text{the sum of the residues of}\right.$$
$$\left.\frac{e^{-\alpha y}}{1-e^{-y}}(-y)^{s-1}\text{ at the points‡ }-y=2\pi i,\ 4\pi i,\ \ldots,\ 2N\pi i\right];$$

---

* We take $\alpha_1\neq 1$; for, if $\alpha_1=1$, $\zeta(s,\alpha_1)$ is the ordinary ζ-function.

† *Messenger of Mathematics*, Vol. XXIX, p. 88.

‡ N is any integer.

the contour C starts from $+\infty$ encircles the origin (but enclosing no other singularities of the integrand) and returns to $+\infty$; initially $\arg(-y) = -\pi$, finally $\arg(-y) = +\pi$; the contour D differs from the contour C in enclosing the points given by $-y = 2\pi i, 4\pi i, \ldots, 2N\pi i$.

Consequently,

$$\zeta(s, a) = \frac{i\Gamma(1-s)}{2\pi}\int_D \frac{e^{-ay}}{1-e^{-y}}(-y)^{s-1}\,dy + \Gamma(1-s)\exp(\tfrac{1}{2}s\pi i)\sum_{n=1}^{N}\frac{\exp(2na\pi i)}{(2n\pi)^{1-s}}. \tag{6}$$

But, for *all* values of N, if $\sigma \leqslant 1$,

$$\begin{aligned}\left|\sum_{n=1}^{N}\frac{\exp(2na\pi i)}{(2n\pi)^{1-s}}\right| &\leqslant \sum_{n=1}^{\infty}\exp[R(2na\pi i)]\\ &< \sum_{n=1}^{\infty}\exp(-2nc\sin\gamma\pi)\\ &= O(1).\end{aligned} \tag{6a}$$

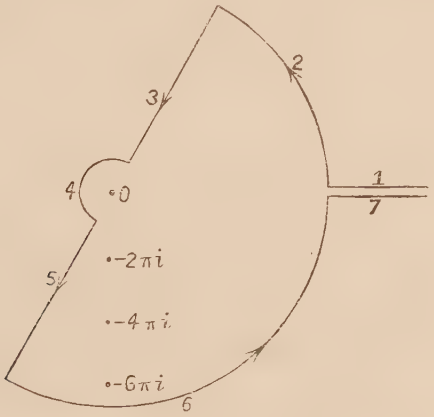
We now take the contour D to consist of the following curves:—

(i) The portions I_1, I_7 of the lines $\arg(-y) = \pm\pi$ for which
$$y \geqslant (2N+1)\pi.$$

(ii) The arcs I_2, I_6 of the circle $|y| = (2N+1)\pi$ terminated by $R(\beta y) = 0$, $\arg(-y) = \pm\pi$.

(iii) The portions I_3, I_5 of the line $R(\beta y) = 0$ for which
$$\pi \leqslant |y| \leqslant (2N+1)\pi.$$

(iv) The semicircle I_4 of $|y| = \pi$ for which $R(\beta y) \leqslant 0$.



Throughout the contour, $(-y)^{s-1}$ is specified by the inequality

$|\arg(-y)| \leqslant \pi$. We shall, in future, use the symbol K to denote any positive number independent of y, s, and N; for brevity we write

$$e^{-\alpha y}(1-e^{-y})^{-1}(-y)^{s-1} = \phi(y),$$

and the integral of $\phi(y)$ taken along I_4 we denote by J_n.

The proofs of the following statements (i)—(vii) may be left to the reader :—

(i) On the whole of D, $|(1-e^{-y})^{-1}| < K$, $|(e^{\frac{1}{2}y}-e^{-\frac{1}{2}y})^{-1}| < K|y|^{-1}$.

(ii) On I_1 and I_7,

$$|\phi(y)| < K|e^{-\alpha y}|\, y^{\sigma-1} e^{\pi|t|} < Ke^{-y} e^{\pi|t|},$$

since $$R(\alpha) > 1, \quad \sigma < 1,$$

so that
$$|J_1| + |J_7| < 2K \int_{(2\lambda+1)\pi}^{\infty} e^{-y} e^{\pi|t|}\, dy$$
$$< 2K \exp\{\pi|t| - (2N+1)\pi\}. \tag{7a}$$

(iii) If, on I_2, $\quad -y = (2N+1)\pi e^{-i(\frac{1}{2}\pi+\gamma+\theta)} \quad (0 \leqslant \theta \leqslant \frac{1}{2}\pi-\gamma)$,

then, on I_2, $|\phi(y)\,dy| < K|e^{-\beta y}(-y)^{s-2}\,dy|$
$$< K \exp\{-(2N+1)\pi c \sin\theta + (\tfrac{1}{2}\pi+\gamma+\theta)t\}\, d\theta.$$

Noting that $\sin\theta \geqslant 2\theta/\pi$, if $0 \leqslant \theta \leqslant \frac{1}{2}\pi$, *and taking* $2c(2N+1) > |t|$, then
$$|\phi(y)\,dy| < K \exp\{-(2N+1)\,2c\theta + (\tfrac{1}{2}\pi+\gamma)t + \theta|t|\}\, d\theta,$$

so that $$|J_2| < K\{2c(2N+1) - |t|\}^{-1} \exp\{(\tfrac{1}{2}\pi+\gamma)t\}. \tag{7b}$$

(iv) If, on I_6, $\quad -y = (2N+1)\pi e^{i(\frac{1}{2}\pi-\gamma+\theta)} \quad (0 \leqslant \theta \leqslant \frac{1}{2}\pi+\gamma)$,

then, in a similar manner, since $\sin\theta/\theta \geqslant \sin(\frac{1}{2}\pi+\gamma)/(\frac{1}{2}\pi+\gamma)$,
$$|J_6| < K\{c\lambda\pi(2N+1) - |t|\}^{-1} \exp\{-(\tfrac{1}{2}\pi-\gamma)t\}, \tag{7c}$$
where $\lambda = \sin(\frac{1}{2}\pi+\gamma)/(\frac{1}{2}\pi+\gamma)$, if *we assume* $(2N+1)c\lambda\pi > |t|$.

(v) If, on I_3, $\quad -y = re^{-(\frac{1}{2}\pi+\gamma)}$,

$$|\phi(y)| < K|e^{-\beta y}(-y)^{s-2}| < Kr^{\sigma-2}\exp\{(\tfrac{1}{2}\pi+\gamma)t\},$$

so that
$$|J_3| < \int_{\pi}^{(2N+1)\pi} Kr^{\sigma-2}\exp\{(\tfrac{1}{2}\pi+\gamma)t\}\, dr$$
$$< K\exp\{(\tfrac{1}{2}\pi+\gamma)t\}, \tag{7d}$$
so long as $\sigma < 1$.

(vi) If, on I_5, $\quad -y = re^{i(\frac{1}{2}\pi-\gamma)}$,

we deduce, similarly, $\quad |J_5| < K\exp\{-(\frac{1}{2}\pi-\gamma)t\}$. (7e)

(vii) On I_4, $\quad |\phi(y)| < K\exp\{\gamma t+\frac{1}{2}\pi|t|\}$,

so that $\quad |J_4| < K\exp\{\gamma t+\frac{1}{2}\pi|t|\}$. (7f)

Choosing N so that

$$(2N+1)\,c\lambda\pi \geqslant |t|+1, \quad (2N+1) \geqslant |t|+1,$$

and collecting the results (7a)—(7f), we get

$$\left|\int_D e^{-ay}(1-e^{-y})^{-1}(-y)^{s-1}\,dy\right|$$

$$< 2K\exp\{\pi|t|-(2N+1)\pi\}+3K\exp(\tfrac{1}{2}\pi|t|+\gamma t)$$

$$+2K\exp\{\tfrac{1}{2}\pi|t|+\gamma t\}\{(2N+1)\,c\lambda\pi-|t|\}^{-1};$$

and hence, when $\sigma < 1$,

$$\left|\int_D e^{-ay}(1-e^{-y})^{-1}(-y)^{s-1}\,dy\right| = O[\exp(\tfrac{1}{2}\pi|t|)]+O[\exp(\tfrac{1}{2}\pi|t|+\gamma t)].$$

Noting that as $|t|\to\infty$,

$$\Gamma(1-s)\to\exp\{(\tfrac{1}{2}\pi-\sigma)\log|t|-\tfrac{1}{2}\pi|t|+\tfrac{1}{2}\log 2\pi\},$$

we deduce from (6) that

$$\xi(s,a) = O(|t|^{\frac{1}{2}-\sigma})\,[O(1)+O(e^{t\arg\beta})].$$

[*We get the same result if γ is negative, by modifying the contour C, not into the contour D, but into a contour whose appearance is that of the contour D reflected in the real axis.*]

Since $|\arg\beta| < |\arg\alpha| < |\arg a|$, we see from the equation

$$\xi(s,a) = \xi(s,\alpha)+O(1)+O(e^{t\arg\alpha}),$$

that $\quad \xi(s,a) = O(|t|^{\frac{1}{2}-\sigma})+O(e^{t\arg a})+O(1) \quad (\sigma<1)$, (8)

and if $t\arg a > 0$, $\quad \xi(s,a) \neq o(e^{t\arg a})$.

By this result, combined with (4), if $t\arg a > 0$,

$$\xi(s,a) = O(e^{t\arg a}) \neq o(e^{t\arg a}) \tag{8a}$$

for every value of σ.

Now, by (4*b*), (5*b*), and (8), if $t \arg a \leqslant 0$,

$$\begin{aligned} \xi(s, a)/\log s &= O(1) && (\sigma \geqslant 1) \\ &= O(|t|^{1-\sigma}) && (0 \leqslant \sigma \leqslant 1) \\ &= O(|t|^{\frac{1}{2}-\sigma}) && (\sigma \leqslant 0). \end{aligned}$$

Hence, by the Phragmen-Lindelöf theorem,*

$$\xi(s, a)/\log s = O(|t|^{\frac{1}{2}(1-\sigma)}),$$

when $0 \leqslant \sigma \leqslant 1$, so that *always*

$$\xi(s, a) = O(|t|^{\tau(\sigma)} \log|t|) + O(e^{t \arg a}), \tag{8b}$$

where

$$\begin{aligned} \tau(\sigma) &= 0 && (\sigma \geqslant 1), \\ \tau(\sigma) &= \tfrac{1}{2}(1-\sigma) && (0 \leqslant \sigma \leqslant 1), \\ \tau(\sigma) &= \tfrac{1}{2}-\sigma && (\sigma \leqslant 0). \end{aligned}$$

The term $\log|t|$ may be dropped when $\sigma < 0$ or $\sigma > 1$, and, if a be not real, when $\sigma = 0$.

It is supposed that σ is fixed throughout; as $\sigma \to -\infty$ the constants, implied in the symbols O, tend to infinity.

All the results contained in the paper are summed up in the formulæ (8*a*), (8*b*).

---

* Landau, *Primzahlen*, § 237; the theorem is that, if

$$f(s) = O(|t|^k) \quad (\sigma = \sigma_1), \qquad f(s) = O(1) \quad (\sigma = \sigma_2),$$
$$f(s) = O(|t|^c) \quad (\sigma_1 \leqslant \sigma \leqslant \sigma_2),$$

where $f(s)$ is analytic when $\sigma_1 \leqslant \sigma \leqslant \sigma_2$, and *either* $t > K$ *or* $t < -K$, then for the range of values of s under consideration $f(s) = O(|t|^{k(\sigma_2-\sigma)/(\sigma_2-\sigma_1)})$.

ON THE CONVERGENCE OF SERIES OF ORTHOGONAL FUNCTIONS

By E. W. HOBSON.

[Received November 30th, 1912.—Read December 12th, 1912.]

A NEW line of investigation in the theory of Fourier series and of other series of orthogonal functions was opened by the discovery made by Liapounoff and Hurwitz that the coefficients of the Fourier series

$$\tfrac{1}{2}a_0+\sum_{n=1}(a_n\cos nx+b_n\sin nx)$$

which corresponds to a function $f(x)$ such that $\{f(x)\}^2$ is integrable are such that $\frac{1}{2}a_0^2+\sum_{n=1}(a_n^2+b_n^2)$ converges to the value $\frac{1}{\pi}\int_{-\pi}^{\pi}\{f(x)\}^2dx$, independently of any knowledge as to the convergence of the Fourier's series. This theorem was established in its most general form by Fatou,* for the case of a function $f(x)$, which, whether limited or not, is such that $\{f(x)\}^2$ is summable, in the sense that it has a Lebesgue integral in the interval $(-\pi, \pi)$, the coefficients being also Lebesgue integrals. The theorem was extended to the case of series of other orthogonal functions, in connection with the theory of integral equations.

If the coefficients $c_1, c_2, \ldots, c_n, \ldots$ of a series $\sum_{n=1} c_n\phi_n(x)$, where the $\{\phi_n(x)\}$ form a sequence of normal orthogonal functions, are given, the question arises whether a function $f(x)$ exists such that the constants c_n are the coefficients $\int_0^1 f(x)\,\phi_n(x)\,dx$, formed in Fourier's manner, corresponding to the function $f(x)$.

It was established† by F. Riesz and Fischer that the necessary and sufficient condition that a function $f(x)$, whose square is summable, should

* *Acta Math.*, Vol. 30 (1906). Other proofs have been given by Lebesgue and by W. H. Young (see his paper on "Successions of Integrals and Fourier Series," *Proc. London Math. Soc.*, Ser. 2, Vol. 11).

† F. Riesz, *Göttinger Nachrichten*, 1907, also *Comptes Rendus* of the French Academy, May 17, 1909; Fischer, *Comptes Rendus* of the French Academy, 1907, Vol. 144, p. 1022.

exist, such that the given constants $c_1, c_2, \ldots, c_n, \ldots$ are the Fourier's constants $\int_0^1 f(x)\,\phi_1(x)\,dx$, $\int_0^1 f(x)\,\phi_2(x)\,dx$, $\ldots$, $\int_0^1 f(x)\,\phi_n(x)\,dx$, $\ldots$ is that the series $c_1^2+c_2^2+\ldots+c_n^2+\ldots$ is convergent.* The further question arises whether, and how far, the series $\sum_{n=1} c_n\phi_n(x)$ is convergent. The earliest result as regards this question was obtained, for the case of the ordinary Fourier's series, by Fatou (*loc. cit.*), who shewed that the series

$$\tfrac{1}{2}a_0+\sum_{n=1}(a_n\cos nx+b_n\sin nx)$$

is convergent at every point of the interval $(-\pi, \pi)$ with the exception at most of a set of points of zero measure, provided

$$\lim_{n=\infty} na_n = 0, \quad \text{and} \quad \lim_{n=\infty} nb_n = 0.$$

It was then proved† by Weyl, by means of an ingenious method originally due to Jerosch, and further developed by Weyl himself, that the series $\sum_{n=1} c_n\phi_n(x)$ converges everywhere in the interval (0, 1), with the possible exception of a set of points of zero measure, in case the series $\sum_{n=1} n^{\frac{1}{2}}c_n^2$ is convergent. Weyl also proved, by a similar method, that, in case the functions $\phi_n(x)$ are such that $|\phi_n(x)|$ is less than some fixed number, for all the values of n and x, the convergence of the series $\Sigma n^{\frac{1}{3}}c_n^2$ is sufficient to ensure the convergence of the series $\sum_{n=1} c_n\phi_n(x)$ in the same sense as before; this condition is clearly satisfied if

$$\lim_{n=\infty} n^{\frac{2}{3}+\lambda}c_n = 0,$$

for some value of $\lambda(>0)$; or, in the case of the ordinary Fourier's series, if

$$\lim_{n=\infty} n^{\frac{2}{3}+\lambda}a_n = 0, \quad \text{and} \quad \lim_{n=\infty} n^{\frac{2}{3}+\lambda}b_n = 0,$$

for some value of $\lambda(>0)$.

In the present communication I have established the wider result that it is sufficient for the convergence of the series $\sum_{n=1} c_n\phi_n(x)$ at all points of (0, 1) with the exception at most of those belonging to an exceptional set of measure zero, that a number $k(>0)$ should exist such that the series $\sum_{n=1} n^k c_n^2$ is convergent. This condition is clearly satisfied, if a number

---

* Various proofs of this theorem are discussed in a paper by W. H. and G. C. Young, *Quarterly Journal*, Vol. XLIV.

† See *Math. Annalen*, Vol. 67, p. 225.

$\lambda\,(>0)$ exists, such that

$$\lim_{n=\infty} n^{\frac{1}{2}+\lambda} c_n = 0,$$

or, in the case of the ordinary trigonometrical series, that

$$\lim_{n=\infty} n^{\frac{1}{2}+\lambda} a_n = 0, \quad \text{and} \quad \lim_{n=\infty} n^{\frac{1}{2}+\lambda} b_n = 0.$$

The proof is developed by a method different from the one employed by Weyl to establish his theorems corresponding to $k = \frac{1}{2}$, and $k = \frac{1}{3}$, in that no use is made of the *Majorantenreihen*. It appears that the restriction that $|\phi_n(x)|$ should be limited for all values of n and x, which Weyl employed for the case $k = \frac{1}{3}$, is entirely unnecessary.

1. Let $\phi_1(x)$, $\phi_2(x)$, ..., $\phi_n(x)$, ... denote a set of normal orthogonal functions for the interval (0, 1); the functions accordingly satisfy the conditions

$$\int_0^1 \{\phi_n(x)\}^2 dx = 1, \qquad \int_0^1 \phi_n(x)\,\phi_{n'}(x)\,dx = 0,$$

for $n \neq n'$.

Let it be assumed that the coefficients c_1, c_2, ..., c_n, of a series

$$c_1\phi_1(x)+c_2\phi_2(x)+\ldots+c_n\phi_n(x)+\ldots,$$

are such that the series $1^k c_1^2+2^k c_2^2+\ldots+n^k c_n^2+\ldots$

is convergent, for some value of k greater than zero. It will then be shewn that the series $\sum_{n=1}^{\infty} c_n\phi_n(x)$ is convergent for all values of x in the interval (0, 1) with the exception at most of those values that belong to a set of points of measure zero.

Let λ be a positive integer so chosen that $1/\lambda \leqslant k$, then it is clear that the series

$$1^{1/\lambda} c_1^2+2^{1/\lambda} c_2^2+\ldots+n^{1/\lambda} c_n^2+\ldots$$

is convergent.

The partial sum $c_1\phi_1(x)+c_2\phi_2(x)+\ldots+c_n\phi_n(x)$ will be denoted by $s_n(x)$. But, for convenience of printing, in the investigation $s(x, n)$ will be written for $s_n(x)$; also $c(n)$ will be written for c_n.

We choose a positive number δ which may be arbitrarily small, and we then choose a positive integer r_1 so large that the series

$$r_1 c^2(r_1^\lambda)+(r_1^\lambda+1)^{1/\lambda}\, c^2(r_1^\lambda+1)+(r_1^\lambda+2)^{1/\lambda}\, c^2(r_1^\lambda+2)+\ldots$$

converges to a sum that is less than δ^3. In view of a later requirement,

r_1 will also be taken to be greater than all the binomial coefficients λ, $\frac{\lambda(\lambda-1)}{2!}$, $\frac{\lambda(\lambda-1)(\lambda-2)}{3!}$, ... in the expanded form of $(1+x)^\lambda$.

Consider the set of indices

$$r_1^\lambda,\ (r_1+1)^\lambda,\ (r_1+2)^\lambda,\ \ldots,\ m^\lambda;$$

we have

$$\int_0^1 \{s(x, m^\lambda)-s(x, t^\lambda)\}^2\,dx = c^2(t^\lambda+1)+c^2(t^\lambda+2)+\ldots+c^2(m^\lambda),$$

where $t<m$. It follows, by letting $t=r_1, r_1+1, \ldots, m-1$, successively, and adding the equations, that

$$\int_0^1 \sum_{t=r_1}^{t=m-1} \{s(x, m^\lambda)-s(x, t^\lambda)\}^2\,dx$$

$$= \sum_{t=r_1^\lambda+1}^{(r_1+1)^\lambda} c^2(t) + 2\sum_{t=(r_1+1)^\lambda+1}^{(r_1+2)^\lambda} c^2(t)+\ldots+(m-r_1)\sum_{t=(m-1)^\lambda+1}^{m^\lambda} c^2(t).$$

Since $m-r_1 < \{(m-1)^\lambda+1\}^{1/\lambda}$, we see that the expression on the right-hand side is less than $\sum_{t=r_1^\lambda+1}^{m^\lambda} t^{1/\lambda}c^2(t)$.

It follows that the set of points at which

$$\sum_{t=r_1}^{t=m-1} \{s(x, m^\lambda)-s(x, t^\lambda)\}^2 \geqslant \delta^2$$

has a measure less than $\frac{1}{\delta^2}\sum_{t=r_1^\lambda+1}^{m^\lambda} t^{1/\lambda}c^2(t)$. From this we see that in a set of points G_m of measure

$$> 1-\frac{1}{\delta^2}\sum_{t=r_1^\lambda+1}^{m^\lambda} t^{1/\lambda}c^2(t) > 1-\delta,$$

all the numbers

$$|s(x, m^\lambda)-s(x, r_1^\lambda)|,\ |s(x, m^\lambda)-s\{x, (r_1+1)^\lambda\}|,\ \ldots,$$

$$|s(x, m^\lambda)-s\{x, (m-1)^\lambda\}|,$$

are less than δ. In this set G_m of points

$$s(x, t^\lambda)-s(x, t'^\lambda)| < 2\delta,$$

for all pairs of values of t and t', such that

$$r_1 \leqslant t \leqslant m, \quad \text{and} \quad r_1 \leqslant t' \leqslant m.$$

Let m have the values $m_1, m_2, m_3, \ldots$ successively, in an increasing sequence of integers. The corresponding sets of points being denoted by $G_{m_1}, G_{m_2}, G_{m_3}, \ldots$, it is clear that each of these sets contains the next, for any point of G_{m_2} belongs to G_{m_1}.

Since the measure of each of the sets is $> 1-\delta$, and since each one contains the succeeding ones, it follows that there exists a set G, of measure $\geqslant 1-\delta$, which consists of those points that belong to all the sets $G_{m_1}, G_{m_2}, G_{m_3}, \ldots$. In this set G, of measure $\geqslant 1-\delta$, we have

$$| s(x, t^\lambda) - s(x, t'^\lambda) | < 2\delta,$$

for all pairs of values of t and t', such that $t \geqslant r_1$, $t' \geqslant r_1$.

Next, let us consider the following groups of indices which are $\geqslant r^\lambda$, and $\leqslant (r+1)^\lambda$, where r is a number $\geqslant r_1$:—

$$\begin{array}{llll}
r^\lambda,\ r^\lambda+1,\ r^\lambda+2, & \ldots,\ r^\lambda+r; & H_1^{(1)}, \\
r^\lambda+r,\ r^\lambda+r+1,\ r^\lambda+r+2, & \ldots,\ r^\lambda+2r; & H_2^{(1)}, \\
\ldots \quad \ldots \quad \ldots \quad \ldots & \ldots \quad \ldots \quad \ldots & \\
r^\lambda+(P_1-1)r,\ r^\lambda+(P_1-1)r+1, & \ldots,\ r^\lambda+P_1 r; & H_{P_1}^{(1)}, \\
r^\lambda+P_1 r,\ r^\lambda+P_1 r+1; & \ldots, & H_{P_1+1}^{(1)},
\end{array}$$

where P_1 denotes $\lambda r^{\lambda-2} + \dfrac{\lambda(\lambda-1)}{2!} r^{\lambda-3} + \ldots + \lambda,$

so that $$r^\lambda + P_1 r + 1 = (r+1)^\lambda.$$

The last index in each group is also the first index in the next, and each group except the last contains $r+1$ indices. Consider the group $H_m^{(1)}$, where $1 \leqslant m \leqslant P_1$. We have

$$\begin{aligned}
\int_0^1 \Big[& \{ s[x, r^\lambda+(m-1)r] - s[x, r^\lambda+(m-1)r+1] \}^2 \\
& + \{ s[x, r^\lambda+(m-1)r] - s[x, r^\lambda+(m-1)r+2] \}^2 + \ldots \\
& + \{ s[x, r^\lambda+(m-1)r] - s(x, r^\lambda+mr) \}^2 \Big] dx \\
= {} & rc^2 \{ r^\lambda+(m-1)r+1 \} + (r-1)c^2 \{ r^\lambda+(m-1)r+2 \} + \ldots + c^2(r^\lambda+mr) \\
< {} & \sum_{t=r^\lambda+(m-1)r+1}^{t=r^\lambda+mr} t^{1/\lambda} c_t^2.
\end{aligned}$$

If $m = P_1+1$, we have

$$\int_0^1 \{s(x, r^\lambda+P_1 r)-s(x, r^\lambda+P_1 r+1)\}^2 dx = c^2(r^\lambda+P_1 r+1) < t^{1/\lambda} c_t^2,$$

where $$t = r^\lambda+P_1 r+1.$$

It follows that in a set of points of measure

$$> 1-\frac{1}{\delta^2} \sum_{t=r^\lambda+(m-1)r+1}^{t=r^\lambda+mr} t^{1/\lambda} c_t^2,$$

we have $$|s[x, r^\lambda+(m-1)r]-s[x, r^\lambda+(m-1)r+k]| < \delta,$$

for all the values 1, 2, 3, ..., r of k. Hence, in this set, we have

$$|s[x, r^\lambda+(m-1)r+t]-s[x, r^\lambda+(m-1)r+t']| < 2\delta,$$

for all pairs of values of t and t', such that $1 \leqslant t \leqslant r$, $1 \leqslant t' \leqslant r$. This holds for each of the groups $H_m^{(1)}$, $m = 1, 2, \ldots, P_1$; and also for the group $H_{P_1+1}^{(1)}$, in a set of measure

$$> 1-\frac{1}{\delta^2} t^{1/\lambda} c_t^2,$$

where $$t = (r+1)^\lambda,$$

we have $$|s(x, r^\lambda+P_1 r)-s[x, (r+1)^\lambda]| < \delta.$$

Taking all the values of m successively, we see that in a certain set E_1 of measure

$$> 1-\frac{1}{\delta^2} \sum_{t=r^\lambda}^{t=(r+1)^\lambda} t^{1/\lambda} c_t^2,$$

the condition $$|s(x, a_1)-s(x, a)| < \delta$$

is satisfied for all integers a, such that

$$r^\lambda \leqslant a < (r+1)^\lambda,$$

when a_1 denotes the first index in that group $H_m^{(1)}$ to which a belongs.

Next, consider the indices r^λ, $r^\lambda+r$, $r^\lambda+2r$, ..., $r^\lambda+r^2$, $r^\lambda+r^2+r$, ..., $r^\lambda+P_1 r$. As before, we define a number of groups of indices, each of which, except the last, contains $r+1$ indices. Thus, if $P_1 = rP_2+\lambda$,

where P_2 denotes $\lambda r^{\lambda-3}+\frac{\lambda(\lambda-1)}{2!}r^{\lambda-4}+\ldots+\frac{\lambda(\lambda-1)}{2}$, the groups are taken to be

$$\begin{aligned}
&r^\lambda,\ r^\lambda+r,\ r^\lambda+2r,\ \ldots,\ r^\lambda+r^2; && H_1^{(2)},\\
&r^\lambda+r^2,\ r^\lambda+r^2+r,\ r^\lambda+r^2+2r,\ \ldots,\ r^\lambda+2r^2; && H_2^{(2)},\\
&\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots &&\\
&r^\lambda+(m-1)r^2,\ r^\lambda+(m-1)r^2+r,\ \ldots,\ r^\lambda+mr^2; && H_m^{(2)},\\
&\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots &&\\
&r^\lambda+(P_2-1)r^2,\ r^\lambda+(P_2-1)r^2+r,\ \ldots,\ r^\lambda+P_2r^2; && H_{P_2}^{(2)},\\
&r^\lambda+P_2r^2,\ \ldots,\ r^\lambda+P_2r^2+\lambda r; && H_{P_2+2}^{(2)}.
\end{aligned}$$

We see that, taking the group $H_m^{(2)}$,

$$\begin{aligned}
\int_0^1\Big[&\{s[x, r^\lambda+(m-1)r^2]-s[x, r^\lambda+(m-1)r^2+r]\}^2\\
&+\{s[x, r^\lambda+(m-1)r^2]-s[x, r^\lambda+(m-1)^2+2r]\}^2+\ldots\\
&+\{s[x, r^\lambda+(m-1)r^2]-s(x, r^\lambda+mr^2)\}^2\Big]dx\\
=\ & r\{c^2[r^\lambda+(m-1)r^2+1]+\ldots+c^2[r^\lambda+(m-1)r^2+r]\}\\
&+(r-1)\{c^2[r^\lambda+(m-1)r^2+r+1]+\ldots+c^2[r^\lambda+(m-1)r^2+2r]\}\\
&+\ldots\\
&+c^2(r^\lambda+mr^2).
\end{aligned}$$

The expression on the right-hand side is

$$<\sum_{t=r^\lambda+(m-1)r^2+1}^{t=r^\lambda+mr^2} t^{1/\lambda}c_t^2.$$

For the group $H_{P_2+1}^{(2)}$, the expression is

$$\begin{aligned}
&\lambda\{c^2(r^\lambda+P_2r^2+1)+\ldots+c^2(r^\lambda+P_2r^2+r)\}\\
&\qquad+(\lambda-1)\{c^2(r^\lambda+P_2r^2+r+1)+\ldots\}+\ldots,
\end{aligned}$$

and since r_1 has been so chosen that $r_1^\lambda>\lambda$, we see that this is less than

$$\sum_{t=r^\lambda+P_2r^2}^{t=r^\lambda+P_2r^2+\lambda r} t^{1/\lambda}c_t^2.$$

As before, we conclude that there exists a set E_2, of measure greater than

$$1-\frac{1}{\delta^2}\sum_{t=r^\lambda}^{t=(r+1)^\lambda} t^{1/\lambda}c_t^2,$$

in which $$|s(x, \alpha)-s(x, \alpha_2)| < \delta,$$

for all values of α which belong to the set r^λ, $r^\lambda+r$, ..., $(r+1)^\lambda$; where α_2 denotes the first index in that set $H_m^{(2)}$ to which α belongs.

We proceed, as before, with the indices

$$r^\lambda,\ r^\lambda+r^2,\ r^\lambda+2r^2,\ \ldots,\ r^\lambda+P_2r^2;$$

and write $$P_2 = rP_3+\frac{\lambda(\lambda-1)}{2}.$$

We define as before the sets $H_1^{(3)}$, $H_2^{(3)}$, ..., $H_{P_2+1}^{(3)}$, each of which contains $r+1$ indices, except that the last contains only $\frac{1}{2}\lambda(\lambda-1)$ indices, and this is by hypothesis $< r_1$. As before, we conclude that in a set E_3 of measure

$$> 1-\frac{1}{\delta^2}\sum_{t=r^\lambda}^{t=(r+1)^\lambda} t^{1-\lambda}c_t^2,$$

the condition $|s(x, \alpha)-s(x, \alpha_3)|$ is satisfied for all values of α, such that α belongs to the set r^λ, $r^\lambda+r^2$, ..., $r^\lambda+P_2r^2$, and where α_3 is the first index of that set $H_m^{(3)}$ to which α belongs.

We continue this process until we have to divide the indices

$$r^\lambda,\ r^\lambda+r^{\lambda-2},\ \ldots,\ r^\lambda+r^{\lambda-1},\ r^\lambda+r^{\lambda-1}+r^{\lambda-2},\ \ldots,\ r^\lambda+\lambda r^{\lambda-1}+\tfrac{1}{2}\lambda(\lambda-1)r^{\lambda-2},$$

into sets $$H_1^{(\lambda-1)},\ H_2^{(\lambda-1)},\ \ldots,\ H_{P_{\lambda-1}+1}^{(\lambda-1)}.$$

As before, there exists a set $E_{\lambda-1}$ of measure greater than

$$1-\frac{1}{\delta^2}\sum_{t=r^\lambda}^{t=(r+1)^\lambda} t^{1/\lambda}c_t^2,$$

in which $$|f(x, \alpha)-f(x, \alpha_{\lambda-1})| < \delta,$$

where α is any of the above indices, and $\alpha_{\lambda-1}$ is the first index of the group $G_m^{(\lambda-1)}$ to which α belongs.

Lastly, we have to consider the set of indices r^λ, $r^\lambda+r^{\lambda-1}$, ..., $r^\lambda+\lambda r^{\lambda-1}$, which we denote by $H^{(\lambda)}$. As before, it is seen that there exists a set of points E_λ of measure

$$> 1-\frac{1}{\delta^2}\sum_{t=r^\lambda}^{t=(r+1)^\lambda} t^{1/\lambda}c_t^2,$$

at every point of which $$|s(x, \alpha)-s(x, \alpha_\lambda)| < \delta,$$

where, in this case, α_λ has the unique value r^λ.

Now the sets E_1, E_2, E_3, ..., E_λ have each a measure

$$> 1-\frac{1}{\delta^2}\sum_{t=r^\lambda}^{t=(r+1)^\lambda} t^{1/\lambda}c_t^2;$$

therefore there exists a set of points F_r of measure

$$> 1-\frac{\lambda}{\delta^2}\sum_{t=r^\lambda}^{t=(r+1)^\lambda} t^{1-\lambda}c_t^2,$$

each point of which belongs to all the sets $E_1, E_2, \ldots, E_\lambda$.

Now let α denote any integer such that $r^\lambda \leqslant \alpha < (r+1)^\lambda$, then α is in one of the sets $H_1^{(1)}, H_2^{(1)}, \ldots, H_{P_1+1}^{(1)}$; therefore there is a number $\alpha^{(1)}$, belonging to the set $r^\lambda, r^\lambda+r, \ldots, r^\lambda+P_1 r$, such that in the set F_r, we have

$$|s(x,\alpha)-s(x,\alpha^{(1)})| < \delta.$$

Again, $\alpha^{(1)}$ belongs to one of the sets $H_1^{(2)}, H_2^{(2)}, \ldots, H_{P_2+1}^{(2)}$; therefore there exists a number $\alpha^{(2)}$ among those numbers which are the first indices of these sets, such that in F_r, we have

$$|s(x,\alpha^{(1)})-s(x,\alpha^{(2)})| < \delta,$$

and so on. It follows since

$$|s(x,r^\lambda)-s(x,\alpha)| \leqslant |s(x,\alpha)-s(x,\alpha^{(1)})| + |s(x,\alpha^{(1)})-s(x,\alpha^{(2)})| + \ldots + |s(x,\alpha^{(\lambda)})-s(x,r^\lambda)|,$$

that

$$|s(x,r^\lambda)-s(x,\alpha)| < \lambda\delta$$

for all points of F_r.

Now give to r the values $r_1, r_1+1, r_1+2, \ldots$, successively; we thus obtain sets of points $F_{r_1}, F_{r_1+1}, F_{r_1+2}, \ldots$; the measure of F_{r_1+s} being

$$> 1-\frac{\lambda}{\delta^2}\sum_{t=(r_1+s)^\lambda}^{t=(r_1+s+1)^\lambda} t^{1/\lambda}c_t^2.$$

There exists a set of points K, of measure

$$> 1-\frac{2\lambda}{\delta^2}\sum_{t=r_1^\lambda}^{\infty} t^{1/\lambda}c_t^2,$$

each point of which belongs to all the sets $F_{r_1}, F_{r_1+1}, \ldots$.

At a point x of this set K, we have

$$|s(x,\alpha)-s(x,r^\lambda)| < \lambda\delta,$$

for all indices α which are $\geqslant r_1^\lambda$, where r^λ is such that

$$r^\lambda \leqslant \alpha < (r+1)^\lambda.$$

In accordance with the choice which we have made of the integer r_1, the measure of K is $> 1-2\lambda\delta$. In the set G which is of measure $\geqslant 1-\delta$, we have

$$|s(x,t^\lambda)-s(x,t'^\lambda)| < 2\delta,$$

for all values of t and t' which are not less than r_1. The sets K and G have in common a set L_δ of measure greater than $1-(2\lambda+1)\,\delta$. This set L_δ has the property that for any two integers α, α', neither of which is $< r_1^\lambda$, the condition

$$|\,s(x,\,\alpha)-s(x,\,\alpha')\,| < 2\,(\lambda+1)\,\delta\,;$$

for if t^λ, t'^λ are such that

$$t^\lambda \leqslant \alpha < (t+1)^\lambda, \quad t'^\lambda \leqslant \alpha' < (t'+1)^\lambda,$$

we have $|\,s(x,\,\alpha)-s(x,\,\alpha')\,| \leqslant |\,s(x,\,\alpha)-s(x,\,t^\lambda)\,| + |\,s(x,\,\alpha')-s(x,\,t'^\lambda)|$

$$+ |\,s(x,\,t^\lambda)-s(x,\,t'^\lambda)\,|$$

$$< 2(\lambda+1)\,\delta.$$

If ϵ be an arbitrarily fixed positive number, we may take

$$\delta = \epsilon/(2\lambda+2)\,;$$

therefore a set D_ϵ of points exists, of measure $> 1-\epsilon$, such that

$$|\,s_n(x)-s_{n'}(x)\,| < \epsilon$$

for all points of the set, and for all values of n and n' which are both greater than some fixed integer i_ϵ. Let ξ be an arbitrarily chosen positive number, then a sequence of values of ϵ can be so chosen that $\epsilon_1+\epsilon_2+\ldots$ converges to a value less than ξ. The sets D_{ϵ_1}, D_{ϵ_2}, ... have in common a set D of points which belong to all of them, and the measure of D is

$$> 1-(\epsilon_1+\epsilon_2+\ldots) > 1-\xi.$$

In this set D the sequence $\{s_n(x)\}$ converges uniformly; for, at all points x of this sequence, $|\,s_n(x)-s_{n'}(x)\,|$ is less than the arbitrarily small number ϵ_m, for all values of n and n' which are not less than some fixed integer dependent on ϵ_m.

It has now been shewn that the series

$$c_1\phi_1(x)+c_2\phi_2(x)+\ldots+c_n\phi_n(x)+\ldots$$

converges uniformly in some set of points of measure $> 1-\xi$, where ξ is an arbitrarily chosen positive number (< 1). Since ξ is arbitrarily small, it follows that the series is convergent at all points of a set of which the measure is equal to that of the interval (0, 1) of representation. This mode of convergence is said to be quasi-uniform convergence in the interval.* The following theorem has now been established:—

* This mode of convergence I have discussed in a previous paper, "On the Representation of a Summable Function by a Series of Finite Polynomials," see *ante*, p. 162.

If $\phi_1(x)$, $\phi_2(x)$, ... *are a sequence of normal orthogonal functions, and if the series* $1^k c_1^2 + 2^k c_2^2 + \ldots + n^k c_n^2 + \ldots$ *converges for some value of* k *that is greater than zero, then the series* $c_1\phi_1(x) + c_2\phi_2(x) + \ldots + c_n\phi_n(x) + \ldots$ *converges at all points of the interval for which the orthogonal functions are defined, with at most the exception of a set of points of which the measure is zero. The convergence is quasi-uniform, in the sense that a set of points of measure less than, but as nearly equal as we please to, that of the interval can be determined so that the convergence of the series is uniform in the set of points.*

It will be observed that no assumption is made in this theorem that the set of normal orthogonal functions is a complete set. Thus, if $\phi_1(x)$, $\phi_2(x)$, ..., $\phi_n(x)$, ... is a complete set, the condition stated in the theorem is sufficient to ensure the convergence, in the sense stated in the enunciation, of all the series of the form

$$c_1\phi_{n_1}(x) + c_2\phi_{n_2}(x) + \ldots + c_r\phi_{n_r}(x) + \ldots,$$

where n_1, n_2, ..., n_r, ... is a sequence of increasing integers defined according to any prescribed law.

The particular case of the above theorem which arises when k has the value $\frac{1}{2}$ was established* by Weyl by a somewhat intricate method depending upon the use of *Majorantenreihen*. Weyl also established the theorem for the case $k = \frac{1}{3}$, on the assumption that the functions $\phi_n(x)$ are less in absolute value than some fixed positive number, for all the values of n and x; this last restriction has been shewn above to be unnecessary.

In the case of the ordinary trigonometrical series

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

the convergence of the two series $\sum_{n=1}^{\infty} n^k a_n^2$, $\sum_{n=1}^{\infty} n^k b_n^2$ for some value of k greater than zero is sufficient to ensure the convergence of the series at every point of the interval $(-\pi, \pi)$ with the exception at most of a set of points of which the measure is zero.

2. In case a number p greater than $\frac{1}{2}$ exists, such that

$$\lim_{n=\infty} (n^{\frac{1}{2}+p} c_n) = 0,$$

we see that

$$c_n < A/n^{\frac{1}{2}+p},$$

* See *Mathematische Annalen*, Vol. 67 (1909), p. 225.

where A is some fixed positive number. Choosing the positive number k less than $2p$, we then see that the series $\sum_{n=1}^{\infty} n^k c_n^2$, being less than the convergent series $\sum_{n=1}^{\infty} \frac{A^2}{n^{1+2p-k}}$, is convergent, and thus the condition contained in the general theorem given above is satisfied. We thus have the following result :—

If, for some value of p greater than zero, $n^{\frac{1}{2}+p} c_n$ converges to zero as n is indefinitely increased, the series $\sum_{n=1}^{\infty} c_n \phi_n(x)$ is convergent at all points of the interval for which the normal orthogonal functions $\phi_n(x)$ are defined, with the exception at most of the points of some set of which the measure is zero.

The conditions

$$\lim (n^{\frac{1}{2}+p} a_n) = 0, \qquad \lim (n^{\frac{1}{2}+p} b_n) = 0,$$

for some value of p greater than zero, are sufficient to ensure that the trigonometrical series

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is convergent, with the same exception as above.

NOTES ON THE DYNAMICAL THEORY OF THE TIDES

By A. E. H. LOVE.

[Received and Read January 9th, 1913.]

1. The chief object of this note is to point out a simple method of obtaining the fundamental results which were originally given by Hough.* It will be sufficient to consider the problem of free tidal oscillations of an ocean of uniform depth covering the whole globe, and to neglect the self-attraction of the ocean. The results in question are the expressions for the elevation ζ at a point of the surface as series of spherical harmonic terms. The coefficients of the series are connected by a sequence equation involving three terms, and practically all Hough's results are reduced from his sequence equation. The method which he used to obtain his sequence equation is very indirect, especially in the case of unsymmetrical modes. It appears that the equation can be obtained by a straightforward method.

The equations of motion are known † to be

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - 2\omega v \cos\theta &= -\frac{g}{a}\frac{\partial \zeta}{\partial \theta} \\ \frac{\partial v}{\partial t} + 2\omega u \cos\theta &= -\frac{g}{a\sin\theta}\frac{\partial \zeta}{\partial \phi} \end{aligned}\right\}, \tag{1}$$

and the equation of continuity takes the form

$$\frac{\partial \zeta}{\partial t} = -\frac{h}{a\sin\theta}\left\{\frac{\partial}{\partial\theta}(u\sin\theta) + \frac{\partial v}{\partial \phi}\right\}. \tag{2}$$

---

* S. S. Hough, "The Application of Harmonic Analysis to the Dynamical Theory of the Tides," *London Phil. Trans. R. Soc.*, Part I, Vol. 189 A, 1897; Part II, Vol. 191 A, 1898.

† H. Lamb, *Hydrodynamics*, 3rd ed., p. 314. The quantity $\bar{\zeta}$, proportional to the potential of the disturbing forces, including the self-attraction, is here omitted from Lamb's equations, as free oscillations only are going to be considered.

Here θ, ϕ denote colatitude measured from the North Pole and longitude measured eastwards from a fixed meridian, ω the angular velocity of the earth's rotation, g the value of gravity at the surface, a the earth's radius, h the depth of the ocean, u and v the southward and eastward components of velocity of the water, ζ the elevation of the free surface above the equilibrium level.

We shall presently suppose the motion to be periodic with period $2\pi/\sigma$, and we shall put, as is usual,

$$f = \frac{\sigma}{2\omega}, \quad \beta = \frac{4a^2\omega^2}{gh}, \quad \mu = \cos\theta.$$

If we introduce two functions $\dot{p}$ and $\dot{q}$ analogous to potential and stream functions, we may write

$$u = -\frac{\partial \dot{p}}{\partial \theta} + \frac{1}{\sin\theta}\frac{\partial \dot{q}}{\partial \phi}, \quad v = -\frac{1}{\sin\theta}\frac{\partial \dot{p}}{\partial \phi} - \frac{\partial \dot{q}}{\partial \theta}. \tag{3}$$

Here the dots denote differentiation with respect to the time, so that the equation of continuity may be written

$$\zeta = \frac{h}{a}\Delta p, \tag{4}$$

where

$$\Delta = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}. \tag{5}$$

If now we eliminate ζ directly between the two equations of motion (1) by differentiating the first with respect to ϕ and the second (after multiplication by $\sin\theta$) with respect to θ, and subtracting the results, we obtain the equation

$$\Delta\dot{q} + 2\omega\frac{\partial \dot{q}}{\partial \phi} + 2\omega\cos\theta\,\Delta\dot{p} - 2\omega\sin\theta\frac{\partial \dot{p}}{\partial\theta} = 0. \tag{6}$$

If we form $\Delta\zeta$ by operating on the right-hand members of (1), and then substitute from (4), we obtain the equation

$$\Delta\ddot{p} + 2\omega\frac{\dot{p}}{\partial\phi} - 2\omega\cos\theta\,\Delta\dot{q} + 2\omega\sin\theta\frac{\partial\dot{q}}{\partial\theta} = \frac{gh}{a^2}\Delta^2 p. \tag{7}$$

Now it is very easy to solve these equations formally by means of series of spherical harmonics.

Put
$$\left.\begin{aligned} p &= \Sigma - \frac{C_n}{n(n+1)} P_n^{(s)}(\mu)\cos(s\phi+\sigma t) \\ q &= \Sigma D_n P_n^{(s)}(\mu)\sin(s\phi+\sigma t) \end{aligned}\right\}, \tag{8}$$

where $P_n^{(s)}(\mu)$ is the "associated function" $(1-\mu^2)^{\frac{1}{2}s}\frac{d^s P_n(\mu)}{d\mu^s}$, s is a positive integer, and the summation extends over positive integral values of n. The first of these combined with (4) gives

$$\xi = \frac{h}{a}\Sigma C_n P_n^{(s)}(\mu)\cos(s\phi+\sigma t). \tag{9}$$

Then (6) gives

$$\sigma^2\Sigma n(n+1)D_n P_n^{(s)} - 2\omega\sigma s\Sigma D_n P_n^{(s)} - 2\omega\sigma\Sigma C_n \mu P_n^{(s)} + 2\omega\sigma\Sigma\frac{C_n}{n(n+1)}(1-\mu^2)\frac{dP_n^{(s)}}{d\mu} = 0, \tag{10}$$

and (7) gives

$$-\sigma^2\Sigma C_n P_n^{(s)} + 2\omega\sigma s\Sigma\frac{C_n}{n(n+1)}P_n^{(s)} + 2\omega\sigma\Sigma n(n+1)D_n\mu P_n^{(s)} - 2\omega\sigma\Sigma D_n(1-\mu^2)\frac{dP_n^{(s)}}{d\mu} + \frac{gh}{a^2}\Sigma n(n+1)C_n P_n^{(s)} = 0. \tag{11}$$

Now we have the formulæ

$$\mu P_n^{(s)} = \frac{n-s+1}{2n+1}P_{n+1}^{(s)} + \frac{n+s}{2n+1}P_{n-1}^{(s)}, \tag{12}$$

and
$$(1-\mu^2)\frac{dP_n^{(s)}}{d\mu} = -\frac{n(n-s+1)}{2n+1}P_{n+1}^{(s)} + \frac{(n+1)(n+s)}{2n+1}P_{n-1}^{(s)}. \tag{13}$$

Hence we get from (10)

$$D_n = -\frac{2\omega}{2\omega s - n(n+1)\sigma}\left[\frac{n+1}{n}\frac{n-s}{2n-1}C_{n-1} + \frac{n}{n+1}\frac{n+s+1}{2n+3}C_{n+1}\right], \tag{14}$$

and, from (11),

$$C_n\left[\frac{gh}{a^2}n(n+1) + \frac{2\omega\sigma s}{n(n+1)} - \sigma^2\right] + 2\omega\sigma\frac{(n-1)(n+1)(n-s)}{2n-1}D_{n-1} + 2\omega\sigma\frac{n(n+2)(n+s+1)}{2n+3}D_{n+1} = 0, \tag{15}$$

leading to the equation

$$\left[\frac{n(n+1)}{\beta}+\frac{sf}{n(n+1)}-f^2\right]C_n$$

$$-f\frac{(n-1)(n+1)(n-s)}{(2n-1)\{s-n(n-1)f\}}\left[\frac{n}{n-1}\,\frac{n-s-1}{2n-3}C_{n-2}+\frac{n-1}{n}\,\frac{n+s}{2n+1}C_n\right]$$

$$-f\frac{n(n+2)(n+s+1)}{(2n+3)\{s-(n+1)(n+2)f\}}\left[\frac{n+2}{n+1}\,\frac{n-s+1}{2n+1}C_n+\frac{n+1}{n+2}\,\frac{n+s+2}{2n+5}C_{n+2}\right]$$

$$=0. \qquad (16)$$

This equation becomes identical with that obtained by Hough [Part II, p. 153, equation (48)] when the effects of self-attraction and external forces are neglected, and it reduces in the case of symmetry ($s = 0$) to the corresponding equation given by him for that case [Part I, p. 221, equation (23)].

There would be no difficulty in adapting the above method to include the effects of self-attraction and external forces.

2. The formulæ of § 1 afford an interesting example of a characteristic feature of oscillations about steady motion. When a system oscillates about a position of equilibrium all the parts are instantaneously in the same phase; but in a system oscillating about a state of steady motion the phase in general varies from point to point. The nodal lines of any type of oscillation about equilibrium are fixed, but in oscillations about steady motion there may be moving nodal lines. For example, in a type specified by $\cos(s\phi+\sigma t)$ there are $2s$ nodal meridians which revolve around the poles. The differing characters of oscillations about equilibrium and oscillations about steady motion may be exemplified by contrasting the types of free tidal oscillation on a non-rotating globe with those on a rotating globe. On a non-rotating globe the nodal lines of a type of oscillation divide the surface into a certain number of compartments. At one instant in a period the water in half the compartments is at its maximum elevation and that in the other half at its minimum elevation. In other words, there is high water all over half the compartments and low water all over the other half at the same instant. After half a period there is low water all over the first set and high water all over the second set. At intermediate instants there is not high water or low water anywhere. On a rotating globe there is high water at any instant at every

point of a certain number of lines traced on the surface, and low water at every point of an equal number of lines. The lines of high water and low water are meridians, and they revolve around the poles. The symmetrical oscillations are exceptional; so far as the distribution of phase is concerned, they have the same character as oscillations on a non-rotating globe.

Now this feature of the revolving lines of high and low water will belong to the oscillations of sheets of water confined within fixed boundaries. In general, we may put

$$\zeta = \zeta_1 \cos \sigma t + \zeta_2 \sin \sigma t, \tag{17}$$

where ζ_1, ζ_2 are functions of μ and ϕ which satisfy the simultaneous partial differential equations.*

$$\left.\begin{aligned} \frac{\partial}{\partial \mu}\left(\frac{1-\mu^2}{f^2-\mu^2}\frac{\partial \zeta_1}{\partial \mu}\right) + \frac{1}{f}\frac{f^2+\mu^2}{(f^2-\mu^2)^2}\frac{\partial \zeta_2}{\partial \phi} + \frac{1}{(1-\mu^2)(f^2-\mu^2)}\frac{\partial^2 \zeta_1}{\partial \phi^2} + \beta \zeta_1 = 0 \\ \frac{\partial}{\partial \mu}\left(\frac{1-\mu^2}{f^2-\mu^2}\frac{\partial \zeta_2}{\partial \mu}\right) - \frac{1}{f}\frac{f^2+\mu^2}{(f^2-\mu^2)^2}\frac{\partial \zeta_1}{\partial \phi} + \frac{1}{(1-\mu^2)(f^2-\mu^2)}\frac{\partial^2 \zeta_2}{\partial \phi^2} + \beta \zeta_2 = 0 \end{aligned}\right\}. \tag{18}$$

They must also satisfy the boundary conditions which express the vanishing of the normal component of the velocity (u, v) at the boundaries of the ocean. Now the times at which it is high or low water at a place are given by

$$-\zeta_1 \sin \sigma t + \zeta_2 \cos \sigma t = 0,$$

high water if ζ is positive, low water if it is negative. This equation represents at time t a curve traced on the surface, this curve being the locus of points at which there is high or low water at the instant in question. These curves all pass through the point or points determined by the two equations

$$\zeta_1 = 0, \quad \zeta_2 = 0.$$

Further, if we put

$$\zeta_1 = R \cos \psi, \quad \zeta_2 = -R \sin \psi, \tag{19}$$

where R is positive, we get

$$\zeta = R \cos (\sigma t + \psi), \tag{20}$$

---

* These are obtained from (1) and (2) by eliminating u and v, the motion being assumed to be simple harmonic and of period $2\pi/\sigma$.

so that the quantity ψ plays the same part in this problem as the quantity $s\phi$ plays in the case of an ocean covering the whole globe. The phase relations of the various parts of the ocean are determined by the quantity ψ.

The only case which has been solved completely is that of an ocean of circular shape and relatively small radius. The curves intersect at the centre of the circle, and the quantity ψ is $s\phi$ if the centre is at a pole, or in general it is $s\phi'$, where ϕ' is the angle which a great circle passing through the centre of the small circle and the point (μ, ϕ) makes with a fixed great circle passing through the same centre.

SOME POINTS IN THE GEOMETRY OF CUBIC SURFACES

By A. B. GRIEVE.

[Received October 11th, 1912. — Read November 14th, 1912. — Revised February 11th, 1913.]

Cubic Surfaces.

REYE (*Math. Ann.*, B. 55, p. 257) has shown that a cubic surface can be generated by means of a network of cubic curves if we are given as fundamental two such cubic curves which intersect in one point only. He shows that there are two conjugate networks of such curves on the surface such that any two of the same network meet in one point only, while two of different networks meet in five points, and are such that a quadric can be described containing them.

There is a certain quadric surface H^2 (principal surface) connected with these curves, such that the quadrics, containing two curves of opposite networks, are "apolar" to H^2, while the five points of section of two curves of opposite networks enjoy the property that the pole of the plane of any three of them with respect to H^2 lies in the line joining the other two, and is, in fact, the point where this line meets the cubic surface again.

Dr. Baker has shewn * that the lines on the cubic surface arise naturally as degenerate cases of the curves which lie on the surface. I proceed first of all in this paper to shew exactly what groups of lines do form degenerate *cubic* curves on the surface, and to use these degenerate cubic curves to deduce, in the light of Reye's paper, some theorems regarding the lines and the cubic curves on the surface.

I also give, in more or less satisfactory form, the equations of the thirty-six Schür quadrics. These equations are, in general, complicated; but there are three which are quite simple, and these I use to deduce some properties of the Schür quadrics.

---

* "On Curves which Lie on a Cubic Surface," *Proc. London Math. Soc.*, Ser. 2, Vol. 11, p. 285.

There ought, I think, to be a large range of theorems connected with these quadrics, but I have only succeeded in getting some elementary properties.

There is one theorem, however (§ 15), which I think is interesting.

1. *Intersection of Straight Lines of the Surface with the Two Networks of Cubic Curves.**

Consider a cubic curve made up of the straight lines a_1, a_2, and c_{12} (in the usual notation). We can describe an infinity of quadrics through these lines, and these will cut the cubic surface in a family of cubic curves. If we consider any two of these quadrics we see that they have two generators a_1, a_2 of one system and one of the other c_{12} in common. Their residual intersection is therefore a generator cutting a_1 and a_2.

This generator cuts the two corresponding cubic curves in one point only, and it cuts the surface F^3 in three points, of which one is on a_1 and one on a_2.

Hence the third must be on each of the two cubic curves.

We see thus that two cubic curves of the network meet in only one point, and hence the network is one of the two conjugate networks described by Reye.

If we start with $b_1\, b_2\, c_{12}$, say, we get the conjugate network of curves on F^3.

Consider now the double-six of lines

$$\left.\begin{matrix} a_1\, a_2\, a_3\, a_4\, a_5\, a_6 \\ b_1\, b_2\, b_3\, b_4\, b_5\, b_6 \end{matrix}\right\}.$$

First of all a_1 and a_2, we know, meet each of the curves of the network obtained above *twice*, for the one-system generators of a quadric meet a cubic curve on the quadric twice, and the other set once.† Again, all the lines of the surface F^3 must meet either $a_1\, a_2\, c_{12}$ or the residual cubic curves, and they meet the quadrics twice only. Now, $a_3\, a_4\, a_5\, a_6$ do not meet $a_1\, a_2$ or c_{12}, and therefore all meet each curve of the network twice.

---

* The ideas contained in the §§ 1–5 were given me in conversation by Dr. Baker.

† If quadric be $x = a\dfrac{\lambda\mu - 1}{\lambda + \mu}$, ... ; and if a_1 and a_2 be λ generators, and c_{12} be a μ generator, the cubic curve is given by a relation of form

$$\lambda = \frac{a\mu^2 + b\mu + c}{d\mu^2 + e\mu + f};$$

and a_1 and a_2 meet such a curve twice, c_{12} once.

b_1 meets a_2 and c_{12}; b_2 meets a_1 and c_{12}; $b_3\,b_4\,b_5\,b_6$ all meet a_1 and a_2, and hence we see that the lines b_r do not meet the curves of the network at all.

Lastly, all the lines $c_{rs}(\neq c_{12})$ meet one of the lines $a_1\,a_2\,c_{12}$, and only one, and therefore all meet each curve once.

We have, therefore, summing up,

$a_1\,a_2\,a_3\,a_4\,a_5\,a_6$ meet each curve *twice*;

$b_1\,b_2\,b_3\,b_4\,b_5\,b_6$ do *not* meet them at all;

c_{rs} meets each curve *once*.

Conversely, for the other conjugate network,

$$\left.\begin{array}{l} a_r \text{ does not meet the curves} \\ b_s \text{ meets each twice} \\ c_{rs} \text{ meets each once} \end{array}\right\}.$$

We see, therefore, that the two conjugate networks are related in this way to a certain double-six of the surface F^3.

This result is, of course, well known.

2. We see from the above that the cubic curve, made up of the lines a_1, a_2, c_{12}, meets the network, to which it gives rise, in five points, while it meets the conjugate network in only one point. It follows, therefore, that it is itself a degenerate curve of the conjugate network.

Similarly all curves made up of lines of the form $a_r\,a_s\,c_{rs}$ are degenerate cases of this network.

Similarly the groups $(b_r\,b_s\,c_{rs})$ are degenerate cases of the first network.

These degenerate curves are, we see, got by taking *two elements in the same line of the associated double-six and the line meeting these two, which does not belong to the double-six.*

The lines of the surface therefore arise in this way from the cubic curves on the surface.

It appears that there are exceptions to the statement that curves of the same network meet in one point, and curves of conjugate networks in five points.

For (1) curves of the form $(a_1\,a_3\,c_{13})\,(a_1\,a_4\,c_{14})$ have a line in common.

In fact, every curve $(a_r\,a_s\,c_{rs})$ has a line in common with eight of the fifteen similar curves.

(2) The curves of conjugate systems $(a_r\,a_s\,c_{rs})$ and $(b_r\,b_s\,c_{rs})$ have also a

line common. In this case the quadric containing the two curves breaks up into two planes.

3. We have incidentally proved in the above work the theorem that the whole system of curves of a network have *six common chords.**

If two curves k_1^3 and k_2^3 have a common chord, then it meets the surface F^3 in four points, and therefore is a line of the surface.

But we have seen that $a_1 \ldots a_6$ are the only lines of the surface which meet the curves k^3 in two points, and therefore these lines are common chords of all the curves of the network. Similarly $b_1 \ldots b_6$ are chords of the conjugate network.

Hence the two groups of six common chords of two conjugate networks form the two halves of the associated double-six.

4. We have seen that the two conjugate networks are related to a particular double-six, and hence we see that we have thirty-six systems of conjugate networks of cubic curves on F^3.

Through every two points on the surface go thirty-six pairs of cubic curves.

I give below the number of intersections of the curves of the various networks as an application of the use of the degenerate curves on the surface (§ 8).

5. We can now quite easily prove that a_1 *and* b_1, ..., *are conjugate pairs of lines with respect to the "Principal Surface"* H^2, which is determined by the conjugate networks k^3 and l^3. We have seen that two curves k^3 and l^3 meet in five points such that the pole of the plane of any three with respect to H^2 lies on the line joining the other two.

Take as a pair of such cubics

$$a_1\, a_2\, c_{12}$$

and

$$b_3\, b_4\, c_{34}.$$

The five intersections are

$$(a_1\, b_3)\,(a_1\, b_4)\,(a_2\, b_3)\,(a_2\, b_4)\,(c_{12}\, c_{34})\ ;$$

---

* Reye states this in his paper, but gives no proof.

therefore the line joining points $(a_1 b_3)$ and $(a_1 b_4)$, *i.e.* a_1, contains the pole of plane of $(a_2 b_3)(a_2 b_4)(c_{12} c_{34})$, *i.e.* the plane of a_2 and c_{12}.

Similarly a_1 contains pole of plane containing a_3 and c_{13}.

But these two planes meet in line b_1, and therefore a_1 and b_1 are conjugate, and similarly for the other pairs.

It appears, therefore, that the surface H^2 is the quadric first discovered by Schür, and usually called the Schür quadric. Just as we have thirty-six pairs of families of cubic curves, so related to every double-six, we have a Schür quadric.

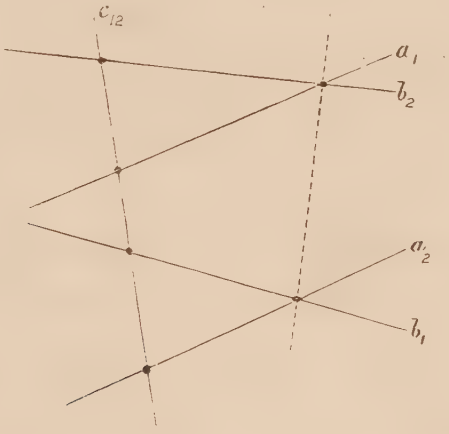
It appears, therefore, that we have *thirty-six such quadrics.*

6. *Property of the Lines.*

Consider the double-six

$$\left\{ \begin{matrix} a_1 a_2 a_3 a_4 a_5 a_6 \\ b_1 b_2 b_3 b_4 b_5 b_6 \end{matrix} \right\},$$

and the corresponding quadric H^2.



The pole of the plane $(a_2 b_1 c_{12})$ is the point $(a_1 b_2)$, and the pole of the plane $(a_1 b_2 c_{12})$ is the point $(a_2 b_1)$.

Therefore the conjugate line of c_{12} is the line joining points $(a_1 b_2)$ and $(a_2 b_1)$.

Similarly the conjugate of c_{34} is $\overline{(a_3 b_4)(a_4 b_3)}$, and the conjugate of c_{56} is $\overline{(a_5 b_6)(a_6 b_5)}$. Hence, since $c_{12} c_{34} c_{56}$ are coplanar, *the three lines* $\overline{(a_1 b_2)(a_2 b_1)}$, $\overline{(a_3 b_4)(a_4 b_3)}$ and $\overline{(a_5 b_6)(a_6 b_5)}$ *meet in a point, the pole of the plane* $(c_{12} c_{34} c_{56})$ with respect to H^2.

The pairs of points to be joined are the intersections of two "opposite" (or conjugate) lines of the double-six with another pair of conjugate lines.

Now we can divide the six pairs of conjugate lines of the double-six into three groups of two in fifteen ways, and corresponding to each division we have three concurrent lines.

Hence we have fifteen ways of joining up pairs of intersections of lines of the double-six so that three lines are concurrent, the fifteen corresponding points being the fifteen poles of the planes $(c_{12}\, c_{34}\, c_{56})$, ...

Each line arises from only two double-sixes, *i.e.* $\overline{\{(a_1\, b_2)\, a_2\, b_1\}}$ comes from the given double-six, and also

$$\begin{Bmatrix} a_1\, b_1\, c_{23}\, c_{24}\, c_{25}\, c_{26} \\ a_2\, b_2\, c_{13}\, c_{14}\, c_{15}\, c_{16} \end{Bmatrix},$$

and so on; therefore the number of such lines $= 18 \times 45 = 810$. These meet in threes in $36 \times 15 = 540$ points. Moreover, on each line lie six of these points (three arising from each of the two corresponding double-sixes).

The thirty points of intersection of the lines of the double-six are the poles of the thirty tritangent planes, containing two lines of the double-six.

The fifteen points, where the lines joining these poles in the manner just described meet, are the poles of the remaining fifteen tritangent planes, which do not contain any of the lines of the double-six.

7. *Method of Generation of the Surface.*

Consider any two curves of a network k^3, say k_1^3 and k_2^3. They meet in a point O say.

Any plane through O cuts the curves k^3 in chords, which meet in a point on the cubic surface.

If we take all planes through an axis g, passing through O, the corresponding points trace out a curve l^3 of the network conjugate to k^3.

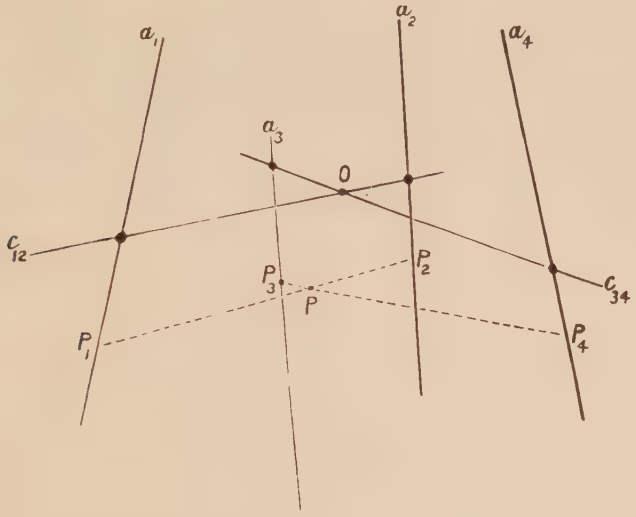
If now we vary g in any plane, the corresponding curves l^3 generate the cubic surface.*

By means of the degenerate cubic curves, we can now get a *much simpler construction.*

Consider any four lines in space, which do not intersect, say $a_1\, a_2\, a_3\, a_4$,

* Reye, *loc. cit.*

and two intersecting lines c_{12} and c_{34}, where c_{12} meets a_1 and a_2, and c_{34} meets a_3 and a_4.



(These six lines are easily seen to be equivalent to the required nineteen conditions of a cubic surface.)

Let c_{12} and c_{34} meet in O.

Through O draw a plane, cutting a_1 and a_2 in the points P_1 and P_2, and cutting a_3 and a_4 in the points P_3 and P_4.

P_1P_2 meets P_3P_4 in a point P on the surface determined by the six lines.

As the plane through O turns round any axis, P traces out a cubic curve on the surface. As the axis now revolves on a plane through O the corresponding curves generate F^3.

Note.—Dr. Baker has pointed out to me that this can easily be proved otherwise. The twenty-seven lines meet any plane section in points whose elliptic parameters may be taken to be

$$a_1 \ldots a_6,\quad a_1-\frac{S}{3},\quad \ldots,\quad a_6-\frac{S}{3},\quad -a_1-a_2+\frac{S}{3},\quad \ldots,\quad -a_5-a_6+\frac{S}{3},$$

where $$S = a_1+\ldots+a_6.$$

The line P_1P_2 cuts the surface again in the point $-(a_1+a_2)$, and P_3P_4 cuts the surface again in points $-(a_3+a_4)$. We have, therefore, only to take $a_1+a_2 \equiv a_3+a_4$.

8. *Intersections of the Various Networks of Cubic Curves on F^3.*

We have already seen that there are thirty-six pairs of conjugate networks of cubic curves on F^3, each related to a particular double-six. We can, by means of the degenerate curves of the various networks, determine how the curves of these different networks intersect. For it is obvious that these degenerate curves (provided they do not have a line in common, as we have seen they may have) will intersect in the same number of points as non-degenerate curves of the appropriate networks.

Suppose we consider the cubic curve

$$(a_1\, a_2\, c_{12}).$$

The various double-sixes may be written as follows:—

$$(1)\quad \left.\begin{matrix} a_1\, a_2\, a_3\, a_4\, a_5\, a_6 \\ b_1\, b_2\, b_3\, b_4\, b_5\, b_6 \end{matrix}\right\} \quad \textit{one},$$

$$(2)\quad \left.\begin{matrix} a_1\ a_2\ a_3\ c_{56}\, c_{64}\, c_{45} \\ c_{23}\, c_{31}\, c_{12}\ b_4\ b_5\ b_6 \end{matrix}\right\} \quad \textit{twenty},$$

$$(3)\quad \left.\begin{matrix} a_1\, b_1\, c_{23}\, c_{24}\, c_{25}\, c_{26} \\ a_2\, b_2\, c_{13}\, c_{14}\, c_{15}\, c_{16} \end{matrix}\right\} \quad \textit{fifteen}.$$

Consider first a double-six of the second group, say

$$\left.\begin{matrix} a_3\ a_4\ a_5\ c_{12}\, c_{26}\, c_{61} \\ c_{45}\, c_{35}\, c_{34}\ b_6\ b_1\ b_2 \end{matrix}\right\}.$$

Then the number of intersections of curve $(a_1\, a_2\, c_{12})$ with $(a_3\, a_4\, b_5)$, one of degenerate curves which is appropriate to this double-six, $= 2$. (The degenerate curves corresponding to a double-six consist, as we have seen, of two lines in the same row and the line meeting them which does *not* occur in the double-six.)

Similarly the number of intersections of $(a_1\, c_2\, c_{12})$ with curve

$$(c_{26}\, c_{61}\, a_6) = 2.$$

Taking degenerate curves of the conjugate network the number of intersections with

$$\left.\begin{matrix} (c_{45}\, c_{53}\, b_5) = 4 \\ (c_{34}\, b_6\, c_{56}) = 4 \end{matrix}\right\},$$

and so on.

These numbers will be found to hold in all cases. We see, therefore, that of the networks corresponding to this set of double-sixes one set meets our old l^3 network (to which $a_1 a_2 c_{12}$ belongs) in *two points*, the other in *four points*.

We see also that all the degenerate cubics considered have *three* or *one* element included in lines of the original double-six.

Secondly, consider a double-six of third group, say

$$\left.\begin{matrix} a_4\, b_4\, c_{15}\, c_{25}\, c_{35}\, c_{56} \\ a_5\, b_5\, c_{14}\, c_{24}\, c_{34}\, c_{46} \end{matrix}\right\}.$$

Intersections of $(a_1 a_2 c_{12})$ with $(a_4 b_4 c_{45}) = 3, \ldots$;

„ „ $(a_5 b_5 c_{45}) = 3, \ldots$.

In this case the degenerate cubics have *two* or *no elements* included in our original double-six.

We have, therefore, the *following theorem* :—

The cubic curves of any network meet (1) those of their own or the conjugate system in five or one point; (2) for those of systems associated with a double-six having *six* of components of the original double-six, one network in four points, and its conjugate network in *two points*; (3) for those systems connected with a double-six having *four* of original elements, each of two conjugate networks, in *three points*. These are obviously all the cases which can arise, for if two cubic curves have six points in common they coincide.

We have also the *following theorem* :—

We know that through two points A and B of the surface go thirty-six pairs of cubic curves. Any of these cubics through A and B meets

$$\left.\begin{matrix} \text{1 cubic in three more points} \\ \text{20 cubics in two more points} \\ \text{20 cubics in } no \text{ points} \\ \left.\begin{matrix} 15 \\ 15 \end{matrix}\right\} \text{ in 1 more point} \end{matrix}\right\}.$$

Again, if we examine the degenerate curves we shall find that the seventy-two systems of cubic curves have no common members, each cubic curve belonging to one, and only one, of the seventy-two kinds. They have at most a straight line in common. These results are also quite easily found by means of the mapping of the surface on a plane.

All the curves considered, of course, are twisted cubics. There are, of course, an ∞ more plane cubic curves which have not been considered.

9. *The Equations of the Thirty-six "Principal Surfaces" (or Schür Quadrics).*

In order to obtain the equations of these quadrics I have used the *Schläffli form* of the equation to the cubic surface,

$$uvw = xyz, \tag{1}$$

where

$$\left.\begin{aligned} \alpha_1 u+\beta_1 v+\gamma_1 w &= \delta_1 x+\epsilon_1 y+\xi_1 z \\ \alpha_2 u+\beta_2 v+\gamma_2 w &= \delta_2 x+\epsilon_2 y+\xi_2 z \end{aligned}\right\}. \tag{2}$$

Then, as is well known, we get the lines on the surface as follows:—

Nine lines are given immediately by equations

$$\left.\begin{aligned} u &= 0 \\ x &= 0 \end{aligned}\right\}, \quad \left.\begin{aligned} u &= 0 \\ y &= 0 \end{aligned}\right\}, \quad \ldots.$$

These may be named as:—

	x	y	z
u	c_{12}	c_{34}	c_{56}
v	c_{36}	c_{25}	c_{14}
w	c_{45}	c_{16}	c_{23}

(I)

Again, from equations (2), we have

$$\Sigma(\alpha_1-\rho\alpha_2)\,u = \Sigma(\delta_1-\rho\delta_2)\,x.$$

Choose ρ so that

$$(\alpha_1-\rho\alpha_2)(\beta_1-\rho\beta_2)(\gamma_1-\rho\gamma_2) = (\delta_1-\rho\delta_2)(\epsilon_1-\rho\epsilon_2)(\xi_1-\rho\xi_2).$$

Let the three roots be ρ_1, ρ_2, ρ_3, and let

$$\left.\begin{aligned} \alpha_1-\rho_1\alpha_2 &= a_1, \ \ldots \\ \alpha_1-\rho_2\alpha_2 &= a_2, \ \ldots \\ \alpha_1-\rho_3\alpha_2 &= a_3, \ \ldots \end{aligned}\right\},$$

so that

$$\begin{aligned} a_1 b_1 c_1 &= d_1 e_1 f_1, \\ a_2 b_2 c_2 &= d_2 e_2 f_2, \\ a_3 b_3 c_3 &= d_3 e_3 f_3. \end{aligned}$$

Then we have the three relations

$$\left.\begin{aligned} a_1 u + b_1 v + c_1 w &= d_1 x + e_1 y + f_1 z \\ a_2 u + b_2 v + c_2 w &= d_2 x + e_2 y + f_2 z \\ a_3 u + b_3 v + c_3 w &= d_3 x + e_3 y + f_3 z \end{aligned}\right\} \quad (3)$$

these three relations not being independent. The other lines are now given by such equations as

$$a_1 u = d_1 x, \quad b_1 v = c_1 y,$$

and, therefore, $$c_1 w = f_1 z,$$

and so on.

These eighteen lines may be arranged as follows :—

	Suff. 1	Suff. 2	Suff. 3
$au = dx,\ bv = ey,\ cw = fz$	a_2	b_2	c_{46}
$au = ey,\ bv = fz,\ cw = dx$	a_4	b_4	c_{26}
$au = fz,\ bv = dx,\ cw = ey$	a_6	b_6	c_{24}
$au = dx,\ bv = fz,\ cw = ey$	b_1	a_1	c_{35}
$au = ey,\ bv = dx,\ cw = fz$	b_3	a_3	c_{15}
$au = fz,\ bv = ey,\ cw = dx$	b_5	a_5	c_{13}

(II)

As I have said, equations (3) contain only two independent equations, and, therefore, there are certain relations between the various constants $a_1 \ldots a_2 \ldots a_3 \ldots$.

By solving the first two for u and v, say, and substituting in the third, and so on, we find immediately that these relations are given by the fact that all the determinants in the matrix

$$\left.\begin{matrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 & f_3 \end{matrix}\right| \text{ vanish.} \quad (4)$$

From (4) we see that we can determine κ_1, κ_2, κ_3, so that all

equations

$$\left.\begin{aligned} \kappa_1 a_1+\kappa_2 a_2+\kappa_3 a_3 &\equiv 0 \\ \kappa_1 b_1+\kappa_2 b_2+\kappa_3 b_3 &\equiv 0 \\ \cdots \quad & \cdots \quad \cdots \\ \kappa_1 f_1+\kappa_2 f_2+\kappa_3 f_3 &= 0 \end{aligned}\right\} \tag{5}$$

are satisfied.

Finally, it follows that, if $(a_1 f_2) = a_1 f_2 - c_2 f_1$, ...,

$$\frac{(a_1 b_2)}{(a_1 b_3)} = \frac{(a_1 c_2)}{(a_1 c_3)} = \ldots = \frac{(e_1 f_2)}{(e_1 f_3)}. \tag{6}$$

[For, if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0,$$

equations $a_1 x+b_1 y+c_1 z = 0$, ..., ...,

are consistent. From the first two of these

$$\frac{x}{(b_1 c_2)} = \frac{y}{(c_1 a_2)} = \frac{z}{(a_1 b_2)},$$

and from first and third

$$\frac{x}{(b_1 c_3)} = \frac{y}{(c_1 a_3)} = \frac{z}{(a_1 b_3)};$$

and therefore $\dfrac{(b_1 c_2)}{(b_1 c_3)} = \dfrac{(c_1 a_2)}{(c_1 a_3)} = \dfrac{(a_1 b_2)}{(a_1 b_3)}$, ...]

10. There are three of the double-sixes for which we can write down the equations of the corresponding quadric immediately. We see that in the Scheme (II), § 9, of the lines the vertical columns, taken in pairs, give three double-sixes.

These are

$$\text{(i)} \quad \left.\begin{matrix} a_1 a_2 a_3 a_4 a_5 a_6 \\ b_1 b_2 b_3 b_4 b_5 b_6 \end{matrix}\right\},$$

for which the corresponding quadric is

$$a_1 a_2 u^2+b_1 b_2 v^2+c_1 c_2 w^2 = d_1 d_2 x^2+e_1 e_2 y^2+f_1 f_2 z^2.$$

$$\text{(ii)} \quad \left.\begin{matrix} a_2 a_4 a_6 c_{35} c_{15} c_{13} \\ c_{46} c_{26} c_{24} b_1 b_3 b_5 \end{matrix}\right\},$$

for which the quadric is

$$a_1 a_3 u^2 + b_1 b_3 v^2 + c_1 c_3 w^2 = d_1 d_3 x^2 + e_1 e_3 y^2 + f_1 f_3 z^2.$$

(iii) $\left.\begin{matrix} a_1\, a_3\, a_5\, c_{46}\, c_{26}\, c_{24} \\ c_{35}\, c_{15}\, c_{13}\, b_2\, b_4\, b_6 \end{matrix}\right\}$,

for which the quadric is

$$a_2 a_3 u^2 + b_2 b_3 v^2 + c_2 c_3 w^2 = d_2 d_3 x^2 + e_2 e_3 y^2 + f_2 f_3 z^2.$$

These are the only three perfectly symmetrical equations, and I shall return to them later to discuss some properties of the quadrics.

Again we see that in Scheme (II), § 9, any two rows of the top three or any two of the bottom three make up half a double-six, the other half being made up of lines from Scheme (I).

We get *six* more double-sixes in this way.

Consider any one of these, say

$$\left.\begin{matrix} a_2\, b_2\, c_{46} & c_{14}\, c_{34}\, c_{45} \\ a_4\, b_4\, c_{26} & c_{12}\, c_{23}\, c_{25} \end{matrix}\right\}.$$

To determine the quadric completely we do not need all six pairs of conjugate lines. The latter three pairs will be more than sufficient. I use these because their equations are much simpler.

The three pairs are

(α) c_{45} and c_{25} whose equations are $\left.\begin{matrix} w = 0 \\ x = 0 \end{matrix}\right\}$ and $\left.\begin{matrix} v = 0 \\ z = 0 \end{matrix}\right\}$,

(β) c_{34} and c_{23} ,, ,, $\left.\begin{matrix} u = 0 \\ y = 0 \end{matrix}\right\}$ and $\left.\begin{matrix} w = 0 \\ z = 0 \end{matrix}\right\}$,

(γ) c_{14} and c_{12} ,, ,, $\left.\begin{matrix} v = 0 \\ z = 0 \end{matrix}\right\}$ and $\left.\begin{matrix} u = 0 \\ x = 0 \end{matrix}\right\}$.

The general equation in four variables of the quadric is, say

$$(abc\,fgh\,ijk\,d\,\mathcal{X}\,xyzw)^2 = 0.$$

The general equation of the polar of $(x'y'z'w')$ is

$$\Sigma(ax' + hy' + gz' + iw')\,x = 0.$$

(α) Take as a point of c_{45},

$$x' = 0, \quad w' = 0, \quad y' = y', \quad z' = z',$$

and we get for the equation of the polar

$$x(hy'+gz')+y(by'+fz')+z(fy'+cz')+w(jy'+kz') = 0. \tag{A}$$

This is to be satisfied for all points on $v = 0$, $y = 0$, Using the first two of equations (3) with $v = 0$, $y = 0$, we get

$$\left.\begin{aligned} y &= 0 \\ (c_1a_2)w &= (d_1a_2)x+(f_1a_2)z \end{aligned}\right\} \tag{7}$$

if we eliminate u.

Equation (A) is to hold for all values of y' and z', and therefore

$$\left.\begin{aligned} hx+by+fz+jw &= 0 \\ gx+fy+cz+kw &= 0 \end{aligned}\right\} \tag{8}$$

for all values of $xyzw$ given by (7).

Getting rid of w and equating coefficients of x and z to zero, we get equations

$$\left.\begin{aligned} h(c_1a_2)+j(d_1a_2) &= 0 \\ f(c_1a_2)+j(f_1a_2) &= 0 \\ g(c_1a_2)+k(d_1a_2) &= 0 \\ c(c_1a_2)+k(f_1a_2) &= 0 \end{aligned}\right\}. \tag{9}$$

Again, consider pair (β), *i.e.*,

$$c_{34} \quad \text{and} \quad c_{23}.$$

We get similarly the equations

$$\left.\begin{aligned} a(c_1b_2)+i(d_1b_2) &= 0 \\ g(c_1b_2)+i(f_1b_2) &= 0 \\ h(c_1b_2)+j(d_1b_2) &= 0 \\ f(c_1b_2)+j(f_1b_2) &= 0 \end{aligned}\right\}. \tag{10}$$

From (9) and (10) we derive immediately

$$f = h = j = 0. \tag{11}$$

Consider now the pair (γ), *i.e.*,

$$c_{14} \quad \text{and} \quad c_{12},$$

given by equations $\left.\begin{matrix} v' = 0 \\ z' = 0 \end{matrix}\right\}$ and $\left.\begin{matrix} u = 0 \\ x = 0 \end{matrix}\right\}$.

From fundamental relations (3), we get for first line

$$a_1 u' + c_1 w' = d_1 x' + e_1 y',$$

$$a_2 u' + c_2 w' = d_2 x' + e_2 y';$$

therefore
$$\left.\begin{matrix} (c_1 a_2)\, w' = (d_1 a_2)\, x' + (e_1 a_2)\, y' \\ z' = 0 \end{matrix}\right\}. \tag{12}$$

And similarly for the second line

$$\left.\begin{matrix} x = 0 \\ (c_1 b_2)\, w = (e_1 b_2)\, y + (f_1 b_2)\, z \end{matrix}\right\}. \tag{13}$$

The equation for the polar plane is (putting $z' = 0$, or remembering that $f = h = j = 0$)

$$x(ax' + iw') + byy' + z(gx' + kw') + w(ix' + dw') = 0. \tag{B}$$

Putting $x = 0$, and substituting for y' from (12), we get on equating coefficients of x' and w' to zero,

$$\left.\begin{matrix} -b\,\dfrac{(d_1 a_2)}{(e_1 a_2)}\, y + gz + iw = 0 \\ b\,\dfrac{(c_1 a_2)}{(e_1 a_2)}\, y + kz + dw = 0 \end{matrix}\right\}, \tag{14}$$

whence using equations (13) we get

$$\left.\begin{matrix} -b\,\dfrac{(d_1 a_2)}{(e_1 a_2)} + i\,\dfrac{(e_1 b_2)}{(c_1 b_2)} = 0 \\ g \quad + i\,\dfrac{(f_1 b_2)}{(c_1 b_2)} = 0 \\ b\,\dfrac{(c_1 a_2)}{(e_1 a_2)} + d\,\dfrac{(e_1 b_2)}{(c_1 b_2)} = 0 \\ k \quad + d\,\dfrac{(f_1 b_2)}{(c_1 b_2)} = 0 \end{matrix}\right\}. \tag{15}$$

Equations (9), (10), and (15) now are more than sufficient to determine all our constants, and we find

$$a = -i\frac{(d_1 b_2)}{(c_1 b_2)}, \qquad b = i\frac{(e_1 b_2)(e_1 a_2)}{(d_1 a_2)(c_1 b_2)},$$

$$c = -\frac{(f_1 a_2)(f_1 b_2)}{(c_1 b_2)(d_1 a_2)}, \qquad d = -i\frac{(c_1 a_2)}{(d_1 a_2)},$$

$$g = -i\frac{(f_1 b_2)}{(c_1 b_2)}, \qquad k = i\frac{(c_1 a_2)(f_1 b_2)}{(d_1 a_2)(c_1 b_2)}.$$

Moreover I have verified that the equations are consistent, and we therefore get the equation

$$(d_1 a_2)(d_1 b_2)x^2 - (e_1 a_2)(e_1 b_2)y^2 + (f_1 a_2)(f_1 b_2)z^2 + (c_1 a_2)(c_1 b_2)w^2$$
$$+ 2(d_1 a_2)(f_1 b_2)zx - 2(d_1 a_2)(c_1 b_2)xw - 2(c_1 a_2)(f_1 b_2)zw = 0.$$

This equation only involves the suffixes 1 and 2, whereas the lines entering into the double-six involve suffixes 1, 2, and 3. We can, however, easily modify the equation so as to make it symmetrical in these suffixes.

We have $$\frac{(a_1 b_2)}{(a_1 b_3)} = \frac{(a_1 c_2)}{(a_1 c_3)} = \ldots;$$

by equations (6); and therefore the equation may be written

(iv) $$\sum_{123}(d_1 a_2)(d_1 b_2)x^2 - \Sigma(e_1 a_2)(e_1 b_2)y^2 + \Sigma(f_1 a_2)(f_1 b_2)z^2 + \Sigma(c_1 a_2)(c_1 b_2)w^2$$
$$+ 2\Sigma(d_1 a_2)(f_1 b_2)zx - 2\Sigma(d_1 a_2)(c_1 b_2)xw$$
$$- 2\Sigma(c_1 a_2)(f_1 b_2)zw = 0.$$

We can still further modify the equation by introducing u and v, for it is obvious that, instead of getting rid of u and v, we might have got rid of any two of letters $u \ldots z$. However, it seems hardly worth while writing down the equation. As the symmetry increases, the equation becomes more involved.

We can now deduce other five of these quadrics.

(v) For $$\left.\begin{matrix} a_4\, b_4\, c_{26}\, c_{16}\, c_{36}\, c_{56} \\ a_6\, b_6\, c_{24}\, c_{14}\, c_{34}\, c_{46} \end{matrix}\right\},$$

we change dx into ey, ey into fz, fz into dx, and we get

$$\Sigma(d_1a_2)(d_1b_2)\,x^2+\Sigma(e_1a_2)(e_1b_2)\,y^2-\Sigma(f_1a_2)(f_1b_2)\,z^2+\Sigma(c_1a_2)(c_1b_2)\,w^2$$
$$+2\Sigma(e_1a_2)(d_1b_2)\,xy-2\Sigma(e_1a_2)(c_1b_2)\,yw$$
$$-2\Sigma(c_1a_2)(d_1b_2)\,xw=0.$$

(vi) For
$$\left.\begin{matrix}a_2\,b_2\,c_{46}\,c_{16}\,c_{36}\,c_{56}\\ a_6\,b_6\,c_{24}\,c_{12}\,c_{23}\,c_{25}\end{matrix}\right\},$$

we get similarly

$$-\Sigma(d_1a_2)(d_1b_2)\,x^2+\Sigma(e_1a_2)(e_1b_2)\,y^2+\Sigma(f_1a_2)(f_1b_2)\,z^2+\Sigma(c_1a_2)(c_1b_2)\,w^2$$
$$+2\Sigma(f_1a_2)(e_1b_2)\,yz-2\Sigma(f_1a_2)(c_1b_2)\,zw$$
$$-2\Sigma(c_1a_2)(e_1b_2)\,yw=0.$$

(vii) For
$$\left.\begin{matrix}a_1\,b_1\,c_{35}\,c_{23}\,c_{34}\,c_{36}\\ a_3\,b_3\,c_{15}\,c_{12}\,c_{14}\,c_{16}\end{matrix}\right\},$$

we have, in (iv), to change by into cz, and we get

$$\Sigma(d_1a_2)(a_1b_2)\,x^2+\Sigma(e_1a_2)(e_1b_2)\,y^2-\Sigma(f_1a_2)(f_1b_1)\,z^2+\Sigma(c_1a_2)(c_1b_2)\,w^2$$
$$+2\Sigma(d_1a_2)(e_1b_2)\,xy-2\Sigma(d_1a_2)(c_1b_2)\,xw$$
$$-2\Sigma(c_1a_2)(e_1b_2)\,yw=0.$$

(viii) and (ix) For
$$\left.\begin{matrix}a_3\,b_3\,c_{15}\,c_{25}\,c_{45}\,c_{56}\\ a_5\,b_5\,c_{13}\,c_{23}\,c_{34}\,c_{36}\end{matrix}\right\},$$

and
$$\left.\begin{matrix}a_1\,b_1\,c_{35}\,c_{25}\,c_{45}\,c_{56}\\ a_5\,b_5\,c_{13}\,c_{12}\,c_{14}\,c_{16}\end{matrix}\right\},$$

we get the equations at once from (vii) by interchanging ax, by, cz cyclically.

12. Consider any of the remaining double-sixes, say

$$\left.\begin{matrix}a_1\,b_1\,c_{23}\,c_{24}\,c_{25}\,c_{26}\\ a_2\,b_2\,c_{13}\,c_{14}\,c_{15}\,c_{16}\end{matrix}\right\}.$$

If we place in Scheme (II), § 9, the lines appearing in this double-six we get

$1'$	$2'$	·
·	·	6
·	·	4
1	2	·
·	·	$5'$
·	·	$3'$

We see at once that *all* the remaining twenty-seven double-sixes give a scheme of this kind, for the number of such similar schemes $= 27$. The quadrics in these cases are more difficult to find, for none of the conjugate pairs belong to the Scheme (I) only.

(x) Consider the above double-six.

Take first the pair c_{23} and c_{13},

i.e.,
$$\left.\begin{aligned} w' &= 0 \\ z' &= 0 \end{aligned}\right\} \text{ and } \left.\begin{aligned} a_3 u &= f_3 z \\ b_3 v &= c_3 y \\ c_3 w &= d_3 x \end{aligned}\right\}.$$

The polar with respect to $(x'y'0\,0)$ is

$$x(ax'+hy')+y(hx'+by')+z(gx'+fy')+w(ix'+jy') = 0\,;$$

and therefore we must have

$$\left.\begin{aligned} ax+hy+gz+iw &= 0 \\ hx+by+fz+jw &= 0 \end{aligned}\right\} \tag{16}$$

satisfied for all points on the second line.

For this line, since $\Sigma a_1 u = \Sigma d_1 x$,

we have
$$\left.\begin{aligned} \frac{(a_1 f_3)}{a_3} z + \frac{(b_1 e_3)}{b_3} y + \frac{(c_1 d_3)}{c_3} x &= 0 \\ c_3 w &= d_3 x \end{aligned}\right\}. \tag{17}$$

From (16) by means of (17) get rid of z and w and equate coefficients of

x and y to zero, and we get

$$\left.\begin{aligned} a - g\,\frac{a_3(c_1d_3)}{c_3(a_1f_3)} + i\,\frac{d_3}{c_3} &= 0 \\ h = g\,\frac{a_3}{b_3}\,\frac{(b_1e_3)}{(a_1f_3)} & \\ h - f\,\frac{a_3}{c_3}\,\frac{(c_1d_3)}{(a_1f_3)} + j\,\frac{d_3}{c_3} &= 0 \\ b = f\,\frac{a_3}{b_3}\,\frac{(b_1e_3)}{(a_1f_3)} & \end{aligned}\right\}. \tag{18}$$

Secondly, consider pair c_{16} and c_{26},

i.e.,
$$\left.\begin{aligned} w' &= 0 \\ y' &= 0 \end{aligned}\right\}, \qquad \left.\begin{aligned} a_3u &= e_3y \\ b_3v &= f_3z \\ c_3w &= d_3x \end{aligned}\right\}.$$

In the same way as above, we get the equations

$$\left.\begin{aligned} a - \frac{b_3}{c_3}\,\frac{(c_1d_3)}{(b_1f_3)}\,g + i\,\frac{d_3}{c_3} &= 0 \\ h = g\,\frac{b_3}{a_3}\,\frac{(a_1e_3)}{(b_1f_3)} & \\ g - c\,\frac{b_3}{c_3}\,\frac{(c_1d_3)}{(b_1f_3)} + k\,\frac{d_3}{c_3} &= 0 \\ f = \frac{b_3}{a_3}\,\frac{(a_1e_3)}{(b_1f_3)}\,c & \end{aligned}\right\}. \tag{19}$$

From (18) and (19) it follows that

$$g = h = 0, \tag{20}$$

while
$$\left.\begin{aligned} a &= -\,i\,\frac{d_3}{c_3}, & & \\ f &= k\,\frac{d_3}{c_3}\,\frac{(a_1e_3)}{(c_1d_3)}, & c &= k\,\frac{d_3}{b_3}\,\frac{(b_1f_3)}{(c_1d_3)} \\ j &= k\,\frac{(a_1e_3)}{(a_1f_3)}, & b &= k\,\frac{d_3}{b_3}\,\frac{(b_1e_3)(a_1e_3)}{(c_1d_3)(a_1f_3)} \end{aligned}\right\}. \tag{21}$$

Consider now the conjugate pair

$$c_{25} \quad \text{and} \quad c_{15},$$

i.e.,

$$\left.\begin{aligned} v' &= 0 \\ y' &= 0 \end{aligned}\right\}. \qquad \left.\begin{aligned} a_3 u &= e_3 y \\ b_3 v &= d_3 x \\ e_3 w &= f_3 z \end{aligned}\right\}.$$

The first is equivalent to

$$\left.\begin{aligned} (a_1 c_3)\, w' &= (a_1 d_3)\, x' + (a_1 f_3)\, z' \\ y' &= 0 \end{aligned}\right\}, \tag{22}$$

and the polar equation of $(x', 0, z', w')$ is, since

$$g = h = 0,$$

$$x(ax'+iw')+y(fz'+jw')+z(cz'+kw')+w(ix'+kz'+dw') = 0. \tag{C}$$

Eliminate w' by means of (22), and equate coefficients of x' and z' to zero, and we get

$$\left.\begin{aligned} &\left\{a+i\frac{(a_1d_3)}{(a_1c_3)}\right\}x+j\frac{(a_1d_3)}{(a_1c_3)}y+k\frac{(a_1d_3)}{(a_1c_3)}z+\left\{i+d\frac{(a_1d_3)}{(a_1c_3)}\right\}w=0 \\ \text{and} \\ &i\frac{(a_1f_3)}{(a_1c_3)}x+\left\{f+j\frac{(a_1f_3)}{(a_1c_3)}\right\}y+\left\{c+k\frac{(a_1f_3)}{(a_1c_3)}\right\}z+\left\{k+d\frac{(a_1f_3)}{(a_1c_3)}\right\}w=0 \end{aligned}\right\}, \tag{23}$$

where $(xyzw)$ is on the line c_{15}, *i.e.*, on the line

$$\left.\begin{aligned} \frac{(a_1e_3)}{a_3}y+\frac{(b_1d_3)}{b_3}x+\frac{(c_1f_3)}{c_3}z &= 0 \\ c_3w = f_3z \end{aligned}\right\}. \tag{24}$$

Therefore, from (23), getting rid of y and w, and equating coefficients of x and z to zero,

$$\left.\begin{aligned} a+i\frac{(a_1d_3)}{(a_1c_3)} &= j\frac{a_3}{b_3}\frac{(b_1d_3)}{(a_1e_3)}\frac{(a_1d_3)}{(a_1c_3)} \\ -j\frac{a_3}{c_3}\frac{(c_1f_3)}{(a_1e_3)}\frac{(a_1d_3)}{(a_1c_3)}+k\frac{(a_1d_3)}{(a_1c_3)}+&\left\{i+d\frac{(a_1d_3)}{(a_1c_3)}\right\}\frac{f_3}{c_3}=0 \\ i\frac{(a_1f_3)}{(a_1c_3)} &= \left\{f+j\frac{(a_1f_3)}{(a_1c_3)}\right\}\frac{a_3}{b_3}\frac{(b_1d_3)}{(a_1e_3)} \\ -\left\{f+j\frac{(a_1f_3)}{(a_1c_3)}\right\}\frac{a_3}{c_3}\frac{(c_1f_3)}{(a_1e_3)}+c+k\frac{(a_1f_3)}{(a_1c_3)}+&\left\{k+d\frac{(a_1f_3)}{(a_1c_3)}\right\}\frac{f_3}{c_3}=0 \end{aligned}\right\}. \tag{25}$$

Now we know that $$a = -i\frac{d_3}{c_3}.$$

Therefore, from the first of (25), we find

$$\left.\begin{aligned} i &= k\,\frac{c_3}{b_3}\,\frac{(a_1d_3)(b_1d_3)}{(c_1d_3)(a_1f_3)} \\ a &= -k\,\frac{d_3}{b_3}\,\frac{(a_1d_3)(b_1d_3)}{(c_1d_3)(a_1f_3)} \end{aligned}\right\}. \tag{26}$$

The third equation of (25) is now *identically* satisfied. I have verified this.

All that remains is to determine d, which we can do from the second equation of (25).

We find from it

$$d = -k\,\frac{(a_1c_3)\{b_3(c_1d_3)+c_3(b_1d_3)\}}{b_3(a_1f_3)(c_1d_3)}; \tag{27}$$

with this value we find that the last equation is now satisfied.

The equation to the quadric is therefore given by

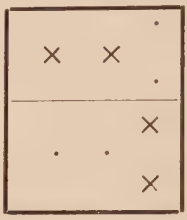
$$\begin{aligned} &-\frac{d_3}{b_3}\,\frac{(a_1d_3)(b_1d_3)}{(c_1d_3)(a_1f_3)}\,x^2+\frac{d_3}{b_3}\,\frac{(a_1e_3)(b_1e_3)}{(c_1d_3)(a_1f_3)}\,y^2+\frac{d_3}{b_3}\,\frac{(b_1f_3)}{(c_1d_3)}\,z^2 \\ &-\frac{(a_1c_3)\{b_3(c_1d_3)+c_3(b_1d_3)\}}{b_3(a_1f_3)(c_1d_3)}\,w^2+2\,\frac{d_3}{b_3}\,\frac{(a_1e_3)}{(c_1d_3)}\,yz \\ &+2\,\frac{c_3}{b_3}\,\frac{(a_1d_3)(b_1d_3)}{(c_1d_3)(a_1f_3)}\,xw+2\,\frac{(a_1e_3)}{(a_1f_3)}\,yw+2zw=0, \end{aligned}$$

or, remembering ratios $\dfrac{(a_1b_2)}{(a_1b_3)}=\dfrac{(a_1c_2)}{(a_1c_3)}=\ldots.$

(x) $$\begin{aligned} &-(a_1d_2)(b_1d_2)\,x^2+(a_1e_2)(b_1e_2)\,y^2+(a_1f_2)(b_1f_2)\,z^2 \\ &-\frac{b_3(a_1c_2)(c_1d_2)+c_3(a_1c_2)(b_1d_2)}{d_3}\,w^2 \\ &+2\,(a_1e_2)(a_1f_2)\,yz+2\,\frac{c_3}{d_3}\,(a_1d_2)(b_1d_2)\,xw \\ &+2\,\frac{b_3}{d_3}\,(a_1e_2)(c_1d_2)\,yw+2\,\frac{b_3}{d_3}\,(c_1d_2)(a_1f_2)\,zw=0, \end{aligned}$$

the required equation.

13. We may now deduce immediately all the remaining equations.
If in Scheme (I), § 9, the lines of the double-six appear as



we deduce the corresponding equation by the change $au \to cw$, $ey \to fz$.

For the scheme

we make the change $au \to bv$, $ey \to fz$.

In each of these now make the cyclical interchanges

$$(1) \quad au \to bv \to cw, \quad dx \to ey \to fz,$$

$$(2) \quad au \to cw \to bv, \quad dx \to fz \to ey,$$

and we get all the remaining schemes in which the third column has four elements. We thus get nine equations.

Each of these, however, gives rise to two others by interchanging suffixes (1) $1 \to 3 \to 2$; (2) $1 \to 2 \to 3$. We therefore get the twenty-seven remaining equations.

14. *Properties of the Schür Quadrics.*

If we examine the three groups of lines in Schemes (I) and (II), we see that we have three sets of nine each :

$$\left\{\begin{matrix} a_2\, b_2\, c_{46} \\ a_4\, b_4\, c_{26} \\ a_6\, b_6\, c_{24} \end{matrix}\right\} \quad \left\{\begin{matrix} b_1\, a_1\, c_{35} \\ b_3\, a_3\, c_{15} \\ b_5\, a_5\, c_{13} \end{matrix}\right\} \quad \left\{\begin{matrix} c_{12}\, c_{23}\, c_{25} \\ c_{14}\, c_{34}\, c_{45} \\ c_{16}\, c_{46}\, c_{56} \end{matrix}\right\}.$$

Each of these sets is given by the intersection of two trihedral angles, each face of the trihedral angles containing three lines. We may call such groups "trihedral groups," and we see that every trihedral group uniquely determines other two such groups.

The number of such groups $= 120$, and these fall into forty sets of three each like the above.

Every two groups out of a set of three give three double-sixes composed of elements from them, *e.g.*, in the first two groups above we get a double-six by taking any two columns in one and the same two columns in the other.

Hence *associated with every set of three allied trihedral groups we have nine Schür quadrics.*

15. Consider the three quadrics which we have already found associated with the groups of Scheme (II), § 9.

Their equations are

$$(1)\quad a_2 a_3 u^2+b_2 b_3 v^2+c_2 c_3 w^2 = d_2 d_3 x^2+e_2 e_3 y^2+f_2 f_3 z^2,$$

$$(2)\quad a_3 a_1 u^2+b_3 b_1 v^2+c_3 c_1 w^2 = d_3 d_1 x^2+e_3 e_1 y^2+f_3 f_1 z^2,$$

$$(3)\quad a_1 a_2 u^2+b_1 b_2 v^2+c_1 c_2 w^2 = d_1 d_2 x^2+e_1 e_2 y^2+f_1 f_2 z^2.$$

Also the lines lie in 6's (the lines in the same columns) on the three quadrics

$$(4)\quad a_1^2 u^2+b_1^2 v^2+c_1^2 w^2 = d_1^2 x^2+e_1^2 y^2+f_1^2 z^2,$$

$$(5)\quad a_2^2 u^2+b_2^2 v^2+c_2^2 w^2 = d_2^2 x^2+e_2^2 y^2+f_2^2 z^2,$$

$$(6)\quad a_3^2 u^2+b_3^2 v^2+c_3^2 w^2 = d_3^2 x^2+e_3^2 y^2+f_3^2 z^2.$$

The polar reciprocal of (4) with respect to (1) is (6), with respect to (3) is (5), and so on, since the lines determining (4), (5), and (6) reciprocate into one another.

Now we have found (p. 326) that we can find $k_1 k_2 k_3$ such that

$$\left.\begin{aligned} k_1 a_1+k_2 a_2+k_3 a_3 &= 0 \\ \cdots \quad \cdots \quad \cdots \\ k_1 f_1+k_2 f_2+k_3 f_3 &= 0 \end{aligned}\right\} \text{ are all satisfied.}$$

Substituting in (1) for a_2, b_2, ..., we get

$$k_2(1)+k_1(2)+k_3(6) = 0,$$

and so on. Hence we see that the quadric (6) passes through the intersection of (1) and (2).

Similarly (5) passes through the intersection of (1) and (3), and (4) goes through the intersection of (2) and (3). Hence the six quadrics (1), ..., (6) *have eight points in common.*

We may state the theorem thus :

The three Schür quadrics associated with two trihedral groups (of the same allied set of three such groups) meet in three quartic curves, of which the eighteen lines are chords in sets of six.

The six chords of any one of these curves are three non-intersecting lines of one group and the three of the other which meet them. The three quadrics on which the lines lie pass through the eight points of intersection of the three Schür quadrics, and each passes through one of the quartic curves of intersection of the Schür quadrics.

16. It is a well known result that if two intersecting lines (a_1 and b_2, say) be taken the cross ratio of the four tritangent planes through the one equals the cross ratio of the four through the other (the common tritangent plane being omitted).

Thus, pencil $a_1(b_3\, b_4\, b_5\, b_6)$ = pencil $b_2(a_3\, a_4\, a_5\, a_6)$. It follows, therefore, that the four lines of intersection of corresponding planes lie, with the axes of the pencils, on a quadric cone.

The corresponding planes $(a_1\, b_3\, c_{13})$ and $(a_3\, b_2\, c_{23})$ meet in the line joining the points $(a_3\, c_{13})$ and $(b_3\, c_{23})$.

Similarly the other lines are the joins of the points

$$(b_4\, c_{24}) \quad \text{and} \quad (a_4\, c_{14}),$$

$$(b_5\, c_{25}) \quad \text{and} \quad (a_5\, c_{15}),$$

$$(b_6\, c_{26}) \quad \text{and} \quad (a_6\, c_{16})\ ;$$

in other words, the joins of the intersections of line pairs lying in corresponding planes through a_2 and b_1.

It is to be noticed that the lines $a_1\, a_2\, b_1\, b_2$ form two pairs of opposite lines common to two double-sixes, *i.e.*

$$\left\{\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{matrix}\right\}$$

and

$$\left\{\begin{matrix} a_1 & b_1 & c_{23} & c_{24} & c_{25} & c_{26} \\ a_2 & b_2 & c_{13} & c_{14} & c_{15} & c_{16} \end{matrix}\right\}.$$

Hence the *theorem:*

If we have four lines, forming elements common to two double-sixes (and, therefore, such that two pairs intersect), the four lines joining intersections of line pairs, which lie in corresponding planes through two of the intersecting lines, meet in a point, which is the point of intersection of the other two intersecting lines, and lie with them on a quadric cone.

Mr. Richmond has pointed out to me that, using Geiser's method, drawing the tangent cone to the cubic surface from the point of meeting of the two intersecting lines (a_2 and b_1 in the above case), we get the well known result that the *cross ratio of the four tangents from the nodes of a binodal quartic curve are equal.*

ON UNIFORM OSCILLATION OF THE FIRST AND SECOND KIND

By W. H. YOUNG, Sc.D., F.R.S.

[Received January 4th, 1913.—Read January 9th, 1913.]

1. The introduction of the concept of uniform convergence marked an epoch in the history of the theory of converging series of functions. If the discovery of far wider conditions which admit of term-by-term integration of series has tended to diminish the importance of the concept in certain branches of the theory, it remains without a rival as affording us a convenient sufficient condition for the continuity of the sum-function of series of continuous functions.

The theory of series has recently undergone important changes. We no longer confine ourselves to converging series, and we have learnt how to distribute all known or knowable functions into classes, and to give simple necessary and sufficient conditions that an assigned function should belong to a particular class.* The question then naturally presents itself as to the possibility of generalising uniform convergence so as to embrace the case of oscillating successions of functions, and at the same time to employ these generalised concepts to obtain sufficient conditions that the upper or lower function of a succession of functions of an assigned type should be of a particular type.

As regards the former desideratum, the generalisation of the concept of uniform convergence, I have already given certain indications. Several modes of generalization are possible, and I have referred to all of them; but their simultaneous use and the adaptation of that portion of the theory I have elaborated, so as to render it available for any kind of functions, are needed. When this has been done the newly devised scheme of classification of functions comes into use and completely justifies its existence by the facility with which it permits of the application of the tests so obtained.

It is convenient, in the first instance, to suppose the functions of the

* Cf. "On Functions and their Associated Sets of Points," in this volume, pp. 260 *seqq*.

succession to be continuous. The upper and lower functions are then necessarily ul and lu-functions respectively. We require, then, tests that the upper function may be (i) both a lu and an ul, (ii) an l, (iii) an u, (iv) both an l and an u, that is a continuous function, and we require corresponding tests that the lower function should belong to these types.

Now we have this remarkable fact that there are two quite distinct modes of generalising the concept of uniform convergence,* so as to make it applicable to an oscillating succession as a whole. Further, we may so generalise as to require either of these new concepts to apply not only to the succession as a whole, but to each sub-succession; we may, as I have expressed it, require our condition to be homogenous.† Whether we do so or not, and whether we adopt the one or other of the two distinct methods alluded to, we have an irreproachable generalisation of the original concept of uniform convergence, which reduces to it when the upper and lower functions of the succession of continuous functions coincide. When the upper and lower functions do not coincide, each of the two modes furnishes us with tests for distinct properties.

Confining our attention still for convenience to successions of continuous functions, we may say that the succession oscillates uniformly above on the right at the point x in the first mode if the peak function, that is the upper double limit of $f_n(x+h)$, as h decreases to zero and n approaches infinity, is at the point x equal to the upper function. If this test is satisfied the upper function is, at the point in question, upper semi-continuous on the right. Similar and corresponding statements, of course, apply to uniform oscillation above on the left, and below on the right and left respectively, in the first mode.

On the other hand, we may say that the succession oscillates uniformly above on the right at the point x in the second mode if, given any quantity e, we can find an n and an h so that, for this and all greater values of n, and for this and all smaller values of h,

$$f_n(x+h)-u(x+h) \leqslant e.$$

If this test is satisfied the upper function $u(x)$ is at the point x in question lower semi-continuous on the right. Corresponding statements apply to uniform oscillation above on the left, and below on the right and left respectively, in the second mode.

* W. H. Young, "Oscillating Successions of Continuous Functions," 1908, *Proc. London Math. Soc.*, Ser. 2, Vol. 6, p. 309 and p. 313. "On Successions whose Oscillation is usually Finite," 1912, *Quarterly Journal of Mathematics*, Vol. 44, pp. 132, 133, and 141.

† W. H. Young, "On Homogeneous Oscillation of Successions of Functions," 1909, *Proc. London Math. Soc.*, Ser. 2, Vol. 8, p. 353.

At a point at which the succession of continuous functions oscillates uniformly and homogeneously in the first mode, both above and below, it oscillates uniformly and homogeneously in the second mode, both above and below. If throughout an interval the succession oscillates uniformly above in the first mode, it oscillates uniformly above in the second mode except at a set of the second category. The same is true if all that we know of the oscillation above in the first mode at each point is that it is uniform on the right or on the left, but not necessarily on both. Corresponding statements also hold for the oscillation below.

On the other hand, whereas the points of uniform oscillation above, below, on the right, on the left, in the first mode necessarily all exist in the case of a succession of continuous functions except at a set of the first category, this is not the case with those of the second kind, which are, in general, entirely absent.

These remarks enable us to obtain sufficient tests in terms of the uniformity of the oscillation in order that the upper (lower) function should belong to any assigned possible type of function.

When the functions of the succession are no longer continuous the definition above given for uniform oscillation of the first kind is no longer applicable. It appears, however, that we may replace it by a precisely equivalent definition. We have, in fact, only to employ the ascending successions for all values of m of which the general term is the function which is the upper bound of all the functions of the succession from the m-th up to the n-th inclusive, and the descending succession of which the general term is the corresponding lower bound; the uniform convergence of all the former successions is then taken to be the definition of uniform oscillation above in the first mode, and that of all the second successions the definition of uniform oscillation below in the first mode. Here the convergence and the corresponding oscillation may be considered as to their uniformity either on the right or on the left or both simultaneously.

It then at once suggests itself that the definition of the second mode of uniform oscillation may be modified in a similar manner. It appears, in fact, that, if we take the descending succession of which the general term is the function which is the upper bound of all the functions of the given succession from the n-th onwards, and the ascending succession formed by the corresponding lower bounds, the uniform convergence of the former succession on the right (left) may be taken to be the definition of uniform oscillation above on the right (left) in the second mode, and that of the second succession as that of uniform oscillation on the right (left) below in the second mode. This modification is convenient for various purposes

but it is not necessary, as it is evident that the original definition is valid for successions of any kind of functions.

We are now able to answer, in terms of the new concepts, questions as to the nature of the upper and lower functions of a succession of functions belonging to any assigned type. For definiteness let us suppose that the functions are *lu*-functions; then uniform oscillation above in the first mode has no simplifying effect on the upper function, which is an *ulu*, but uniform oscillation below in the first mode ensures the lower function being a *lu* instead of an *ulu*, as it would be in the general case. On the other hand, uniform oscillation in the second mode, when above, makes the upper function a *lu*, and if below makes the lower function an *ulu*, in each case depressing the type by one. If the functions of the successions are all *lu*'s and *ul*'s simultaneously, the first mode of uniform oscillation above makes the upper function an *ul*, and uniform oscillation below in the first mode makes the lower function a *lu*. In the second mode these are interchanged, the upper function being a *lu* and the lower function an *ul*.

These results, in fact, and others of a similar nature follow from the considerations I have just exposed, and the use of the theorem that a uniformly converging sequence of functions of any type has a function of that type for limit, a theorem of which the proof will be found below (§ 19).

2. The remarks I have made as to the importance of the concepts of uniform oscillation of the first and second kind are further enforced when we come to adapt an important theorem recently obtained by Egoroff * so as to render it available for oscillating successions. Egoroff finds in terms of the concept of uniform convergence the necessary and sufficient condition that a succession should converge except at a set of content zero. The corresponding problem in oscillating successions is to express in terms of uniform oscillation the necessary and sufficient condition that the upper function should not assume the value $+\infty$, or that the lower function should not assume the value $-\infty$, in each case except at a set of content zero. The result we obtain is as follows :—

It is a necessary and sufficient condition that the upper (lower) function should be nowhere $+\infty$ $(-\infty)$ *except at a set of content zero, that, given any positive quantity e, however small, we should be able*

* D. Egoroff, "Sur les suites de fonctions mesurables," 1911, *Comptes Rendus*. See also "On Successions whose Oscillation is usually Finite," § 4, *loc. cit.*, *supra*.

to find a perfect set whose content differs from that of the fundamental segment by less than e, with respect to which the succession oscillates uniformly above (below) in either the first mode or the second mode or both, as we please.

It is clear that the most convenient test to apply is that one of the three which relates to uniform convergence in the second mode; but all three tests are equally necessary and sufficient.

3. A series of functions is, as is well known, said to be uniformly convergent at the point P, if given any quantity e, we can find an integer m and an interval d containing the point x, so that, for all values of $n \geqslant m$ and all points x_1 in d,

$$-e \leqslant f_n(x_1) - f(x_1) \leqslant e.$$

Here m and d both depend on e, the one in general increasing indefinitely and the other decreasing indefinitely as e approaches zero. We distinguish between uniform convergence pure and simple, uniform convergence on the right, and uniform convergence on the left, according as the interval is unrestricted or has P for left-hand or for right-hand end-point respectively.

This definition is, when the generating functions $f_n(x)$ are continuous, as I have pointed out, precisely equivalent to the following: the series is said to be uniformly convergent at P if, as $n \to \infty$ and $h \to 0$, $f_n(x+h)$ has at the point x an unique double limit, this being uniform convergence pure and simple, or uniform convergence on the right (left), according as h assumes all values or only all positive (negative) values.

We may in the case of either definition introduce the distinction between uniform convergence above and below. In the first definition it suffices to divide the double inequality (1) into two inequalities, taking the upper inequality

$$f_n(x_1) - f(x_1) \leqslant e,$$

to characterize uniform convergence above, and the lower inequality,

$$-e \leqslant f_n(x_1) - f(x_1),$$

to characterize uniform convergence below. In the case of the second definition, uniform convergence above is determined by the equality of the upper double limit (peak function) of $f_n(x_1)$ at the point x with the limiting function $f(x)$, and uniform convergence below by the equality of $f(x)$ and the lower double limit (chasm function) of $f_n(x_1)$ at the point x.

The moment, however, we have done this, our two definitions no longer agree. Uniform convergence above defined in the one way does not agree with that defined in the other way, nor does it agree with uniform convergence below defined in the other way. Uniform convergence above in the one way so far agrees with uniform convergence below in the other way that the existence of either property ensures the limiting function being of the same type, the same for both. The one condition, however, is necessarily fulfilled at some point of every perfect set, while the other need not be fulfilled at all.

Moreover, when we proceed to analyse our concept still further and to replace the converging succession of continuous functions by an oscillating one, so that the limiting function has to share its properties between two functions, the upper function and the lower function of the succession, even this point of contact between the two modes of splitting up the concept of uniform convergence into components disappears. We are evidently in the presence of four distinct conceptions, even when we make no difference between right and left, which together make up the equivalent of the single concept of uniform convergence, properly so called.

4. Let us continue to confine our attention to continuous functions in the first instance, and suppose that we have an oscillating succession of such functions. Then we give the following definitions :—

A succession of continuous functions $f_1(x)$, $f_2(x)$, ..., *is said to oscillate uniformly above on the right in the first mode at the point* x *if*

$$u(x) = \pi_R(x),$$

where $u(x)$ *is the upper function and* $\pi_R(x)$ *the right-hand peak function, that is*

$$u(x) = \operatorname*{upper\ limit}_{n\to\infty} f_n(x),$$

$$\pi_R(x) = \operatorname*{upper\ double\ limit}_{n\to\infty \quad h\to 0} f_n(x+h). \qquad (0 < h)$$

The same definition applies on the left instead of on the right if we take h to be negative instead of positive, and replace $\pi_R(x)$ by $\pi_L(x)$. If, on the other hand, we allow h to assume both positive and negative values, and write $\pi(x)$, the peak function, instead of $\pi_R(x)$ or $\pi_L(x)$, the definition is that of uniform oscillation above in the first mode at the point x. Similar definitions apply to uniform oscillations below, using the chasm funtions in place of the peak function and changing upper into lower. Thus, for instance, disregarding the distinction of right and left, we have the definition

A succession of continuous functions $f_1(x)$, $f_2(x)$, ..., *is said to oscillate*

uniformly below in the first mode at the point x if $l(x) = \chi(x)$, where $l(x)$ is the lower function and $\chi(x)$ is the chasm function, that is

$$l(x) = \text{lower } \lim_{n\to\infty} f_n(x),$$

$$\chi(x) = \text{lower double } \lim_{n\to\infty,\ h\to 0} f_n(x+h).$$

In the definition of uniform oscillation of the second kind we need not insert the word "continuous," as the definition, though applied in the first instance to continuous functions, is applicable for all types of functions. The definition is given disregarding the distinction of right and left. If we wish to insert this distinction we only have to restrict h to be positive or to be negative.

A succession of functions $f_1(x), f_2(x), \ldots$, is said to oscillate uniformly above in the second mode at the point x if, given any positive quantity e, we can find an interval d containing the point x as internal point and a corresponding integer m, so that for all integers $n \geqslant m$ and all points $x+h$ in the interval d $\quad f_n(x+h) - u(x) \geqslant e,$

$u(x)$ being the upper function.

Under the same circumstances if, $l(x)$ being the lower function,

$$-e \leqslant f_n(x+h) - l(x),$$

the succession is said to oscillate uniformly below in the second mode.

5. THEOREM.—*If the succession of continuous functions $f_1(x), f_2(x), \ldots$, oscillates uniformly above (below) in the first mode, the upper function $u(x)$ is upper (lower) semi-continuous.*

For, by the definition of uniform oscillation above in the first mode,

$$u(x) = \text{upper double limit } f_n(x+h). \tag{1}$$

Now, since any repeated limit is a double limit,

$$\text{upper } \lim_{h\to 0} u(x+h) = \text{a certain double } \lim_{n\to\infty,\ h\to 0} f_n(x+h)$$

$$\leqslant \text{upper double } \lim_{n\to\infty,\ h\to 0} f_n(x+h). \tag{2}$$

The comparison of (1) and (2) shows that $u(x)$ is upper semi-continuous. Similarly, since

$$\text{lower double } \lim_{h\to 0} l(x+h) \geqslant \text{lower double } \lim_{n\to\infty,\ h\to 0} f_n(x+h), \ldots, \tag{2'}$$

the equation defining uniform oscillation below in the first mode, viz.,

$$l(x) = \text{lower double } \lim_{n\to\infty,\ h\to 0} f_n(x+h),$$

shows that in this case $l(x)$ is lower semi-continuous.

This proves the theorem.

The inequality (2) of the above proof holds without any special assumption as to the nature of the functions $f_n(x)$ or the mode of oscillation of the succession.

The left-hand side of (2) is the associated upper semi-continuous function of the upper function, and the right-hand side may still be defined to be the peak function. Thus we may express (2) by saying that *the associated upper limiting function of the upper function is always less than or equal to the peak function.* Similarly we may express (2′) by saying that *the associated lower limiting function of the lower function is always greater than or equal to the chasm function.* If the functions $f_n(x)$ are continuous, (2) remains true when we replace the left-hand side by $u(x)$. The same is, of course, true if $u(x)$ is lower semi-continuous. In the general case, since a function lies between its upper and lower associated semi-continuous functions except at a countable set of points, we have, except at a countable set,

$$\chi(x) \leqslant l(x) \leqslant u(x) \leqslant \pi(x),$$

an inequality which has no exceptional points if the functions $f_n(x)$ are continuous, or if $u(x)$ is lower semi-continuous and $l(x)$ is upper semi-continuous.

6. THEOREM.—*If the succession of continuous functions $f_1(x), f_2(x), \ldots,$ oscillates uniformly above (below) in the second mode, the upper (lower) function $u(x)$ is lower (upper) semi-continuous.*

For

$$u(x)-u(x+h) = \{f_n(x+h)-u(x+h)\}+\{u(x)-f_n(x)\}+\{f_n(x)-f_n(x+h)\}.$$

Now, if the f_n-succession oscillates uniformly above in the second mode, we can find m so that the first term on the right is less than $e/3$ for all values of h considered and all values of $n \geqslant m$. Also, since $u(x)$ is one of the limits of $f_n(x)$, we can find a value of $n \geqslant m$ for which the second term is less than $e/3$. Finally, since $f_n(x)$ is continuous for this value of n, we can find H so that for $h \leqslant H$ the last term is less than $e/3$. Thus, the left-hand side of the preceding identity is less than e, provided $h \leqslant H$. This proves the theorem as far as the oscillation above is concerned. Similarly it may be proved in the alternative case.

COR.—*The theorem remains true when the functions $f_n(x)$ are not continuous, but only lower (upper) semi-continuous.*

In fact, we have not used the upper semi-continuity of $f_n(x)$ in the proof of the lower semi-continuity of the upper function nor the lower

semi-continuity of $f_n(x)$ in the proof of the upper semi-continuity of the lower function.

7. THEOREM.—*If the upper (lower) function is continuous, then the succession of continuous functions* $f_1(x), f_2(x), \ldots$, *oscillates uniformly above (below) in both modes or in neither mode.*

It will be sufficient to prove the first half of the theorem. Suppose, then, that the succession oscillates uniformly above in the first mode. Then, taking any point y, we can find an interval, or tile, d_y and an integer m_y, so that, for all points x in the tile and all integers $n \geqslant m_y$,

$$f_n(x) \leqslant u(y)+e.$$

Hence, if $u(y)$ is continuous at the point y so that we can so choose the tile that for all points x in it

$$u(y) \leqslant u(x)+e,$$

$$f_n(x) \leqslant u(x)+2e,$$

which shows that the succession oscillates uniformly above in the second mode.

Again suppose the succession oscillates uniformly above in the second mode; then, corresponding to the point y, we can find a tile d_y and a corresponding integer m_y so that, for all points x in the tile and all integers $n \geqslant m_y$,

$$f_n(x) \leqslant u(x)+e,$$

and, if $u(x)$ is continuous at the point y, we can so choose the tile that, for all points x in it,

$$u(x) \leqslant u(y)+e,$$

and therefore

$$f_n(x) \leqslant u(y)+2\,e.$$

Since e is as small as we please this shows that the upper double limit of $f_n(x)$ at the point y is not greater than $u(y)$. But, since $u(y)$ is continuous, the upper double limit of $f_n(x)$ cannot be less than $u(y)$, as pointed out at the end of § 5. Hence it must be equal to $u(y)$, which shows that the succession oscillates uniformly above in the first mode.

Combining these two results, we see that the theorem is true.

COR.—*Uniform and homogeneous oscillation of the first kind involves uniform and homogeneous oscillation of the second kind.*

In the above proof we have not used the continuity of the functions $f_n(x)$ except in so far as the definition of uniform oscillation of the first kind is concerned. It follows, therefore, that, whatever type of functions

generate the given succession, provided only the upper function is continuous, if the succession oscillates uniformly above in the second mode,

$$u(x) = \pi(x),$$

and conversely, if this equation holds, the succession oscillates uniformly above in the second mode. Similarly, provided the lower function is continuous, if the succession oscillates uniformly below in the second mode,

$$l(x) = \chi(x),$$

and conversely.

8. Before proceeding further it will be convenient to prove certain properties of monotone ascending and descending successions. The following theorem is slightly more general than that usually given.

THEOREM.—*A monotone ascending succession of l-functions whose limit is continuous converges uniformly. A monotone descending succession of u-functions whose limit is continuous, converges uniformly.*

Let $f_1(x) \leqslant f_2(x) \leqslant \ldots \to f(x)$, be a monotone ascending succession of l-functions whose limit is continuous. Then, taking any point x, we can determine H, so that

$$f(x+h) - f(x) \leqslant e \quad (h \leqslant H).$$

Also, since the succession converges at the point x, we can find M so that

$$f(x) - f_M(x) \leqslant e.$$

Also, since f_M is lower semi-continuous, we can find $H_M \leqslant H$, so that

$$f_M(x) - f_M(x+h) \leqslant e \quad (h \leqslant H_M \leqslant H).$$

Hence, adding the three inequalities,

$$f(x+h) - f_M(x+h) \leqslant 3e \quad (h \leqslant H_M).$$

But the succession is monotone ascending, therefore

$$0 \leqslant f(x+h) - f_n(x+h) \leqslant 3e \quad (h \leqslant H_M,\ n \geqslant M).$$

This proves that at the point x the sequence converges uniformly. Since this is true at every point x, this proves the theorem in the first case, and similarly it may be proved in the second case.

COR.—*If a monotone ascending (descending) succession of functions each of which is lower (upper) semi-continuous at the point P, has a limiting function which is continuous at the point P, the convergence of the succession at the point P is uniform.*

9. It will be remarked that, in proving the theorem of the last article, we employed the second definition of uniform convergence, and consequently we had only to prove that the lower half of the condition was satisfied in the case of the ascending succession, and the upper half in the case of a descending succession; the other halves are satisfied of themselves. It is of some importance to note that, in the case of ascending successions of continuous functions, it is the lower half of the first definition of uniform convergence which is necessarily satisfied, and it is the upper half in the case of a descending succession. This appears from the following theorem.

THEOREM.—*If* $f_1 \leqslant f_2 \leqslant \ldots$ *is a monotone ascending succession of functions, the lower double limit of* $f_n(x+h)$, *as* $n \to \infty$ *and* $h \to 0$, *is equal to the limiting function* $f(x)$, *at any point* x *where all the functions* $f_n(x)$ *are continuous.*

Since $f_n(x)$ has $f(x)$ as unique limit, we can determine $M_{x,e}$ so that, for $n \geqslant M_{x,e}$,

$$0 \leqslant f(x) - f_n(x) \leqslant e.$$

Taking any one of these values of n, we can determine $H_{n,x,e}$ so that, for $h \leqslant H_{n,x,e}$,

$$-e \leqslant f_n(x) - f_n(x+h) \leqslant e.$$

Therefore
$$-e \leqslant f(x) - f_n(x+h) \leqslant 2e \quad (n \geqslant M_{x,e},\ h \leqslant H_{n,x,e}). \quad (1)$$

A fortiori, if $m \geqslant n$, $\quad f(x) - f_m(x+h) \leqslant 2e.$

Letting m describe such an increasing succession of integers, and h such a monotone descending sequence with zero as limit, that $f_m(x+h)$ has its lower double limit $\chi(x)$ as unique limit, we get

$$f(x) - \chi(x) \leqslant 2e. \quad (2)$$

Letting e approach zero, we get from (2),

$$f(x) - \chi(x) \leqslant 0. \quad (3)$$

But, as e approaches zero, $M_{x,e}$ will in general increase without limit, and we may make n in (1) assume values which increase without limit; at the same time $H_{n,x,e}$ will in general decrease down to zero, and we may in any case take in (1) a succession of values of h approaching zero; thus, as $e \to 0$, $f_n(x+h)$ in (1) approaches a certain double limit, or limits, which are certainly $\geqslant \chi(x)$. Thus we get from the first inequality in (1), by making $e \to 0$,

$$0 \leqslant f(x) - \chi(x). \quad (4)$$

By (3) and (4), $\quad f(x) = \chi(x).$

Similarly we have the alternative theorem :—

THEOREM.—*If* $f_1 \geqslant f_2 \geqslant \ldots$ *is a monotone descending succession of functions the upper double limit of* $f_n(x+h)$, *as* $n \to \infty$ *and* $h \to 0$, *is equal to the limiting function* $f(x)$ *at any point* x *where all the functions* $f_n(x)$ *are continuous.*

10. We now return to oscillating successions of functions and consider the monotone ascending and descending successions associated with it.

The first monotone ascending succession associated with the given succession $f_1(x), f_2(x), \ldots$, we denote by

$$v_{1,1}(x) \leqslant v_{1,2}(x) \leqslant v_{1,3}(x) \leqslant \ldots \to v_1(x),$$

where $v_{1,r}(x)$ is the upper bound of $f_1(x), f_2(x), \ldots, f_r(x)$. There is a countably infinite set of these sequences, the second being defined in the same way from the given succession omitting its first member, and so on, the r-th being got by omitting $f_1(x), f_2(x), \ldots, f_{r-1}(x)$. These monotone sequences lead up to the functions $v_1(x), v_2(x), \ldots$ of the first monotone descending sequence. Here, of course, $v_1(x)$ is the upper bound of the f_n-succession at the point x, and $v_r(x)$ is the upper bound of the same, omitting $f_1, f_2, \ldots, f_{r-1}$. The limit of the monotone descending v_r-succession is therefore the upper function $u(x)$ of the f_n-succession.

Thus we have, for the set of the first monotone ascending sequences, and the first monotone descending sequence, the following scheme :—

$$\begin{array}{l} v_{1,1} \leqslant v_{1,2} \leqslant \ldots \to v_1, \\ \qquad\qquad\qquad\qquad \vee \\ v_{2,1} \leqslant v_{2,2} \leqslant \ldots \to v_2, \\ \qquad\qquad\qquad\qquad \vee \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \qquad\qquad\qquad\qquad \vee \\ v_{r,1} \leqslant v_{r,2} \leqslant \ldots \to v_r, \\ \qquad\qquad\qquad\qquad \vee \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \qquad\qquad\qquad\qquad \downarrow \\ \qquad\qquad\qquad\qquad u. \end{array}$$

Similarly, taking the lower instead of the upper bounds, we get the corresponding scheme for the second monotone descending sequences, and

the second monotone ascending succession associated with the given succession :—

$$\begin{array}{l} w_{1,1} \geqslant w_{1,2} \geqslant w_{1,3} \geqslant \ldots \to w_1, \\ \qquad\qquad\qquad\

Writing $\pi(x) =$ upper double limit of $f_n(x+h)$, $n\to\infty, h\to 0$

$$\pi'(x) = \text{upper double limit of } v_n(x+h),$$

we have evidently $$\pi(x) \leqslant \pi'(x), \tag{1}$$

since $v_n(x+h)$ being the upper bound of $f_n(x+h)$, $f_{n+1}(x+h)$, ...,

$$f_n(x+h) \leqslant v_n(x+h).$$

Take such a succession of integers n, and such a corresponding succession of values of h, that $v_n(x+h)$ has as unique limit its upper double limit. We can then, corresponding to each of these values $v_n(x+h)$, take a lesser quantity V_n, so that, as n describes its succession of values, V_n has the same unique limit as $v_n(x+h)$, namely $\pi'(x)$. Then, corresponding to each of these values of n, we can, since $v_n(x+h)$ is the upper bound of $f_{n+p}(x+h)$ as p varies, find an integer m, depending on n, so that

$$f_m(x+h) \geqslant V_n \quad (m \geqslant n).$$

Now, letting $n\to\infty$, the left-hand side of the last inequality has a limit, or limits, not exceeding $\pi(x)$; therefore

$$\pi(x) \geqslant \pi'(x).$$

Hence, by (1), $$\pi(x) = \pi'(x),$$

which proves the first statement in the enunciation, that the peak functions of the two successions are the same.

12. THEOREM.—*The necessary and sufficient condition that a given succession of continuous functions $f_1(x), f_2(x), \ldots$ should oscillate uniformly above (below) in the first mode, is that the first monotone ascending (descending) successions associated with the given succession should converge uniformly.*

First, to show that the condition is sufficient, we assume that the succession of continuous functions

$$v_{r,1} \leqslant v_{r,2} \leqslant \ldots \to v_r$$

converges uniformly for each integer r. Therefore all the functions $v_r(x)$ are continuous, so that, by the second theorem of § 9, the limiting function of the v_n-succession is its peak function. Therefore, by the theorem of § 11,

$$u(x) = \pi(x); \tag{1}$$

this proves the sufficiency of the condition.

Next, to prove the necessity of the condition, we assume that (1) holds.

Therefore, since $\pi(x)$ is the upper double limit of $f_n(x+h)$, we can find $M > r$ and H, depending on x and on e, so that for $n \geqslant M$ and $h \leqslant H$,

$$f_n(x+h) \leqslant u(x)+e \leqslant v_r(x)+e, \ \ldots .$$

Also, since the functions $f_r(x), f_{r+1}(x), \ldots, f_{M-1}(x)$ are continuous, we may suppose $H_M \leqslant H$ so small that, for $r \leqslant n < M$,

$$f_n(x+h) \leqslant f_n(x)+e \leqslant v_r(x)+e.$$

By the last two inequalities

$$f_n(x+h) \leqslant v_r(x)+e, \ \ldots \quad (n \geqslant r, \ h \leqslant H_M).$$

Hence, since $v_r(x)$ is the upper bound of the left-hand side,

$$v_r(x+h) \leqslant v_r(x)+e \quad (h \leqslant H_M).$$

This shows $v_r(x)$ to be upper semi-continuous. But since the functions $f_n(x)$ are continuous, $v_r(x)$ is necessarily lower semi-continuous (§ 10), and is therefore continuous; therefore, by § 8, the convergence of the monotone succession leading up to $v_r(x)$ is uniform. This proves the necessity of the condition as far as the upper oscillation is concerned. Similarly the theorem may be proved for the lower oscillation.

13. It should be remarked that the result of the last article is equally true whether we are considering a particular point or a whole interval. It is also equally true whether we are concerned with uniform oscillation in general or restrict it to be on the right or on the left. The type of oscillation corresponds precisely to the type of the convergence of the corresponding monotone successions. The same remarks apply equally in all similar places in our investigations, and in particular they apply to the theorem which we are about to prove which establishes the connection between uniform oscillation of the second kind and the monotone successions which ascend to the upper and descend to the lower functions.

THEOREM.—*The necessary and sufficient condition that a given succession of functions* $f_1(x), f_2(x), \ldots$, *should oscillate uniformly above (below) in the second mode is that the second monotone descending (ascending) succession associated with the given succession should converge uniformly to its limiting function, the upper (lower) function of the given succession.*

For, since $v_m(x)$ is the upper bound of the quantities $f_m(x), f_{m+1}(x), \ldots$, the inequality

$$f_n(x)-u(x) \leqslant e, \quad (n \geqslant m \geqslant M)$$

has the inequality

$$v_m(x)-u(x) \leqslant e, \quad (m \geqslant M)$$

as a consequence. Also, since $f_n(x)$ cannot be greater than $v_n(x)$, the latter inequality entails the former as a consequence. Hence the theorem is true as regards the upper oscillation. Similarly it is true as regards the lower oscillation.

14. The results of the theorems of §§ 12 and 13 at once furnish us with new proofs of the theorems of §§ 5 and 6. They also enable us to obtain with ease a number of further results. Thus it follows from the known theory of uniform convergence of successions of continuous functions that the points at which there is uniform oscillation above, below, on the right, and on the left of the first kind all form sets complementary to sets of the first category, and this equally whether we consider the continuum or any perfect set, always provided that the functions of the succession considered are continuous. We have, indeed, the more general result that this statement is true for the upper oscillation on the right and on the left and in general when the functions of the succession are lower semi-continuous, and that the corresponding statement is true for the lower oscillation when they are upper semi-continuous. In fact, in the former case the functions $v_1, v_2, \ldots$, are the limiting functions of monotone ascending successions of l-functions, and are therefore themselves l-functions; hence, bearing in mind that the set composed of a countable infinite set of sets of the first category is a set of the first category, so that all these l-functions $v_r(x)$ are continuous except at points of a single set of the first category, the required result follows by means of the corollary to the theorem of § 8.

Thus we have the theorem:

THEOREM.—*A succession of lower semi-continuous functions necessarily oscillates uniformly above in the first mode except at a set of points which is of the first category.*

Similarly we have the corresponding theorem:

THEOREM.—*A succession of upper semi-continuous functions necessarily oscillates uniformly below in the first mode except at a set of points which is of the first category.*

Both these statements remain, of course, true if we replace the continuum by any perfect set and consider the uniformity of oscillation with respect to this perfect set.

15. There is no corresponding theorem for the points of uniform

oscillation of the second kind. Such points may be entirely absent. We have, however, the following theorem.

THEOREM.—*When the functions of the succession are continuous and every point is a point of uniform oscillation above (below) on one side at least in the first mode, then, except at a set of the first category, there is uniform oscillation above (below) in the second mode.*

Take the first of these alternative theorems.

Then the upper bound $v_n(x)$ of all but the first $(n-1)$ functions of the given succession, being the limit of a succession of continuous functions which converges uniformly on one side at every point, is a function which is continuous on one side at least at every point, and is, therefore, continuous except at a countable set of points. The upper function $u(x)$ of the original succession, being the limiting function of the monotone descending v_n-succession, is then a pointwise discontinuous function.* Therefore, except at a set of the first category, consisting of the points of discontinuity of $u(x)$ and the countable set of points at which at least one of the functions $v_n(x)$ is discontinuous, all the functions $v_n(x)$, as well as the limiting function $u(x)$, are continuous, and therefore the convergence of the v_n-succession is uniform, so that the original succession oscillates uniformly above in the second mode.

This proves the first of the two alternative theorems. Similarly the other may be proved.

16. We can now answer the question which naturally suggests itself as to sufficient tests that the upper and lower functions of a succession of continuous functions should have an assigned form. It will be sufficient to confine our attention to the upper function, as the corresponding results for he lower function follow immediately. The conditions that the upper function should be upper semi-continuous or lower semi-continuous or both, that is continuous, are obvious. As the upper function is necessarily an *ul*-function the only remaining condition that we have to investigate is that it should be a *lu* as well as an *ul*-function. Moreover, since a descending succession of *l*-functions converges to a *lu*-function when the points of uniform convergence exist in every perfect set, a sufficient condition is that there should be no perfect set free from points of uniform oscillation above of the second kind.†

* W. H. Young, "On Sequences of Asymmetrically Continuous Functions," 1909, *Quarterly Journal*, Vol. XL, pp. 374–380.

† It is sufficient to test the oscillation with respect to each perfect set.

17. We may also prove the following theorem :—

THEOREM.—*If the upper (lower) function of a succession of continuous functions is lower (upper) semi-continuous, the points of uniform oscillation above (below) of the second kind form a set complementary to one of the first category.*

In fact, the succession $v_1(x) \geqslant v_2(x) \geqslant \ldots$ leading down to the upper function are in the former of the alternative theorems l-functions. Hence, by the argument already used more than once, the statement follows. Similarly the alternative statement follows.

18. We now pass to the consideration of successions of functions other than continuous functions. We have already so far extended our results as to consider the case when the generating functions are upper or lower semi-continuous; but in doing so we only considered those points where these functions are continuous, at least so far as the theory of uniform oscillation of the first kind is concerned. We have up to the present point, in fact, not yet formulated a definition of uniform oscillation of the first kind available at points or in intervals where the generating functions $f_n(x)$ of the succession are not continuous.

We now proceed to consider this more general case. We begin by making two remarks. The first is that it follows, from the reasoning of §§ 12 and 13, that, if a succession of functions converges uniformly, all the monotone successions considered in those articles converge uniformly. This at once suggests that we may replace the definition already given of uniform oscillation of the first kind by that which is identical with it in the case when the functions are continuous, namely, the property proved in § 12. The second is that the limiting function of a uniformly convergent series of functions of a certain type is always a function of the same type. As this latter remark may not seem obvious, and certainly requires proof, we now give such a proof.

19. THEOREM. — *The limiting function of a uniformly convergent succession of functions of a certain type is a function of the same type.*

It will be sufficient to treat at length the case when the functions of the succession are *ul*-functions.

Let $e_1 > e_2 > \ldots \to 0$ be any monotone descending sequence with zero as limit. Let us denote some function of the given succession which differs numerically by less than e_1 from the limiting function $f(x)$, by $f_1(x)$,

another which differs numerically by less than e_2 from $f(x)$ by $f_2(x)$, and so on. This is possible since the convergence is uniform.

Let G_1 denote the set of points at which

$$f_1(x) \geqslant k - e_1 ;$$

this is an i-set, since $f_1(x)$ is an ul-function.

Let G_2 denote the set of points at which both

$$f_1(x) \geqslant k - e_1,$$

and

$$f_2(x) \geqslant k - e_2 ;$$

this is the common sub-set of two i-sets, and is therefore an i-set.

Similarly we go on; thus let G_n denote the set of points at which, for $r = 1, 2, \ldots, n$,

$$f_r(x) \geqslant k - e_r,$$

then G_n is an i-set.

Each of these i-sets G_n evidently contains the next G_{n+1}; they have therefore in common a set G, which is the inner limiting set of i-sets, and is therefore itself an i-set.

At every point x belonging to G, we have, for all integers n,

$$f_n(x) \geqslant k - e_n.$$

But $f(x)$ is the limit of $f_n(x)$, when $n \to \infty$ and accordingly e_n approaches zero, therefore

$$f(x) \geqslant k.$$

But, on the other hand, if x is a point at which

$$f(x) \geqslant k,$$

we must have, since $f_n(x) - f(x)$ is numerically less than e_n,

$$f_n(x) \geqslant k - e_n,$$

for all integers n, so that x is a point of G_n for every n, and is therefore a point of G. Thus the i-set G is precisely the set of points at which

$$f(x) \geqslant k,$$

which proves that $f(x)$ is itself an ul-function.

If the functions $f_n(x)$ are u-functions, the argument is the same, only instead of i-sets, we have closed sets; if they are ulu-functions, we only have io-sets; if $ulul$-functions, we have ioi-sets, and so on, but in all cases the same argument leads to the required result.

If, on the other hand, the functions $f_n(x)$ are l-functions, or lu-functions, or lul-functions, and so on, we merely have to change the sign $\geqslant$ into $\leqslant$, and the argument applies. In all cases we use the theorem of

§ 24 of my paper "On Functions and their Associated Sets of Points," which characterises the type of a function by the type of set of points at which it is $\geqslant k$, or $\leqslant k$.

20. We have now to frame a definition of uniform oscillation of the first kind above and below in accordance with the remark of § 18. A little consideration shows us that the requisite that the ascending succession whose limiting function is $v_1(x)$ should be uniformly convergent is precisely equivalent to the following:—

Given e, we can find an integer m_e, so that, whatever point x be taken, there is a corresponding integer $r_x \leqslant m_e$, so that, for all values of n,

$$f_n(x) - f_{r_x}(x) \leqslant e.$$

First to show that this is sufficient, suppose the condition to hold. Then, if $p > m_e$, the function $f_{r_x}(x)$ will, for every point x, be among the functions of which $v_{1,p}(x)$ is the upper bound. Hence

$$f_n(x) - v_{1,p}(x) \leqslant e. \tag{1}$$

But since $v_1(x)$ is the upper bound at the point x of $f_1(x), f_2(x), \ldots$, we can at each point x find an integer n_x so that

$$f_{n_x}(x) \geqslant v_1(x) - e. \tag{2}$$

Taking $n = n_x$ in (1), we get therefore at the point x,

$$v_1(x) - v_{1,p}(x) \leqslant 2e, \quad \ldots \quad (p > m_e).$$

Since m_e is independent of x, and this inequality holds for every point x, this shows that the monotone ascending $v_{1,p}$-succession converges uniformly. Thus the condition is sufficient.

Next to show that it is necessary, we assume that, given e, we can find m_e so that for all values of $p \geqslant m_e$ and all points x,

$$v_1(x) - v_{1,p}(x) \leqslant e.$$

Now $v_{1,p}(x)$ is the upper bound at the point x of $f_1(x), f_2(x), \ldots, f_p(x)$, therefore there is an integer r_x, such that

$$v_1(x) - f_{r_x}(x) \leqslant e, \quad \text{and} \quad r_x \leqslant p.$$

Taking $p = m_e$, we see that the given condition is necessary.

Hence we see that the definition of uniform oscillation above in the first mode of a succession of any kind of function $f_1(x), f_2(x), \ldots$ may be given in the following form:—

The succession $f_1(x), f_2(x), \ldots$ is said to oscillate uniformly above, in the first way if, given e, we can, corresponding to each integer r, find an

integer $m_{r,e}$, so that, whatever point x be taken, there is a corresponding integer $n_x \leqslant m_{r,e}$, so that, for all values of n,

$$f_{r+n}(x) - f_{r+n_x}(x) \leqslant e.$$

The succession is said to oscillate uniformly below in the first mode if, given e, we can find $m_{r,e}$, so that, for a value $n_x \leqslant m_{r,e}$,

$$f_{r+n_x}(x) - f_{r+n}(x) \leqslant e,$$

for all values of n.

This same condition may be expressed as follows:—

The succession $f_1(x), f_2(x), \ldots$ is said to oscillate uniformly above in the first mode if, given any small positive quantity e and any integer r, we can divide the continuum into a finite number $m_{r,e}$ of sets of points, such that, if the point x is any point of the k-th set,

$$f_{r+n}(x) - f_{r+k}(x) \leqslant e \quad (n = 1, 2, \ldots).$$

The succession is said to oscillate uniformly below in the first mode if, under the same circumstances,

$$f_{r+k}(x) - f_{r+n}(x) \leqslant e \quad (n = 1, 2, \ldots).$$

Indeed there is only a finite number of integers $n_x \leqslant m_{r,e}$, therefore we may group the points x together in a set, if $n_x = k$, where k is any integer $\leqslant m_{r,e}$. If there are no such points, the corresponding set will be the null-set, or, if we prefer, is absent, so that the corresponding integer k will not occur, and the number of sets will be less than $m_{r,e}$.

21. We have now given definitions, in terms of the functions themselves, of uniform oscillation at a point above and below of the first and of the second kinds. The definition of that of the first kind has just been given (§ 20), while that of the second kind is the one we originally took. When the fundamental region is the straight line, we may if we please insert the expressions "on the right" or "on the left" with corresponding generalisations for higher space.

We have seen that, when the functions are continuous, uniform oscillation of the first kind always exists except at a set of the first category. The requirement that the succession should oscillate uniformly in the second mode is more stringent, and points of uniform oscillation in the second mode will in general not be present, even when the functions are continuous. When the functions are continuous, however, and the succession is a sequence, uniform oscillation both above and below throughout the continuum of either kind necessitates uniform oscillation

throughout the continuum of the other kind and ensures the uniformity of the convergence of the original sequence.

When the functions are not continuous it is, however, only the more stringent of the two kinds of uniform oscillation which ensures the uniformity of the convergence of the original sequence, and with it the uniformity of the oscillation in the less stringent of the two modes. To see this last it suffices to suppose the sequence to consist of functions $f_1(x), f_2(x), \ldots$, of which the odd terms form a monotone descending sequence $f_1(x) \geqslant f_3(x), \ldots$, and the even terms a monotone ascending sequence $f_2(x) \leqslant f_4(x) \leqslant \ldots$, both converging non-uniformly to the same limiting function $f(x)$. This is always possible when the functions of the succession are not continuous. But in this case we have clearly uniform and even homogeneous oscillation both above and below of the first kind, combined with non-uniform convergence of the sequence itself.

The justification of the definitions I have given is to be found in the applications, but also in the circumstance, already alluded to, which is fundamental in this connection, that, if a succession converges uniformly, all the monotone successions associated with it do so also, so that the conditions for uniform oscillation, both above and below, of the first and of the second kind are all fulfilled. For completeness I repeat the proof of this fact which I have already given elsewhere.*

22. THEOREM.—*If a succession of functions converge uniformly all its associated monotone successions converge also uniformly.*

Since the given succession $f_1(x), f_2(x), \ldots$, converges uniformly we can, after assigning e, find M, so that, for all values of m and n greater than M,

$$|f_m(x) - f_n(x)| < e,$$

whatever point x be considered, M being independent of x. Therefore if

$$M < m < n,$$

we have, since

$$f_m(x) \leqslant v_{1,m}(x);$$

also

$$f_n(x) \leqslant v_{1,m}(x) + e, \quad (n = m+1, m+2, \ldots)$$

But $v_{1,n}$ is the upper bound of $v_{1,m}, f_{m+1}, f_{m+2}, \ldots, f_n$ at the point x. Therefore

$$v_{1,n} \leqslant v_{1,m}(x) + e, \quad (M < m < n).$$

This proves the uniform convergence of the first monotone ascending

* "On Successions whose Oscillation is usually Finite," *loc. cit.*, p. 139.

sequence. Similarly the remaining monotone ascending sequences formed from the functions $v_{r,s}(x)$, for successive integers r, converge uniformly.

The uniform convergence of the monotone descending v_r-succession follows as in § 13, since this succession, like the original succession, now has $f(x)$ as limit, and the relations

$$f_n(x) - f(x) \leqslant e, \quad (n \geqslant m \geqslant M)$$

and

$$v_m(x) - u(x) \leqslant e, \quad (n \geqslant M)$$

are consequences of each other.

This proves the theorem as far as the $v_{r,s}$ and v_r-successions are concerned. Similarly, making the appropriate changes of ascending into descending, and the like result follows for the $w_{r,s}$ and w_r-successions.

23. We can now completely answer the question as to the effect of uniform oscillation of the two kinds, above and below, on the right and on the left, on the upper and lower functions of the succession of functions belonging to an assigned class.

THEOREM.—*If the name of the class of functions of a certain succession begins with an l, uniform oscillation of the first kind above prefixes an u to the name of the class in the case of the upper function, so that the class is the same as if the oscillation were not uniform, and, in the case of the lower function, uniform oscillation of the first kind below makes the lower function of the same class as the functions of the succession. On the other hand, uniform oscillation of the second kind above makes the upper function of the same class as the functions of the succession, while uniform oscillation of the second kind below prefixes an u in the case of the lower function.*

Similarly we have the alternative theorem :

THEOREM.—*If the name of the class of functions of a certain succession begins with an u, uniform oscillation of the first kind above makes the upper function belong to the same class, while uniform oscillation of the first kind below prefixes an l in the case of the lower function, so that the class of the lower function is the same as if the oscillation were not uniform below. On the other hand, uniform oscillation of the second kind above prefixes an l in the case of the upper function, while uniform oscillation of the second kind below makes the lower function belong to the same class as the functions of the succession.*

The proofs of these theorems will be evident to the reader acquainted with the theory of monotone sequences as exposed in my previously published papers.

From the theorem of § 19 follows still more directly that *if a succession oscillates uniformly above (below) in both ways its upper (lower) function belongs to the same class as the original functions.* In fact, the associated monotone successions all converge uniformly, whence the result must be true.

24. We now pass to the generalisation for oscillating successions of the theorem of Egoroff. I have already obtained a generalisation by a method which is itself the extension of Egoroff's method. It is, however, still simpler to assume Egoroff's theorem and to employ the considerations I have just exposed as to the identity of uniform oscillation with uniform convergence of certain associated monotone successions. If the upper function has the value $+\infty$ at most at a set of zero content, all the monotone successions associated with the upper function are bound to converge except at a set of content zero. Moreover, the convergence, except at a set of content zero, of the successions connected with oscillation of the first kind above, and the convergence, except at a set of content zero, of those connected with oscillation of the second kind above is in either case sufficient to ensure that the points, if any, at which the upper function has the value $+\infty$, form a set of zero content.

Consider first the successions connected with uniform oscillation of the first kind above, namely, the successions of functions $v_{r,s}(x)$. There is a countably infinite number of these successions, each of which converges usually if we suppose, as we shall do, the hypothesis with regard to the upper function to hold. Applying Egoroff's theorem we can separate off a set of content less than $\frac{1}{2}e$ corresponding to the first of these successions, viz., the $v_{1,s}$-succession, a set of content less than $\frac{1}{4}e$ corresponding to the second of these successions, and so on, in such a manner that in the remaining portion of the fundamental region we have uniform convergence of the succession in question with respect to that portion.

Combining the exceptional sets into a single set of content less than e, we have with respect to the complementary set uniform convergence of all the $v_{r,s}$-successions, for every value of r.

Similar consideration holds for the lower oscillation. Thus we have the following theorem.

THEOREM.—*If the upper (lower) function of a given succession* is* $+\infty$ $(-\infty)$ *at the points of a set of zero content only, then, given e, we can find a set of content less than e such that with respect to the complementary set the succession oscillates uniformly in the first way. The converse is also true.*

In the case of uniform oscillation of the second kind the corresponding result is even more easily perceived, since there is only one monotone succession to be considered.

THEOREM.—*If the upper (lower) function of a given succession is* $+\infty$ $(-\infty)$ *at the points of a set of zero content only, then, given e, we can find a set of content less than e, such that with respect to the complementary set the succession oscillates above (below) uniformly in the second way. The converse is also true.*

25. The results of the preceding article are of great importance, but the reader must be warned not to *exaggerate* the signification of a theorem such as that of Egoroff. A set even of zero content may be constructed in the continuum in such a manner as to leave over a set of the first category only,† that is, a set of a kind which does so little towards filling up the continuum that we may subtract a countably infinite number of such sets from the continuum and leave it, in a certain sense, pretty much as it was before. Still more may we say that, when we have been obliged, for the purposes in hand, to remove a set of positive content from the continuum, whether this is a fixed or a variable set, we may have very seriously affected the character of the remaining portion of the continuum from every point of view except that of content. This is illustrated by the fact pointed out by Lusin, that it follows from Egoroff's theorem that *any* function of any of Baire's classes, and therefore any function of monotone class, is continuous with respect to the complementary set when a suitable set of content as small as we please has been removed from the continuum.

* In the present article the functions of the given succession are supposed finite except at a set of content zero.

† See, for instance, W H. Young, "On the Construction of a Pointwise Discontinuous Function, all of whose Continuities are Infinities and which has a Finite Generalised Integral," § 2 (1908), *Quarterly Journal*, Vol. XXXIX, p. 218.

ON THE SUMMABILITY OF FOURIER'S SERIES

By G. H. HARDY.

[Received January 27th, 1913.—Read February 13th, 1913.]

1. The investigations contained in this paper were suggested to me by a very interesting theorem established by Prof. W. H. Young,* viz., that *if*

$$\tfrac{1}{2}a_0+\Sigma(a_n\cos nx+b_n\sin nx)\equiv\tfrac{1}{2}a_0+\Sigma A_n$$

is the Fourier's series of a summable function $f(x)$, then the series

$$\Sigma n^{-\delta}A_n,$$

where δ is any positive number, converges almost everywhere.†

It has been proved by Dr. Marcel Riesz that, *if a series ΣA_n is summable $(C\delta)$, then $\Sigma n^{-\delta}A_n$ is convergent.* The proof of this theorem (in a considerably more general form) will appear in the *Cambridge Tract* "The General Theory of Dirichlet's Series" which Dr. Riesz and I are preparing in collaboration. This theorem of Riesz shows that Young's result above referred to could be deduced as a corollary from the theorem which follows.

THEOREM 1. — *The Fourier's series of any summable function is summable $(C\delta)$, for any positive value of δ, almost everywhere.*

That this theorem should be true appears *a priori* as highly probable. Fejér‡ proved that the series is summable $(C1)$ whenever

$$\tfrac{1}{2}\{f(x+0)+f(x-0)\}$$

* *Comptes Rendus*, December 23rd, 1912. For the moment I state a part only of Young's result. If we insert the additional condition that $f(x)$ has its *square* summable, we obtain a theorem equivalent to one given by Prof. Hobson (*Proc. London Math. Soc.*, Ser. 2, Vol. 12, pp. 297–308). In so far as Fourier's series are concerned, Prof. Hobson's theorem is a special case of Prof. Young's; but Prof. Hobson's is much more general in another respect, viz. in that it is applicable to all series of normal functions, and not merely to Fourier's series.

† *I.e.*, with the possible exception of a set of values of x of measure zero.

‡ *Math. Ann.*, Vol. 58, pp. 51–69.

exists, and Lebesgue* that it is summable almost everywhere; and as Riesz† and Chapman‡ have shown that, in Fejér's theorem, the index 1 may be replaced by any positive δ, it is natural to expect Lebesgue's result to be capable of a similar generalisation.

2. I proceed now to the proof of Theorem 1. In proving it I shall use, not the definition of summability $(C\delta)$ employed by Chapman§ and Knopp, but a different definition, due to Riesz,¶ and shown by him** to be equivalent to the former. I shall say that a series ΣA_n is summable $(C\delta)$, to sum s, if

$$\sum_{n<\omega}\left(1-\frac{n}{\omega}\right)^{\delta}A_n\to s,$$

as the continuous variable ω tends to infinity.

Now suppose that $f(x)$ is a summable function with period 2π, and that, in the customary notation of the theory of Fourier's series††

$$f(x)\sim\tfrac{1}{2}a_0+\Sigma(a_n\cos nx+b_n\sin nx)\equiv\tfrac{1}{2}a_0+\Sigma A_n.$$

Further, let

$$C_q(t)=\frac{t^q}{\Gamma(q+1)}\left\{1-\frac{t^2}{(q+1)(q+2)}+\frac{t^4}{(q+1)(q+2)(q+3)(q+4)}-\dots\right\},$$

so that $C_0(t)=\cos t, \quad C_1(t)=\sin t.$

Then Prof. Young‡‡ has established the formula

$$(1)\quad \tfrac{1}{2}a_0+\sum_{n<\omega}\left(1-\frac{n}{\omega}\right)^{\delta}A_n = \frac{\Gamma(1+\delta)}{\pi}\int_0^{\infty}t^{-1-\delta}C_{1+\delta}(t)\left\{f\left(x+\frac{t}{\omega}\right)+f\left(x-\frac{t}{\omega}\right)\right\}dt.$$

Here δ is any positive number. The integral is absolutely convergent at

---

* *Math. Ann.*, Vol. 61, pp. 251–280.

† *Comptes Rendus*, Nov. 22, 1909.

‡ *Proc. London Math. Soc.*, Ser. 2, Vol. 9, pp. 369–409.

§ *L.c. supra.*

‖ *Dissertation*, Berlin, 1907.

¶ *Comptes Rendus*, Nov. 22, 1909.

** *Ibid.*, June 12, 1911.

†† Hobson, *Theory of Functions of a Real Variable*, p. 715.

‡‡ *Quarterly Journal*, Vol. 43, pp. 161–177. See also Young, *Leipziger Berichte*, Vol. 63, pp. 369–387, where methods of summation are considered of which Cesàro's method (as modified by Riesz) is a special case. Young considers only integral values of ω, but his proof is perfectly general.

infinity; for Prof. Young has proved that

$$t^{-q}C_q(t) = O(t^{-q}) \quad or \quad O(t^{-2}),$$

according as $0 \leqslant q \leqslant 2$ or $2 \leqslant q$. It is, in fact, easy, by the use of methods similar to those which have been used by Dr. Barnes and myself in investigations concerning the asymptotic expansions of functions defined by Taylor's series,* to obtain the much more precise equation

$$C_q(t) = \frac{t^{q-2}}{\Gamma(q-1)}\{1+o(1)\}+\cos(t-\tfrac{1}{2}q\pi)+o(1).$$

But this equation is not required for our present purpose.

3. It is a simple deduction from the formula (1) that the Fourier's series is summable $(C\delta)$, and has the sum

$$\tfrac{1}{2}\{f(x+0)+f(x-0)\},$$

wherever this limit exists. This deduction is made by Prof. Young, and constitutes a new and simpler proof of the theorem of Riesz-Chapman referred to in § 1. In order to prove Theorem 1, we have to generalise this result.

For any value of x for which $f(x)$ is defined (and therefore almost everywhere) we have

$$(2)\quad \tfrac{1}{2}a_0+\sum_{n<\omega}\left(1-\frac{n}{\omega}\right)^{\delta}A_n-f(x) = \frac{\Gamma(1+\delta)}{\pi}\int_0^{\infty} t^{-1-\delta}C_{1+\delta}(t)\,\phi\left(\frac{t}{\omega}\right)dt,$$

where $$\phi(\lambda) = f(x+\lambda)+f(x-\lambda)-2f(x).$$

We write $$\Phi(\lambda) = \int_0^{\lambda}|\,\phi(\mu)\,|\,d\mu,$$

and we suppose that, as $\lambda \to 0$,

$$(3)\qquad \Phi(\lambda) = o(\lambda).$$

This is a condition of which Lebesgue has made great use, and which he has shown to be satisfied almost everywhere.†

We shall prove that, when the condition (3) is satisfied,

$$(4)\qquad J = \int_0^{\infty} t^{-1-\delta}C_{1+\delta}(t)\,\phi\left(\frac{t}{\omega}\right)dt \to 0,$$

as $\omega \to \infty$. From this Theorem 1 will follow at once.

* Barnes, *Phil. Trans. Roy. Soc.*, (A), Vol. 206, pp. 249–297; Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 2, pp. 401–431.

† See Lebesgue, *Leçons sur les séries trigonométriques*, pp. 15–16; de la Vallée-Poussin, *Cours d'Analyse Infinitésimale*, 2nd edition, Vol. 2, pp. 115–162, 163.

It is evident that we may, without loss of generality, suppose $\delta < 1$. When $\delta \geqslant 1$ the theorem is a mere corollary of Lebesgue's.

We write $$J = \int_0^1 + \int_1^\omega + \int_\omega^\infty = J_1 + J_2 + J_3.$$

In the first place, since $t^{-1-\delta} C_{1+\delta}(t)$ is bounded for $0 < t < 1$, we have

$$J_1 = O\int_0^1 \left|\phi\left(\frac{t}{\omega}\right)\right| dt = O\left\{\omega\Phi\left(\frac{1}{\omega}\right)\right\} = o(1). \tag{5}$$

Secondly, since $C_{1+\delta}(t)$ is bounded* for $1 \leqslant t$, we have

$$\begin{aligned} J_2 &= O\int_1^\omega t^{-1-\delta}\left|\phi\left(\frac{t}{\omega}\right)\right| dt \\ &= O\left\{\omega^{-\delta}\int_{1\ \omega}^{1} \xi^{-1-\delta}\,|\,\phi(\xi)\,|\,d\xi\right\} \\ &= O\left\{\omega^{-\delta}\Phi(1) - \omega\Phi\left(\frac{1}{\omega}\right) - (1+\delta)\,\omega^{-\delta}\int_{1/\omega}^{1} \xi^{-2-\delta}\Phi(\xi)\,d\xi\right\} \\ &= o(1) + o(1) + O\left\{\omega^{-\delta}\int_{1/\omega}^{1} o(\xi^{-1-\delta})\,d\xi\right\} \\ &= o(1) + o(1) + o(1) = o(1). \end{aligned} \tag{6}$$

Finally,

$$\begin{aligned} J_3 &= O\int_\omega^\infty t^{-1-\delta}\left|\phi\left(\frac{t}{\omega}\right)\right| dt \\ &= O\left\{\omega^{-\delta}\int_1^\infty \xi^{-1-\delta}\,|\,\phi(\xi)\,|\,d\xi\right\} \\ &= O(\omega^{-\delta}) = o(1). \end{aligned} \tag{7}$$

From (5), (6), and (7) it follows that $J = o(1)$; and so the proof is completed.

4. Prof. Young's result is deducible from Theorem 1, whereas the converse is not the case.† But Prof. Young has gone further, and shown that the series

$$\Sigma \frac{A_n}{(\log n)^{2+\delta}}, \quad \Sigma \frac{A_n}{\log n\,(\log\log n)^{2+\delta}}, \quad \ldots$$

are convergent almost everywhere. These results cannot be deduced from

* Young, *Quarterly Journal*, *l.c.*, pp. 163–164.

† The series $\Sigma n^{-1-\delta-ti}$ is convergent (absolutely), but Σn^{-1-ti} is not summable by any of Cesàro's means (Riesz, *Comptes Rendus*, June 21, 1909; Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 8, pp. 301–320).

Theorem 1, for $n^{-\delta}$ is the *least* convergence factor which will always convert a series summable $(C\delta)$ into a convergent series.*

It is naturally suggested that summability $(C\delta)$ is not the *most* that can (almost always) be asserted about a Fourier's series, and that such a series must be almost always summable by methods less powerful than any method $(C\delta)$. For this purpose the methods of "infinitely small" or "functional" order, considered by Mr. Chapman and myself in a recent paper,† appear to be appropriate. I have obtained certain results in this direction: thus, for example, the Fourier's series of any periodic and continuous function is uniformly summable $(C\kappa)$ for certain functional forms of κ. But my results are not as yet sufficiently definite or final to justify publication.

5. I have, however, found, in another direction, a theorem from which it is possible, in an extremely simple manner, to deduce all Prof. Young's results and even go a little further.

THEOREM 2.—*If s_n is the sum of the first n terms A_n‡ of the Fourier's series of a summable function $f(x)$, then*

$$s_n = o(\log n)$$

almost everywhere.

The theorem will evidently be proved if we can show that, for some positive η,

$$\int_0^\eta \phi(t)\frac{\sin nt}{t}\,dt = o(\log n), \tag{8}$$

whenever the condition (3) of § 3 is satisfied. Here n is to be regarded as a continuous variable which tends to infinity.

$$\text{But}\quad \int_\sigma^\eta = \int_0^{1/n} + \int_{1/n}^\eta = O\left\{n\int_0^{1/n}|\phi(t)|\,dt + \int_{1/n}^\eta \frac{|\phi(t)|}{t}\,dt\right\}$$

$$= o(1) + O\left\{\frac{\Phi(\eta)}{\eta} - n\Phi\left(\frac{1}{n}\right) + \int_{1/n}^\eta \frac{\Phi(t)}{t^2}\,dt\right\}$$

$$= o(1) + O(1) + o(1) + O\int_{1/n}^\eta o\left(\frac{1}{t}\right)dt$$

$$= O(1) + o(\log n) = o(\log n);$$

and the theorem is therefore proved.

* This may be shown by means of the series $1^\delta - 2^\delta + 3^\delta - \dots$.

† *Quarterly Journal*, Vol. 42, pp. 181–215.

‡ It is convenient to ignore the term $\frac{1}{2}a_0$.

Now let a_n be a sequence of positive numbers which tend steadily to zero as $n \to \infty$, and satisfy the conditions that $a_n = O(1/\log n)$ and that the series

$$\Sigma (a_n - a_{n+1}) \log n$$

is convergent. Then

$$\sum_1^n a_\nu A_\nu = \sum_1^{n-1} s_\nu \Delta a_\nu + s_n a_n.$$

The last term is of the form

$$o(\log n)\, O(1/\log n) = o(1),$$

and the series $\Sigma s_\nu \Delta a_\nu$ is absolutely convergent. Hence $\Sigma a_\nu A_\nu$ is convergent. The conditions imposed on a_ν are satisfied, if, *e.g.*,

$$a_n = \frac{1}{(\log n)^{1+\delta}}, \quad \frac{1}{\log n\, (\log \log n)^{1+\delta}}, \quad \ldots .$$

Hence we obtain, from Theorem 2, a slightly generalised form of Prof. Young's result, viz.,

The series $\quad \Sigma \dfrac{A_n}{(\log n)^{1+\delta}}, \quad \Sigma \dfrac{A_n}{\log n\, (\log \log n)^{1+\delta}}, \quad \ldots$

converge almost everywhere.

It is, however, possible to assert rather more than this.

THEOREM 3.—*The series* $\quad \Sigma \dfrac{A_n}{\log n}$

*converges almost everywhere.**

For ΣA_n is, by Lebesgue's theorem, summable $(C1)$ almost everywhere, and therefore

$$\Sigma \frac{A_n}{\log n}$$

is so also.†

Now the necessary and sufficient condition that a series ΣB_n, known to

* Prof. Young had conjectured the truth of this theorem, and, by a curious coincidence, a letter of his suggesting it to me reached me just as I had completed my own proof. The theorem was also discovered independently by Dr. Marcel Riesz.

† The first and second differences of $a_n = 1/\log n$ are ultimately positive. The summability of the second series therefore follows from a theorem proved by Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 4, pp. 247–265. Much more general forms of this theorem were given later by Hardy, *Math. Ann.*, Vol. 64, pp. 77–94; and still more general forms, which may be regarded as final, by Bromwich, *Math. Ann.*, Vol. 65, pp. 350–369; Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 6, pp. 255–264, and Vol. 8, pp. 277–294; Bohr, *Bidrag til de Dirichlet'ske Raekkers Theorie*, Copenhagen, 1910. That $\Sigma A_n/(\log n)$ is summable $(C1)$ almost everywhere may also be deduced from the fact that it is a Fourier series, since $\Sigma (\cos nx)/(\log n)$ is one: see Young, *Proc. London Math. Soc.*, Ser. 2, Vol. 12, pp. 41–71.

be summable ($C1$), should be convergent, is that

$$B_1+2B_2+\ldots+nB_n = o(n).*$$

Taking $B_n = A_n/(\log n)$, we find

$$\sum_1^n \nu B_\nu = \sum_1^n \frac{\nu A_\nu}{\log \nu} = \sum_1^{n-1} s_\nu \Delta \frac{\nu}{\log \nu} + \frac{n s_n}{\log n}$$

$$= \sum_1^{n-1} o(\log n)\, O\left(\frac{1}{\log n}\right) + o(n)$$

$$= \sum_1^{n-1} o(1) + o(n) = o(n).$$

Hence ΣB_n, being almost everywhere summable, and also almost everywhere convergent if summable, must be almost everywhere convergent. It should be observed that the series

$$\Sigma \log n\, \Delta \frac{1}{\log n}$$

is divergent, so that our previous reasoning would not apply.

6. The question naturally arises as to whether the equation

$$s_n = o(\log n)$$

of Theorem 2 is the best possible of its kind. I have not proved rigorously that this is so, but it seems to me very probable.

Fejér† has given an exceedingly simple and ingenious method for the construction of trigonometrical series which are the Fourier's series of continuous functions, but which cease to converge for isolated values of x, or for an enumerable, everywhere dense, set of values of x. Thus, for example, he has constructed a pure cosine series

$$\Sigma a_n \cos nx,$$

such that the $(g_1+g_2+\ldots+g_{\nu-1}+\frac{1}{2}g_\nu)$-th partial sum of the series Σa_ν is greater than

$$\frac{1}{\nu^2} \log \tfrac{1}{2} g_\nu.‡$$

Fejér takes $g_\nu = 2 \cdot 2^{\nu^3}$, and in his example s_n is sometimes of order

* See, *e.g.*, Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 8, pp. 301–320, where the analogous condition, which enables us to infer summability (Ck) from summability $(C, k+1)$, is given.

† *Crelle's Journal*, Vol. 138, pp. 22–53; *Annales de l'École Normale*, Vol. 28, pp. 63–103.

‡ *L.c. supra*, p. 85.

$\sqrt[3]{(\log n)}$. It is easy, by choosing still more rapidly increasing forms of g_ν, to replace $\sqrt[3]{(\log n)}$ by $(\log n)^{1-\delta}$, or indeed by any function of the form $o(\log n)$: and the same remarks apply to his examples which cease to converge for values of x which are everywhere dense. Inasmuch as these values form a set of measure zero, Fejér's examples do not prove that the equation of Theorem 2 is the best possible; but, at any rate, they suggest it very forcibly.

Another class of non-convergent Fourier's series of continuous functions was defined by Schwarz and du Bois-Reymond, and has been further discussed by Prof. Hobson.* The oscillation of these series is of order $\log \log n$, and it does not seem possible, at any rate without a much more delicate discussion, to modify the functions so as to obtain oscillation of any notably more pronounced type.

---

* *Theory of Functions of a Real Variable*, pp. 701-707 ; see also *Proc. London Math. Soc.*, Ser. 2, Vol. 3, pp. 48–61.

ON A GRAPHICAL DEMONSTRATION OF THE FUNDAMENTAL PROPERTIES OF QUADRATIC RESIDUES

By P. J. HEAWOOD.

[Received January 28th, 1913.—Read February 13th, 1913.]

SOME time ago I attempted to show * how the chief properties of the convergents to a continued fraction $= b/a$ can be exhibited by taking a circle of circumference $= a$, and measuring off continually round it arcs $= b$. A like method seems peculiarly adapted for dealing with certain questions in the theory of numbers. If a circle be taken whose circumference (on any scale) $= p$, a given prime number; and if, starting from a fixed point X (Fig. 1) we measure round the circumference a length representing any other integer N (greater or less than p) and terminating at A; the (simple) arc XA, measured in the same direction, will represent the least positive residue of N, mod p (which may be N itself); and the *shortest* arc XA will represent the absolutely least residue, positive or negative according to the direction of XA. In particular, if on a circle of circumference p we measure off continually arcs XA, AB, BC, ... each equal to another odd prime q, the resulting figure will exhibit the least residues (mod p) of q, $2q$, $3q$, ... After p such arcs have been measured off we shall get back, in q circuits, to the starting point X; and with the insertion of the $(p-1)$-th point, at a distance q behind X, the circumference will be completely divided into p equal parts.

If we stop at the $\frac{1}{2}(p-1)$-th point K (Fig. 1), we shall be at a distance $\frac{1}{2}q$ behind Y, the point diametrically opposite to X (the complete number of circuits q being odd), and the remaining $\frac{1}{2}(p-1)$ points from this stage onwards will be *the reflections in XY of those already inserted;* since the m-th point from the end, of the complete series, will be as far behind X as the m-th point from the beginning is in front of it. At this stage, then, the "blanks" in the one semicircle will exactly match the "insertions" in the other, and the absolutely least residues will just include *positively or negatively* all the integers from 1 to $\frac{1}{2}(p-1)$ inclusive.

* *Mathematical Gazette*, January 1903.

This shows that the product $q \times 2q \times 3q \ldots \frac{1}{2}(p-1)q$, *i.e.*,

$$[\tfrac{1}{2}(p-1)]!\, q^{\frac{1}{2}(p-1)} \equiv (-1)^r [\tfrac{1}{2}(p-1)]! \pmod{p},$$

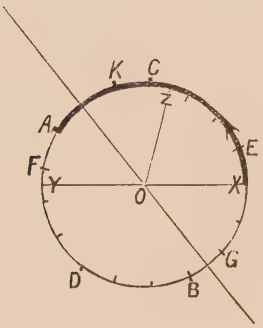


FIG. 1.

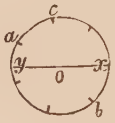


FIG. 5.

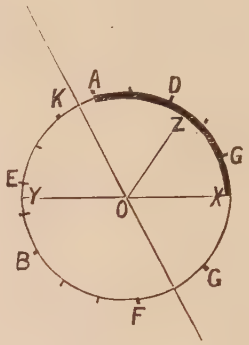


FIG. 2.

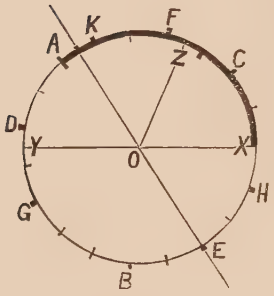


FIG. 3.

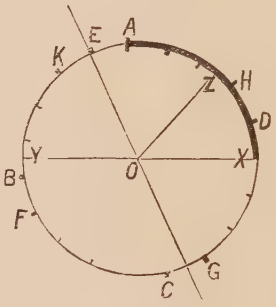


FIG. 4.

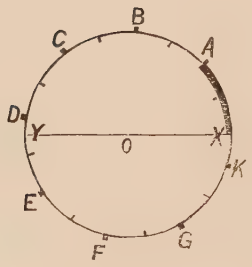


FIG. 6.

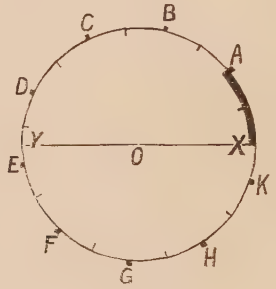


FIG. 7.

where r is the number of points on the "lower" semicircle; or, p being prime,

$$q^{\frac{1}{2}(p-1)} \equiv (-1)^r \pmod{p}.$$

Therefore *q is or is not a quadratic residue of p, according as the number of points on the "lower" semi-circumference at the stage in question is*

even or odd; or, with the usual notation,

$$(q/p) = (-1)^r.$$

Fig. 1 is drawn to exemplify the case where $p > q$ ($p = 17$, $q = 7$); but the result holds equally when this is reversed, as shown in Fig. 5. The former figure indicates that 7 is *not* a quadratic residue of 17, there being an odd number of points (3) on the lower semi-circumference; the latter indicates in like manner that 17 is not a quadratic residue of 7, there being just one point of the three (a, b, c) which lies below xy.

A little consideration, however, will show that for two such related figures, indicating the values of (q/p) and (p/q) respectively, the division of the complete circumference $= q$, in Fig. 5, will exactly correspond to that of the first arc $XA = q$, in Fig. 1, *reckoning backwards from A*. In fact, if the r-th point inserted in Fig. 5 comes at a distance δ in advance of x (so that $rp - sq = \delta$), the r-th *circuit* in Fig. 1 will be completed by the insertion of a point at a distance δ behind X, which will be followed by a point in XA at a distance δ behind A (and so successively in the other q-divisions). Since we stop at the $\frac{1}{2}(q-1)$-th point in Fig. 5, and after q half-circuits in Fig. 1, the divisions of AX and of the smaller circle, at the respective stages, will correspond. (In the case shown we have $xa = -AC$, $xb = -AE$, $xc = -AK$.) We may therefore dispense with Fig. 5 altogether, as it is virtually involved in Fig. 1. The residuary character of p with respect to q will depend on whether the number of points in the *first* half-division XZ is even or odd (this answering to the *second* semi-circumference in Fig. 5), just as the character of q with respect to p depends on the number of points in YX.

It follows that (q/p) *and* (p/q) *will be the same or different, according as the number of points in the whole arc* YXZ *is even or odd.* But further, the diameter bisecting the arc AK between the first and last points inserted, *which will be an axis of symmetry of all the inserted points,* will also bisect the arc YZ, since

$$KY = \tfrac{1}{2}q = ZA.$$

We have, in fact, $\frac{1}{2}(XA + XK) = \frac{1}{2}(XY + XZ) = \frac{1}{4}(p+q)$.

The points in YXZ on one side of the axis of symmetry will therefore answer to points within the same arc on the other side of the axis; and the number on the whole arc will therefore be even, except only when one of the points inserted, and that necessarily the middle point in order of insertion, lies at the intersection of the axis with YXZ. Since the number of *points inserted* $= \frac{1}{2}(p-1)$, the condition that there should be such a middle point is that $\frac{1}{2}(p-1)$ should be odd. Since the number of *com-*

plete circuits from X to $K = \frac{1}{2}(q-1)$, the condition that when midway from A to K we shall be on the lower part of the circumference, is that $\frac{1}{2}(q-1)$ should be odd. It is only when both are fulfilled that there will be an odd number of points in YXZ. We have thus proved the reciprocal property of quadratic residues, that it is only when p and q are both of the form $4m+3$, that (p/q) and (q/p) will be different. In other cases *each* is a quadratic residue, or *neither* is a quadratic residue of the other; or in the usual notation

$$(p/q)(q/p) = (-1)^{\frac{1}{4}(p-1)(q-1)}.$$

Figs. 1, 2, 3, 4, drawn for the cases $p = 17$, $q = 7$; $p = 17$, $q = 5$; $p = 19$, $q = 7$; $p = 19$, $q = 5$ illustrate the different possibilities. In Fig. 3 only does an inserted point lie on the axis within the arc YXZ. In Figs. 1 and 2* there is no middle point; in Fig. 4 there is a middle point, but it lies in ZY. Thus, in Fig. 3 alone have we the divergent results $(p/q) = -1$, $(q/p) = +1$. In Figs. 1 and 2, $(p/q) = (q/p) = -1$. In Fig. 4, $(p/q) = (q/p) = +1$.

A similar method leads very readily to the rule for the residuary character of 2 with respect to p. Taking on a circle, of circumference $= p$, $\frac{1}{2}(p-1)$ points at intervals

$$XA = AB = BC = \ldots = 2,$$

we shall conclude with a point K at a distance 1 behind X (Figs. 6, 7); and 2 will be a quadratic residue if, and only if, the number of points inserted in the *second* semi-circumference is even, or if YK contains 2 an odd number of times; *i.e.*, if $I[\frac{1}{4}(p-2)]$ is odd. This will be the case when p is of the form $8m \pm 1$, but not when it is of the form $8m \pm 3$. Hence the rule

$$(2/p) = (-1)^{\frac{1}{8}(p^2-1)}.$$

Such methods take a little time to describe, but the results involve hardly any calculation. The figures, when their principle is understood, make the theorems almost "intuitively" manifest, in a way which is rather remarkable when we consider the complexity of some of the analytical proofs.†

---

* In Fig. 2 the intersection of the axis with YXZ is an ultimate *division-point*, of the total number $p-1$, but it has not been filled up at the stage in question.

† Since writing the above it has been pointed out to me that methods involving the consideration of equidistant points on a circle, taken in certain orders, were recommended by Poinsot, in his *Réflexions sur les Principes Fondamentaux de la Théorie des Nombres* (Paris, 1845), as throwing light on the essential properties of numbers. He applies the method to prove the generalised forms of Fermat's Theorem and Wilson's Theorem, and to solve the indeterminate equation $Lx - My = N$; and remarks that "la théorie de l'ordre est la source naturelle des propriétés des nombres, et des principes fondamentaux de l'analyse: vérités qui recevront sans cesse un plus grand jour."

SOME NON-PRIMARY PERPETUANT SYZYGIES OF THE SECOND KIND

By H. T. H. PIAGGIO.

[Received February 11th, 1913.—Read February 13th, 1913.]

CONTENTS.

Introduction.
I. Notation and Preliminary Details.
II. Syzygies of the Second Kind of Degrees less than Six.
III. Generating Functions for Syzygies of the Second Kind of Degree Six.
IV. Formation of Primary Syzygies.
V. The Three Types of Non-Primary Syzygies.
VI. Proof that all the Linearly Independent Syzygies of the Second Kind of Degree Six have been obtained.

IN a previous paper* syzygies of the n-th "kind" were shewn to exist among perpetuant types for all degrees greater than $(n+1)$. It was also shewn that for degrees less than $(2n+2)$, these syzygies were all formed by one simple method, namely, symbolical multiplication of the single syzygy of the $(n+2)$-th degree by perpetuant products. We called these syzygies primary.

It was proved that syzygies not so formed (*non-primary*) certainly do arise for degree $(2n+2)$. For the first "kind" $(n = 1)$, we have

$$2n+2 = 4;$$

thus non-primary syzygies first occur for degree 4. These are the well-known Stroh syzygies. For the second kind, $2n+2 = 6$, so non-primary syzygies first occur for degree 6. The present paper deals with these.

---

* "Perpetuant Syzygies of the n-th Kind" (*Proc. London Math. Soc.*, Ser. 2, Vol. 8), which see for definitions and a general account of the subject.

It may be pointed out that one object of this paper and of similar work is to find some general method of forming syzygies of all "kinds" that will apply to all possible cases. In the previous paper it was shewn that for the n-th kind we can obtain by "symbolical multiplication" all those of the first n degrees, as well as many of higher degrees. A paper by Young and Wood ("Perpetuant Syzygies," *Proc. London Math. Soc.*, Ser. 2, Vol. 2) may be summed up as establishing that for the first kind the syzygies of all degrees up to eight are derived by symbolical multiplication from either the Jacobian identity or the Stroh syzygies.

It is found that the non-primary syzygies of the second kind of degree 6 consist of three different types. One of these is formed according to the general method explained in the last section of the previous paper. The second* is a generalisation of the Stroh syzygies. The third is rather more complex.

It is proved finally that these three types of non-primary syzygies, together with certain specified primary ones, are such that all others (of the second kind of degree 6) can be expressed as linear functions of them.

I. *Notation and Preliminary Details.*

Notation.—The Stroh syzygy

$$\sum_0^w \binom{w}{r} [(a_1a_2)^{w-r}(a_4a_3)^r - (a_1a_3)^{w-r}(a_4a_2)^r] \equiv 0,$$

we shall denote by $\{a_1a_2a_4a_3\}_w$ or even by $\{1243\}_w$, writing only the suffixes.

The result of multiplying symbolically $\{a_1a_2a_4a_3\}_w$ by the perpetuant product $(a_4a_5)^u(a_3a_6)^v$ will be denoted by

$$\{(a_1a_2a_4a_3)_w(a_4a_5)_u(a_3a_6)_v\},$$

or by

$$\{(1243)_w(45)_u(36)_v\}.$$

The result of multiplying the Jacobian identity

$$(a_1a_2a_3) \equiv A_1(a_2a_3) - A_2(a_1a_3) + A_3(a_1a_2) \equiv 0,$$

by the product $(a_1a_4)^u(a_2a_5)^v$ will be written

$$J[(a_1a_4)^u(a_2a_5)^v(a_1a_2a_3)],$$

or

$$J[(14)^u(25)^v(123)],$$

but, for the sake of brevity, $J[(14)^{w-1}(123)]$ will be replaced by $J_w(1423)$.

If $w \geqslant 3$, from four given symbols six essentially distinct J_w's can be formed, since

$$J_w(abcd) = -J_w(abdc) = (-1)^{w-1}J_w(bacd) = (-1)^w J_w(badc).$$

We shall define $J_2(abcd)$ as the form into which $(ab)(cd)-(ab)(ad)+(ab)(ac)$ degenerates, *i.e.*, as

$$(ab)(cd)-\tfrac{1}{2}(ad)^2+\tfrac{1}{2}(db)^2+\tfrac{1}{2}(ac)^2-\tfrac{1}{2}(cb)^2=\tfrac{1}{4}[\{abcd\}_2+\{acdb\}_2-\{adbc\}_2].$$

* This second type was made known to me by Dr. A. Young.

From four given symbols only three J_2's can be formed, as in addition to the relations given above

$$J_2(abcd) = J_2(cdab).$$

For reasons connected with the "resolution" of compound syzygies, it is convenient to use J_2's instead of Stroh syzygies of weight 2. The three J_2's formed from four given symbols are expressible linearly in terms of the three Stroh syzygies, and conversely.

Arrangement Scheme for Compound Syzygies of Degree 6.

(1) The order of precedence is

(i) C_3S_3, (ii) C_2S_4, (iii) C_1S_5, (iv) $C_1C_2S_3$, (v) $C_1^2S_4$, (vi) $C_1^3S_3$.

(S_p is used to denote a syzygy of degree p).

(2) If we have two compounds C_2S_4 and C_2S_4', the first precedes the second if (α) S_4 is a Stroh syzygy of weight $\geqslant 4$ and S_4' is a J_w; or if (β) S_4 is a J_w ($w \geqslant 3$) and S_4' is a J_2.

(3) (α) $(a_p a_q)^\lambda S_4$ precedes $(a_r a_s)^\mu S_4'$, where the above is not sufficient to establish precedence, if the first of the differences $p-r$, $q-s$ that does not vanish is *negative*. Here p is supposed to be less than q, and r less than s.

(β) The order of the compounds $(a_p a_q)^\lambda S_4$, where S_4 is a Stroh syzygy, is given later (in Section VI). It should be added that $(a_p a_q)^\lambda S_4$ precedes $(a_p a_q)^\mu S_4'$ if both S_4 and S_4' are Stroh syzygies and $\lambda < \mu$.

The above is merely the application to syzygies of the first kind of degree 6 of the more detailed scheme for those of the n-th kind given in the previous paper, with some variations in the arrangement of Stroh syzygies. As Stroh syzygies were not used in the last paper (except incidentally in the last section), these variations do not affect the results given there.

Use of Uncanonical Forms.

Theoretically the irreducible syzygies of the first kind whose compounds appear in the syzygies of the second kind should all be *canonical*, that is, members of a more or less arbitrarily chosen set in terms of which all others can be expressed linearly. But this would destroy the symmetry

of the work. We therefore keep the uncanonical forms, with the understanding that they are to be regarded *merely as a short way of writing their canonical equivalents.* The scheme of arrangement has been chosen in such a way that this does not affect the reasoning in the following work. The compound syzygies resolved are always such as to have their irreducible parts canonical. The apparently eccentric arrangement of Stroh syzygies in Section VI is due to the necessity of satisfying this condition.

Similar remarks apply to the use of such reducible forms as $(a_1a_4)^3(a_1a_3)$, which is to be understood as being merely an abbreviation for the equivalent sum of irreducible perpetuants.*

II. *Syzygies of the Second Kind of Degrees less than* 6.

These are needed to obtain the compound syzygies of the second kind of degree 6.

Putting $n = 2$ in the results of the previous paper, we see that :—

(i) None exist for degrees less than 4.

(ii) For degree 4 there is only one, namely,

$$(a_1a_2a_3a_4) \equiv A_1(a_2a_3a_4) - A_2(a_1a_3a_4) + A_3(a_1a_2a_4) - A_4(a_1a_2a_3).$$

(iii) For degree 5 they are all given by the generating function

$$4x + 10x^2/(1-x).$$

The irreducible ones are all of the form

$$\begin{aligned} J^{(2)}\left[(a_pa_t)^\lambda (a_pa_qa_ra_s)\right] \equiv (a_pa_t)^\lambda (a_qa_ra_s) &- A_qJ\left[(a_pa_t)^\lambda (a_pa_ra_s)\right] \\ &+ A_rJ\left[(a_pa_t)^\lambda (a_pa_qa_s)\right] \\ &- A_sJ\left[(a_pa_t)^\lambda (a_pa_qa_r)\right], \end{aligned}$$

where p, q, r, s, t are the five suffixes 1, 2, 3, 4, 5 in any order. These correspond to the part $10x^2/(1-x)$. The part $4x$ corresponds to the *compound* ones of form $A_p(a_qa_ra_sa_t)$, of which 4 (out of 5) are linearly independent.

---

* We are using Grace's form for a perpetuant, viz.,

$$(a_1a_2)^\lambda (a_1a_3)^\mu \ldots (a_1a_n)^\omega,$$

where $\lambda \geqslant 2^{p-2}$, $\mu \geqslant 2^{p-3}$, ..., $\omega \geqslant 1$.

III. *Generating Functions for Linearly Independent Syzygies of the Second Kind of Degree* 6.

The method of procedure is that of the previous paper. If from the *total* number of compound syzygies of a given weight and degree we subtract the number of those that are linearly independent, we are left with the number of linearly independent relations. But a linear relation between compound syzygies is a syzygy of the second kind. Hence we get the number of linearly independent syzygies of the second kind of that weight and degree.

Now each compound syzygy resolves* a product of at least *three* perpetuant factors, so we shall start by writing down all the possible forms of these products of total degree 6. They are

$$\text{(i) } C_2^3, \quad \text{(ii) } C_1C_2C_3, \quad \text{(iii) } C_1^2C_4, \quad \text{(iv) } C_1^2C_2^2, \quad \text{(v) } C_1^3C_3,$$

$$\text{(vi) } C_1^4C_2, \quad \text{(vii) } C_1^6,$$

where C_p denotes a perpetuant of degree p.

The generating functions for the syzygies of the first kind resolving these may be deduced from Young and Wood's paper. They give there the generating functions for *irresolvable* products of various forms. Subtracting these from the corresponding functions† for *all* products of these forms, we get the functions for the syzygies resolving them.

We shall use $[C_lC_m]$ to denote the syzygy resolving the product C_lC_m and $C_k[C_lC_m]$ for the compound syzygy formed by multiplying the previous one by C_k. This compound syzygy will clearly resolve a product $C_kC_lC_m$.

(i) C_2^3. The generating function for all syzygies of form $C_2[C_2^2]$

$$= \binom{6}{2} \times \text{function for } C_2 \times \text{function for } [C_2^2]$$

$$= \binom{6}{2} x/(1-x) \times \{\text{function for } \textit{all} \text{ products of form } C_2^2 \text{ less function for } \textit{irresolvable} \text{ products of this form}\}$$

$$= \binom{6}{2} \frac{x}{1-x} \left\{ \frac{3x^2}{(1-x)^2} - \frac{x^4-x^7}{(1-x)^3} \right\} = \frac{15x}{1-x} \cdot \frac{3x^2+3x^3+2x^4+x^5}{1-x}.$$

---

* For the definition of "resolution," see Section I of the previous paper.

† Which can be easily written down from the principles of permutations and Grace's Perpetuant Type Theorem.

The function for *all* products of form C_2^3 is $15x^3/(1-x)^3$, and that for *irresolvable* products of this form is $(x^6-x^9-x^{11}+x^{14})/(1-x)^5$; therefore that for linearly independent syzygies resolving products of form C_2^3 is the difference of these two expressions. Now from Young and Wood's paper, we can see that all these syzygies are compound. Hence the function for linearly independent syzygies of the second kind resolving such compound syzygies is

$$\frac{15x}{1-x}\,\frac{3x^2+3x^3+2x^4+x^5}{1-x}-\frac{15x^3}{(1-x)^3}+\frac{x^6-x^9-x^{11}+x^{14}}{(1-x)^5}$$

$$=30x^3+90x^4+165x^5+241x^6+305x^7+360x^8+409x^9+\frac{455x^{10}}{1-x}+\frac{45x^{11}}{(1-x)^2}.$$

(ii) Similarly, starting with the product $C_1C_2C_3$, we can obtain the generating function $20x^4/(1-x)^2$ for the associated syzygies of the second kind.

(iii) $C_1^2C_4$. All products of this form are irresolvable, so no syzygies arise. Similarly for (v) $C_1^3C_3$ and (vii) C_1^6.

(iv) $C_1^2C_2^2$. The function is $60x^2/(1-x)$.

(vi) $C_1^4C_2$. The function is $10x$.

Adding the results for these seven cases, we see that the generating function for all the linearly independent syzygies of the second kind of degree 6 is

$$10x+60x^2/(1-x)+20x^4/(1-x)^2+30x^3+90x^4+165x^5$$
$$+241x^6+305x^7+360x^8+409x^9+455x^{10}/(1-x)+45x^{11}/(1-x)^2.$$

IV. *Formation of Primary Syzygies.*

Primary syzygies are those formed by symbolical multiplication of the single syzygy of the $(n+2)$-th degree by perpetuant products.

(A) Multiply symbolically the simplest syzygy of the second kind, *i.e.*,

$$(1234)\equiv A_1(234)-A_2(134)+A_3(124)-A_4(123) \text{ by } (15)^\lambda(16)^\mu,$$

where $$\lambda\geqslant 2,\quad \mu\geqslant 1.$$

We get a new irreducible syzygy of the second kind, of degree 6:—

$$(15)^\lambda(16)^\mu(234)-A_2J[(15)^\lambda(16)^\mu(134)]+A_3J[(15)^\lambda(16)^\mu(124)]$$
$$-A_4J[(15)^\lambda(16)^\mu(123)],$$

which we call $J^{(2)}[(15)^{\lambda}(16)^{\mu}(1234)]$.

It resolves $(15)^{\lambda}(16)^{\mu}(234)$, of form $C_3 S_3$, in terms of compound syzygies of form $C_1 S_5$.

(B) (α) Multiply symbolically (1234) by $(16)^{\lambda}(25)^{\mu}$, where λ and μ are both equal or greater than 2. We get a new irreducible syzygy of the second kind, of degree 6 :—

$$(16)^{\lambda} J_{\mu+1}(2534) - (25)^{\mu} J_{\lambda+1}(1634) + A_3 J[(16)^{\lambda}(25)^{\mu}(124)] - A_4 J[(16)^{\lambda}(25)^{\mu}(123)],$$

which we call $J^{(2)}[(16)^{\lambda}(25)^{\mu}(1234)]$.

It resolves $(16)^{\lambda} J_{\mu+1}(2534)$, of form $C_2 S_4$, in terms of $(25)^{\mu} J_{\lambda+1}(1634)$ and compound syzygies of form $C_1 S_5$.

(β) Let $\mu = 1$ and $\lambda > 1$. In accordance with our arrangement scheme we consider that $J^{(2)}[(16)^{\lambda}(25)(1234)]$ resolves $(25) J_{\lambda+1}(1634)$ and not $(16)^{\lambda} J_2(2534)$.

(γ) But, if $\lambda = 1$, as well as $\mu = 1$, we have two J_2's, and so $J^{(2)}[(16)(25)(1234)]$ resolves $(16) J_2(2534)$ in terms of $(25) J_2(1634)$ and other compound syzygies.

V. *The Three Types of Non-Primary Syzygies.*

(a) *First type.**

If A and B are two syzygies of the first kind, involving one symbol in common, and if a and b are the corresponding sums of perpetuant products, then, denoting by aB and bA the results of multiplying symbolically B by a and A by b respectively, we see that $bA - aB$ must be a syzygy of the second kind, for when expanded as a sum of perpetuant products it is $ba - ab$, which vanishes identically.

For example, take $A = \{1243\}_{w-1}$ and $B = (456)$, with the symbol 4 in common. Then

$$a \equiv \sum_{0}^{w-1} \binom{w-1}{r} [(12)^{w-r-1}(43)^{r} - (13)^{w-r-1}(42)^{r}],$$

* This is a case of the general method of Section XII of the previous paper.

and $$b \equiv A_4(56) - A_5(46) + A_6(45),$$

and we get the syzygy of the second kind

$$(56)\{1243\}_{w-1} - A_5\{(1243)_{w-1}(46)\} + A_6\{(1243)_{w-1}(45)\} - A_1 A_2 J_w(4356)$$
$$+ A_1 A_3 J_w(4256) - A_3(12)^{w-1}(456) + A_2(13)^{w-1}(456)$$
$$- \sum_1^{w-2} \binom{w-1}{r} [(12)^{w-r-1} J_{r+1}(4356) - (13)^{w-r-1} J_{r+1}(4256)],$$

which we shall denote by

$$\begin{pmatrix} (1243)_{w-1} \\ (456) \end{pmatrix}.$$

It resolves $(56)\{1243\}_{w-1}$ of form $C_2 S_4$.

(*b*) *Second type.**

Consider $$Y_w(12, 14, 36, 56)$$

$$\equiv \sum_0^{w-1} \binom{w}{r} [(12)^r \{3654\}_{w-r} + (14)^r \{3256\}_{w-r} + (36)^r \{1452\}_{w-r}$$
$$+ (56)^r \{1234\}_{w-r}].$$

Replace the syzygies of the first kind in this expression by their equivalent sums of perpetuant products. We get a set of terms of form

$$\Sigma C_2^3 + \Sigma C_1^2 C_2^2 + \Sigma C_1^4 C_2.$$

The coefficient of

$$(12)^\lambda (36)^\mu (54)^{w-\lambda-\mu} = \binom{w}{\lambda}\binom{w-\lambda}{\mu} - \binom{w}{\mu}\binom{w-\mu}{\lambda} = 0.$$

Similarly all the other terms of form C_2^3 cancel.

The coefficient of

$$(36)^r (54)^{w-r} = \binom{w}{0}\binom{w}{r} - \binom{w}{r}\binom{w}{0} = 0,$$

and the coefficient of $$(36)^w = 1 - 1 = 0.$$

Similarly all the other terms of form $C_1^2 C_2^2$ or $C_1^4 C_2$ cancel.

Thus the expanded form of Y_w (12, 14, 36, 56) is identically zero, so we have a syzygy of the second kind. It resolves a syzygy of the first kind of form $C_2 S_4$.

* Due to Dr. A. Young.

(*c*) *Third type.*

Let

$$V_w(123, 456) \equiv \sum_0^{w-1} \left\{ 2\binom{w-1}{r-1} - \binom{w-1}{r} \right\}$$

$$\times [\ (14)^r \{2536\}_{w-r} + (15)^r \{2634\}_{w-r} + (16)^r \{2435\}_{w-r}$$

$$+ (24)^r \{3516\}_{w-r} + (25)^r \{3614\}_{w-r} + (26)^r \{3415\}_{w-r}$$

$$+ (34)^r \{1526\}_{w-r} + (35)^r \{1624\}_{w-r} + (36)^r \{1425\}_{w-r}],$$

understanding $\binom{w-1}{-1}$ as 0 and $\binom{w-1}{0}$ as 1.

The coefficient of $(14)^\lambda (25)^\mu (36)^{w-\lambda-\mu}$

$$= 2\left[\binom{w-1}{\lambda-1}\binom{w-\lambda}{\mu} + \binom{w-1}{\mu-1}\binom{w-\mu}{\lambda} + \binom{w-1}{w-\lambda-\mu-1}\binom{\lambda+\mu}{\lambda}\right]$$

$$-\left[\binom{w-1}{\lambda}\binom{w-\lambda}{\mu} + \binom{w-1}{\mu}\binom{w-\mu}{\lambda} + \binom{w-1}{w-\lambda-\mu}\binom{\lambda+\mu}{\lambda}\right].$$

In this expression λ, μ, and $w-\lambda-\mu$ may have any values from 1 to $w-2$. It can be easily verified that when the binomial coefficients are expressed in terms of factorials, the result reduces to zero. Similarly the coefficients of all other terms of form C_2^3 are zero.

The coefficient of

$$(14)^r (25)^{w-r} = \left[2\binom{w-1}{r-1} - \binom{w-1}{r}\right] + \left[2\binom{w-1}{w-r-1} - \binom{w-1}{w-r}\right]$$

$$+ \binom{w}{r}\left[2\binom{w-1}{-1} - \binom{w-1}{0}\right],$$

which also reduces to zero for all values of r from 1 to $w-1$. Similarly for all other terms of form $C_1^2 C_2^2$.

Finally, the coefficient of

$$(14)^w = \left[2\binom{w-1}{-1} - \binom{w-1}{0}\right][1-1-1+1] = 0,$$

and similarly for all other terms of form $C_1^4 C_2$.

Hence in the expanded form all terms vanish, so $V_w(123, 456)$ is a syzygy of the second kind.

VI. *Resolution of Compound Syzygies by means of Syzygies of the Second Kind (being the proof that all the linearly independent syzygies of the second kind of degree* 6 *are of the types obtained above).*

The only possible forms of compound syzygies of degree 6 are:—

(i) C_3S_3, (ii) C_2S_4, (iii) C_1S_5, (iv) $C_1C_2S_3$, (v) $C_1^2S_4$, (vi) $C_1^3S_3$.

(i) C_3S_3. Every such compound is resolvable by a syzygy of the second kind of the form

$$J^{(2)}[(ae)^\lambda (af)^\mu (abcd)],$$

where $$\lambda \geqslant 2, \quad \mu \geqslant 1.$$

The corresponding generating function is

$$\binom{6}{3} x^4/(1-x)^2 = 20x^4/(1-x)^2.$$

(ii) C_2S_4. S_4 can be a Stroh syzygy (of weight $\geqslant 4$), a J_w of weight $\geqslant 3$, or a J_2.

We consider it a resolution to express a compound of a Stroh syzygy in terms of J's or a compound of a $J_w (w \geqslant 3)$ in terms of a J_2.

For the resolution of compounds of Stroh syzygies we use such syzygies of the second kind as

$$\left\{ \begin{matrix} (1243)_{w-1} \\ (456) \end{matrix} \right\}, \quad Y_w(12, 14, 36, 56), \quad \text{and} \quad V_w(123, 456).$$

For compounds of the J's we use syzygies of the second kind of form

$$J^{(2)}[(af)^\lambda (be)^\mu (abcd)].$$

Lemma.—To prove that

$$\left\{ \begin{matrix} (5463)_{w-1} \\ (612) \end{matrix} \right\}, \quad Y_w(12, 13, 54, 64), \quad Y_w(21, 23, 54, 64), \quad V_w(134, 256),$$

and $$V_w(156, 234),$$

together resolve the five compounds $(12)^r \{5463\}_{w-r}$, where r has the values 1, 2, 3, 4, 5. This holds good for $w \geqslant 10$.

Denote $(12)^r \{5463\}_{w-r}$ by $F(r)$.

$F(1)$ is always resolved by a syzygy of the second kind of type

$$\left\{ \begin{matrix} (5463)_{w-1} \\ (612) \end{matrix} \right\}.$$

Now

$Y_w(12, 13, 54, 64)$, $= A$ say, contains $\Sigma \binom{w}{r} F(r)$,

$Y_w(21, 23, 54, 64)$, $= B$ say, contains $\Sigma \binom{w}{r} (-1)^r F(r)$,

$V_w(134, 256)$, $= C$ say, contains* $-\Sigma\left[2\binom{w}{r} - 3\binom{w-1}{r}\right](-1)^{w-r} F(r)$,

$V_w(156, 234)$, $= D$ say, contains $-\Sigma\left[2\binom{w}{r} - 3\binom{w-1}{r}\right] F(r)$;

therefore $\frac{1}{2}(A+B)$ contains $\binom{w}{2} F(2) + \binom{w}{4} F(4) + \binom{w}{6} F(6) + \ldots$,

and $\frac{1}{2}(A-B)$ contains $\binom{w}{3} F(3) + \binom{w}{5} F(5) + \ldots$.

Also $\frac{1}{3}\{2B + (-1)^w C\}$, $= C'$ say, contains

$$\binom{w-1}{2} F(2) - \binom{w-1}{3} F(3) + \binom{w-1}{4} F(4) - \ldots,$$

and $\frac{1}{3}\{2A + D\}$, $= D'$ say, contains

$$\binom{w-1}{2} F(2) + \binom{w-1}{3} F(3) + \binom{w-1}{4} F(4) + \ldots.$$

Therefore $\frac{1}{2}(C'+D')$ contains $\binom{w-1}{2} F(2) + \binom{w-1}{4} F(4) + \ldots$,

and $\frac{1}{2}(D'-C')$ contains $\binom{w-1}{3} F(3) + \binom{w-1}{5} F(5) + \ldots$;

therefore $\frac{1}{2}\binom{w-1}{2}(A+B) - \frac{1}{2}\binom{w}{2}(C'+D')$

contains $\left[\binom{w}{4}\binom{w-1}{2} - \binom{w}{2}\binom{w-1}{4}\right] F(4)$,

and $\frac{1}{2}\binom{w-1}{3}(A-B) - \frac{1}{2}\binom{w}{3}(D'-C')$

contains $\left[\binom{w}{5}\binom{w-1}{3} - \binom{w}{3}\binom{w-1}{5}\right] F(5)$.

---

* Since $V_w(134, 256)$ contains $\Sigma\left\{2\binom{w-1}{r-1} - \binom{w-1}{r}\right\}(12)^r \{3546\}_{w-r}$.

The coefficient $= \left\{2\binom{w}{r} - 3\binom{w-1}{r}\right\}$,

and $\{3546\}_{w-r} = (-1)^{w-r}\{5364\}_{w-r} = -(-1)^{w-r}\{5463\}_{w-r}$.

Now $\left[\binom{w}{4}\binom{w-1}{2}-\binom{w}{2}\binom{w-1}{4}\right]$

$$= \frac{1}{24}w(w-1)^2(w-2)(w-3) \neq 0, \text{ if } w \geqslant 4;$$

and $\left[\binom{w}{5}\binom{w-1}{3}-\binom{w}{3}\binom{w-1}{5}\right]$

$$= \frac{2}{3!\,5!}w(w-1)^2(w-2)^2(w-3)(w-4) \neq 0, \text{ if } w \geqslant 5.$$

The 3 Stroh syzygies formed from 4 given symbols are all canonical for weights $\geqslant 5$. For weight 4, only 2 out of the 3 are canonical, and for lower weights none are so.

Therefore, if $w \geqslant 10$, $w-5 \geqslant 5$, and none of the Stroh syzygies in $F(1) \ldots F(5)$ are uncanonical, so we get 5 resolutions.

If $w = 9$, we get all 5 resolutions with 2 of the 3 Stroh syzygies and only 4 with the 3rd. We discard here the resolution of

$$F(5) = (12)^5 \{5463\}_{w-5},$$

as there are only two canonical Stroh syzygies of weight $w-5\,(=4)$.

If $w = 8$, we get, in 2 cases out of 3, 4 resolutions, and only 3 in the 3rd, discarding also $F(4)$, as there are no canonical Stroh syzygies of weight $w-5\,(=3)$, and only 2 of weight $w-4\,(=4)$.

Similarly, if $w = 7$, we get 3 resolutions in 2 cases and 2 in the 3rd; if $w = 6$, we get 2 resolutions in 2 cases and 1 in the 3rd; if $w = 5$, we get 1 resolution in 2 cases and none in the 3rd; if $w < 5$, there are no resolutions at all.

We resume the consideration of the compound syzygies of form C_2S_4.

Case (α).—S_4 a Stroh syzygy. We shall arrange the compound syzygies in the following order:—

(i) $(12)^\lambda S_4$, (ii) $(23)^\lambda S_4$, (iii) $(34)^\lambda S_4$, (iv) $(45)^\lambda S_4$, (v) $(56)^\lambda S_4$,

(vi) $(13)^\lambda S_4$, (vii) $(24)^\lambda S_4$, (viii) $(35)^\lambda S_4$, (ix) $(46)^\lambda S_4$, (x) $(14)^\lambda S_4$,

(xi) $(25)^\lambda S_4$, (xii) $(36)^\lambda S_4$, (xiii) $(15)^\lambda S_4$, (xiv) $(26)^\lambda S_4$, (xv) $(16)^\lambda S_4$.

It is a resolution to express any one of these in terms of forms further on in the list, or in terms of compound syzygies of any of the types

(i) C_2S_4 (where S_4 is a J), (ii) C_1S_5, (iii) $C_1C_2S_3$, (iv) $C_1^2S_4$,

(v) $C_1^3S_3$.

If $\lambda = 1$, we have 45 forms.* All are resolved by a syzygy of the second kind of the type of $\begin{Bmatrix} (1243)_{w-1} \\ (456) \end{Bmatrix}$, which resolves $(56)\{1243\}_{w-1}$.

For values of $\lambda > 1$, we have to examine each form separately. The result is shown in the following table.

				Least value of w for the Stroh syzygies involved to be canonical.
$(12)^\lambda$	$Y_w(12, 13, 54, 64)$	resolves	$(12)^2\{5463\}_{w-2}$	6
	$Y_w(12, 13, 45, 65)$	,,	$(12)^2\{4563\}_{w-2}$	6
	$Y_w(12, 13, 46, 56)$	,,	$(12)^2\{4653\}_{w-2}$	7
	$Y_w(21, 23, 54, 64)$	,,	$(21)^3\{5463\}_{w-3}$	7
	$Y_w(21, 23, 45, 65)$	,,	$(21)^3\{4563\}_{w-3}$	7
	$Y_w(21, 23, 46, 56)$	,,	$(21)^3\{4653\}_{w-3}$	8

Similarly for $(23)^\lambda$, $(34)^\lambda$, and $(45)^\lambda$.

$(56)^\lambda$	$Y_w(56, 52, 31, 41)$	resolves	$(56)^2\{3142\}_{w-2}$	6
	$Y_w(65, 62, 31, 41)$	,,	$(65)^3\{3142\}_{w-3}$	7

We do not get the set of six here. For example,

$$Y_w(56, 52, 13, 43)$$

cannot be used to resolve $(56)^\lambda$ as it involves $(34)^\lambda$, which precedes $(56)^\lambda$.

$(13)^\lambda$ Full set of six as for $(12)^\lambda$.

$(24)^\lambda$	$Y_w(24, 26, 15, 35)$	resolves	$(24)^2\{1536\}_{w-2}$	6
	$Y_w(42, 46, 15, 35)$	,,	$(42)^3\{1536\}_{w-3}$	7
$(35)^\lambda$	$Y_w(53, 52, 41, 61)$	,,	$(53)^2\{4162\}_{w-2}$	6
	$Y_w(53, 52, 14, 64)$	,,	$(53)^2\{1462\}_{w-2}$	6
	$Y_w(53, 52, 16, 46)$	,,	$(53)^2\{1642\}_{w-2}$	7
	$Y_w(53, 51, 26, 46)$	,,	$(53)^3\{2641\}_{w-3}$	7
$(46)^\lambda$	$Y_w(64, 63, 15, 25)$	,,	$(64)^2\{1523\}_{w-2}$	6
$(14)^\lambda$	$Y_w(14, 15, 26, 36)$	,,	$(14)^2\{2635\}_{w-2}$	6

* $45 = 15 \times 3$; 3 Stroh syzygies can be formed from the same 4 symbols, for all weights $\geqslant 5$.

These resolutions are obtained by using the $\left\{\begin{matrix}(1243)_{w-1}\\(456)\end{matrix}\right\}$ set, and the Y_w set.

Using the V_w set in conjunction with the above, we get (from the Lemma)

$(12)^\lambda$	$V_w(134, 256)$	resolves	$(12)^4\{5463\}_{w-4}$	8
	$V_w(135, 246)$	,,	$(12)^4\{4563\}_{w-4}$	8
	$V_w(136, 245)$	,,	$(12)^4\{4653\}_{w-4}$	9
	$V_w(156, 234)$	,,	$(12)^5\{3645\}_{w-5}$	9
	$V_w(146, 235)$	,,	$(12)^5\{3654\}_{w-5}$	9
	$V_w(145, 236)$	,,	$(12)^5\{3564\}_{w-5}$	10
$(23)^\lambda$	$V_w(126, 345)$	,,	$(23)^4\{6415\}_{w-4}$	8
	$V_w(125, 346)$	,,	$(23)^4\{5416\}_{w-4}$	8
	$V_w(124, 356)$	,,	$(23)^4\{4516\}_{w-4}$	9
$(34)^\lambda$	$V_w(123, 456)$	,,	$(34)^4\{1526\}_{w-4}$	8

All these resolutions are possible if $w \geqslant 10$, giving $45+40+10 = 95$ linearly independent syzygies of the second kind of each weight. The corresponding generating function is $95x^{10}/(1-x)$.

If $w = 9$, one resolution of a Stroh syzygy of weight 4 is lost, so the generating function for this weight is $94x^9$.

If $w = 8$, 4 more resolutions are lost, and the function is $90x^8$.

If $w = 7, 10$,, ,, ,, $80x^7$.

If $w = 6, 19$,, ,, ,, $61x^6$.

If $w = 5$, *all* those in the above table are lost. The resolutions by such syzygies of the second kind as $\left\{\begin{matrix}(1243)_{w-1}\\(456)\end{matrix}\right\}$ remain, but there will be only two-thirds of the former number, *i.e.*, two-thirds of $45 = 30$, so the function is $30x^5$.

If $w < 5$, there are no resolutions at all.

Hence the total generating function for Case (α) is

$$30x^5+61x^6+80x^7+90x^8+94x^9+95x^{10}/(1-x).$$

Case (β).—S_4 a J_w, with $w \geqslant 3$.

As shewn in Section IV, $J^{(2)}[(16)^\lambda(25)^\mu(1234)]$ resolves $(16)^\lambda J_{\mu+1}(2534)$ in terms of $(25)^\mu J_{\lambda+1}(1634)$ and syzygies of form $C_1 S_5$.

Thus it follows from the order of arrangement that $(12)^\lambda J_{\mu+1}(abcd)$, where a, b, c, d are 3, 4, 5, 6 in any order, is always resolvable. There are 6 in this set, since from 4 symbols 6 essentially different J's of the same weight can be formed.

Similarly for $(13)^\lambda$, $(14)^\lambda$, $(15)^\lambda$, and $(16)^\lambda$.

But from our arrangement scheme, there are only 3 resolutions for $(23)^\lambda J_{\mu+1}$, and not the full set of 6. Similarly for $(24)^\lambda$, $(25)^\lambda$, and $(26)^\lambda$. For $(34)^\lambda$, $(35)^\lambda$, and $(36)^\lambda$, there is only one resolution apiece.

To sum up, this gives 45 resolutions for each weight $\geqslant 5$ (as $\lambda \geqslant 2$ and $\mu \geqslant 2$), the corresponding generating function being $45x^5/(1-x)^2$.

As shewn also in Section IV, $J^{(2)}[(16)^\lambda(25)(1234)]$ resolves $(25)J_{\lambda+1}(1634)$ in terms of $(16)^\lambda J_2(2534)$ and syzygies of form C_1S_5.

Every form $(ab)J_{\lambda+1}$ is thus resolvable when $\lambda \geqslant 2$.

We therefore get an extra generating function $\binom{6}{2} \times 6x^4/(1-x)$ making a total for Case (β) of

$$90x^4/(1-x)+45x^5/(1-x)^2.$$

Case (γ).—S_4 a J_2.

As shewn in Section IV, $J^{(2)}[(16)(25)(1234)]$ resolves $(16)J_2(2534)$ in terms of $(25)J_2(1634)$ and other compound syzygies. This case differs from (β) in that from 4 symbols only 3 essentially different J_2's can be formed, as $J_2(abcd) = J_2(cdab)$.

We get similar resolutions for every other syzygy of form $(ab)J_2(cdef)$, if (ab) is (12), (13), (14), (15), (16), (23), (24), (25) or (26). This gives $9 \times 3 = 27$ resolutions.

Also $(34)J_2(5612)$ can be resolved by $J^{(2)}[(34)(56)(3512)]$, and similarly for $(35)J_2(4612)$ and $(36)J_2(4512)$. There are no other resolutions.

Hence the generating function for Case (γ) is $30x^3$.

Resolution of the other Forms of Compound Syzygies of Degree 6.

(iii) C_1S_5 *and* (v) $C_1^2S_4$. All such compounds are irresolvable, so there are no corresponding syzygies of the second kind.

(iv) $C_1C_2S_3$. All these are resolvable by a form of the type of $C_1J^{(2)}[(ab)^\lambda(adef)]$, where $\lambda \geqslant 1$.

The corresponding generating function is

$$6 \times \binom{5}{2} \frac{x}{1-x} x = \frac{60x^2}{1-x}.$$

(vi) $C_1^3 S_3$. These are resolvable (if, and only if, an A_1 is among the three C_1's) by such a compound syzygy of the second kind as $C_1^2(pqrs)$, where p is 1. The q, r, and s can be chosen in $\binom{5}{3} = 10$ ways, so the corresponding generating function is $10x$.

Adding together the generating functions for all the forms discussed in this section, we obtain a total of

$$\frac{20x^4}{(1-x)^2}+30x^5+61x^6+80x^7+90x^8+94x^9+\frac{95x^{10}}{1-x}+\frac{90x^4}{1-x}+\frac{45x^5}{(1-x)^2}+30x^3$$
$$+\frac{60x^2}{1-x}+10x.$$

Now this can be easily shown to be equal to

$$10x+\frac{60x^2}{1-x}+\frac{20x^4}{(1-x)^2}+30x^3+90x^4+165x^5+241x^6+305x^7+360x^8$$
$$+409x^9+\frac{455x^{10}}{1-x}+\frac{45x^{11}}{(1-x)^2},$$

which is the theoretical generating function found in Section III.

Hence we have proved that all the linearly independent syzygies of the second kind of degree 6 *have been obtained.*

To sum up, the irreducible syzygies found are :—

(A) The *primary* syzygies

(α) $J^{(2)}[(ae)^\lambda (af)^\mu (abcd)]$, and ($\beta$) $J^{(2)}[(ab)^\lambda (cd)^\mu (acef)]$.

(B) The non-primary syzygies

(α) $\left\{\begin{matrix}(abcd)_{w-1}\\(def)\end{matrix}\right\}$, ($\beta$) $Y_w(ab, ac, de, fe)$,

and (γ) $V_w(abc, def)$.

THE CLASSIFICATION OF IDEAL NUMBERS THAT DEPEND ON A CUBIC IRRATIONALITY

By W. E. H. BERWICK.

[Received December 19th, 1912.—Read January 9th, 1913.—Revised March 27th, 1913.]

IT was first proved by Lagrange that the expansion of a real quadratic surd in the form of a simple continued fraction must become periodic. The determination of rational approximations to an irrational number of the third degree by means of a periodic expansion is a problem which was attempted and partially solved by Jacobi in 1868. Jacobi devised an extension of continued fractions which enabled him to determine rational approximations to the mutual ratios of three numbers and showed that every approximation could be expressed in terms of the three preceding ones and the coefficients of the expansion. He further showed that if the three numbers are taken to be 1, θ, $p+q\theta+r\theta^2$, where θ is the real root of a rational cubic equation of negative discriminant, the expansion may become periodic, and in certain numerical cases actually does do so. It has been proved by Bachmann, however, that this periodicity cannot exist unless a certain limiting inequality is satisfied. The object of the present paper is to show that the expansion can be made in such a way, different from Jacobi's, that periodicity must ensue in every case. The coefficients required are found by means of Minkowski's theorem on convex surfaces, and considerable use is made of Poincaré's method of nets. It is then shown that an ideal in a cubic corpus (of negative discriminant) can be linked up with an equivalent ideal by a substitution derived from the coefficients of the expansion, that every ideal is equivalent to one of a finite number of "reduced ideals," and that this equivalence may be exhibited in a finite number of operations. Finally, it is shown that the reduced ideals in each class are linked up in one closed cycle which gives rise to the primitive unit of the corpus.

The memoir being based on considerations of a strictly arithmetical nature, no attempt is made to obtain an analytical expression for the class number. Dedekind has already solved this problem for such corpora as can be defined by an equation of the form $\theta^3 = n$, and the extension

of his method to the general corpus does not appear to present much difficulty.

A method of obtaining a minimalbasis of any cubic corpus has been given by Woronoj, and numerical examples of cubic integers, ideals, and units by Mathews. Tables have been calculated by Markoff and Reid, but in neither are all the units given fundamental. Mention should also be made of Minkowski's work on periodic substitutions derived from algebraic numbers. A further method of approximation, based on Minkowski's algorithm, is due to Kollros. In this last the expansion becomes periodic for some numerical cases of corpora both of negative and positive discriminant, and the geometry used is two-dimensional throughout.

The following list of the memoirs referred to above includes all previous papers on cubic corpora known to the author at the time of writing :—

C. G. J. Jacobi.—"Allgemeine Theorie der Kettenbruchähnlichen Algorithmen in welchen jede Zahl aus drei vorhergehenden gebildet wird," *Crelle*, Bd. 69 (1868), S. 29–64; *Ges. Werke*, Bd. 6, S. 385–426.

P. Bachmann.—"Zur Theorie von Jacobi's Kettenbruch Algorithmen," *Crelle*, Bd. 75 (1873), S. 25–34.

H. Poincare.—"Sur un mode nouveau de représentation géométrique des formes quadratiques définies ou indéfinies," *Journal de l'Ecole Polytechnique*, Cah. 47 (1880), pp. 177–245.

R. Dedekind.—"Ueber die Anzahl der Idealklassen in reinen kubischen Zahlkörpern," *Crelle*, Bd. 121 (1900), S. 40–123.

G. T. Woronoj.—"On the Algebraical Integers derived from an Irreducible Cubic Equation" (translation of Russian title), Petersburg (1894).

G. B. Mathews.—"On the Algebraical Integers derived from an Irreducible Cubic Equation," *Proc. London Math. Soc.*, Vol. xxiv (1893), pp. 327–336.

——— "On the Complex Integers connected with the Equation $\theta^3-2=0$," *Proc. London Math. Soc.*, Vol. xxiv (1893), pp. 319–327.

——— "On the Arithmetical Theory of the Form $x^3+ny^3+n^2z^3-3nxyz$," *Proc. London Math Soc.*, Vol. xxi (1890), pp. 280–287.

A. Markoff.—"Sur les nombres entiers dépendants d'une Racine cubique d'un nombre entier ordinaire," *Mém. de l'Acad. Imp. des Sciences de St. Pétersbourg*, viie Série, Tome 38, No. 9, pp. 1–37 (1892).

L. W. Reid.—"Tafel der Klassenanzahlen für kubische Zahlkörper," Inaugural-Dissertation, Göttingen (Kaestner, 1899); also in Abstract with table, *Amer. Jour. Math.*, Vol. 23 (1901), pp. 68–84.

H Minkowski.—"Ueber periodische Approximationen algebraischer Zahlen," *Acta Math.*, Bd. 26 (1903), S. 333–351; *Ges. Abhandlungen*, Bd. 1, S. 357–371.

——— "Zur Geometrie der Zahlen," *Verhandlungen des III Internationalen Mathematiker-Kongresses*, Heidelberg (1904), S. 164–173; *Ges. Abhandlungen*, Bd. 2, S. 43–52.

L. Kollros.—"Un Algorithme pour l'Approximation simultanée de deux Grandeurs," Inaugural-Dissertation, Zürich (Genève, Soullier, 1905).

1. In the following investigations we shall suppose that Θ is the real root of a rational irreducible cubic equation

$$A(\Theta) \equiv A_0\Theta^3+A_1\Theta^2+A_2\Theta+A_3 = 0,$$

the discriminant of which is negative, so that the two conjugate roots Θ', Θ'' are conjugate complex quantities. We shall denote by $k(\Theta)$ the corpus of numbers derived from the rational corpus, R, by adjunction of the number Θ. The norm of Θ, the rational number $\Theta\Theta'\Theta''$, will be denoted by $N(\Theta)$, and its discriminant, the negative rational number $(\Theta-\Theta')^2(\Theta-\Theta'')^2(\Theta'-\Theta'')^2$, by $d(\Theta)$. For convenience we shall write $-\Delta(\Theta)$ for $d(\Theta)$. The discriminant of the corpus will be denoted by d or $-\Delta$.

A cubic corpus is always primitive, *i.e.*, it cannot be a sub-corpus (*Unterkörper*) of a corpus of lower degree. For if $k(\Theta)$ were a sub-corpus of degree m of a corpus $k(\Theta_1)$ of degree n we should have $mn = 3$, so that $(m, n) = (1, 3)$ or $(3, 1)$, from which it follows that $k(\Theta_1)$ is coincident either with $k(\Theta)$ itself or with the corpus, R, of rational numbers. The corpus $k(\Theta)$ is therefore identical with $k(\theta)$, where θ is any number other than a rational number contained in it. Now, since every algebraical corpus contains complex integers, no generality is lost in supposing that θ is an integer. The equation defining the corpus may thus be taken to be

$$a(\theta) \equiv \theta^3+a_1\theta^2+a_2\theta+a_3 = 0,$$

wherein a_1, a_2, a_3 are rational integers.

Every complex integer of the corpus is of the form

$$\frac{c_0+c_1\theta+c_2\theta^2}{\Delta(\theta)},$$

where c_0, c_1, c_2 are rational integers. The aggregate of complex integers in $k(\theta)$ forms a triple modulus $(\omega_0, \omega_1, \omega_2)$, the notation indicating that the modulus is to include all elements of the type $m_0\omega_0+m_1\omega_1+m_2\omega_2$ for rational integral and zero values of m_0, m_1, m_2. The choice of the triad of integers ω_0, ω_1, ω_2, which form a base of the modulus and a minimalbasis of the corpus is not unique, for if

$$\begin{pmatrix} p_0, & q_0, & r_0 \\ p_1, & q_1, & r_1 \\ p_2, & q_2, & r_2 \end{pmatrix}$$

is any integral unit matrix and

$$\Omega_i = p_i\omega_0+q_i\omega_1+r_i\omega_2, \quad (i = 0, 1, 2),$$

the integers Ω_0, Ω_1, Ω_2 also form a minimalbasis, and, conversely, every base can be derived from every other by a substitution of this form. It has been shown by Woronoj that three integers defined in the following way form a minimalbasis of $k(\theta)$:—

Let D be the greatest rational integer such that

$$d(\theta) \equiv 0 \pmod{D^6},$$

and that the three congruences

$$3x+a_1 = f_1(x) \equiv 0 \pmod{D},$$

$$3x^2+2a_1x+a_2 = f_2(x) \equiv 0 \pmod{D^2},$$

$$x^3+a_1x^2+a_2x+a_3 = f_3(x) \equiv 0 \pmod{D^3},$$

have a common solution $x \equiv A' \pmod{D^3}$;

further let D_1 be the greatest integer such that

$$d(\theta) \equiv 0 \pmod{D^6 D_1^2},$$

and that the two congruences

$$f_2(x) \equiv 0 \pmod{D^2 D_1}, \qquad f_3(x) \equiv 0 \pmod{D^3 D_1^2},$$

have a common solution $x \equiv A \pmod{D^3 D_1^2}$,

which is necessarily also a solution of the former three congruences. A minimalbasis of $k(\theta)$ is then $(1, \omega_1, \omega_2)$, where

$$\omega_1 = \frac{-A+\theta}{D}, \qquad \omega_2 = \frac{A^2+a_1A+a_2+(A+a_1)\theta+\theta^2}{D^2D_1}.$$

From this result it is seen that, if $d(\theta)$ has no squared factor, the numbers 1, θ, θ^2 form a minimalbasis.

A particular case is that of the corpus defined by the equation $\theta^3-n=0$. For convenience we suppose that n contains no cubed factor l^3 for the corpora $k(\theta)$ and $k(\theta/l)$ are identical. If m^2 is the greatest square which divides n, the sign of m being chosen so that $m \equiv 0, 1 \pmod 3$, the minimalbasis is $(1, \omega_1 = \theta, \omega_2)$, where

$$\omega_2 = \frac{\theta^2}{m}, \quad \frac{m+m\theta+\theta^2}{3m} \quad \text{or} \quad \frac{m-m\theta+\theta^2}{3m},$$

according as $n \not\equiv \pm 1$, $\equiv 1$ or $\equiv -1 \pmod 9$.

The discriminant, d or $-\Delta$, of the corpus $k(\theta)$ is the negative integer

$$\begin{vmatrix} \omega_0, & \omega_1, & \omega_2 \\ \omega_0', & \omega_1', & \omega_2' \\ \omega_0'', & \omega_1'', & \omega_2'' \end{vmatrix}^2 \quad \text{which} \quad = \frac{d(\theta)}{D^6 D_1^2}.$$

For the particular case of $\theta^3 - n = 0$ above, we have

$$d = -27n^2/m^2 \quad \text{or} \quad -3n^2/m^2,$$

according as $$n \not\equiv \pm 1 \quad \text{or} \quad \equiv \pm 1 \pmod 9.$$

2. It has already been remarked that the corpus $k(\theta)$ is identical with $k(\theta_1)$ where θ_1 is any number other than a rational number contained in $k(\theta)$. Now, if θ_1 is a complex integer, it must satisfy an equation

$$a'(\theta_1) \equiv \theta_1^3 + a_1'\theta_1^2 + a_2'\theta_1 + a_3' = 0,$$

where a_1', a_2', a_3' are rational integers. The corpora defined by $a(\theta) = 0$, $a'(\theta) = 0$ are, therefore, identical, so that the equation $a(\theta) = 0$ only gives a particular mode of representation of the corpus $k(\theta)$. Now many problems connected with $k(\theta)$, such as the calculation of the class number and the determination of the equivalence or non-equivalence of two ideals, depend essentially on the corpus itself, and not on any particular mode of representing it. Thus, if the class number of $k(\theta)$, when defined by $a(\theta) = 0$ has been found, it is only repetition to calculate the class number of the same corpus when defined by a different cubic equation $a'(\theta) = 0$.

We accordingly show how to discover whether or not two corpora $k(\theta)$, $k(\theta_1)$ defined by the equations $a(\theta) = 0$, $a'(\theta_1) = 0$ are identical, and, when they are identical, to exhibit the rational relation connecting them.

In the first place the discriminants must be equal, for the discriminant of an algebraical corpus is an arithmetical invariant and unaltered by any rational Tschirnhausen transformation of the equation defining it. Assuming then that the discriminants of $k(\theta)$, $k(\theta_1)$ are equal (which cannot be true unless $\Delta(\theta)/\Delta(\theta_1)$ is a square), let $(\omega_0, \omega_1, \omega_2)$ be a minimal-basis, such as Woronoj's, of $k(\theta)$ and Ω any irrational integer contained in $k(\theta_1)$. If the corpora are identical Ω must be an integer of $k(\theta)$, and must therefore belong to the modulus $(\omega_0, \omega_1, \omega_2)$ so that the equations

$$\Omega = m_0\omega_0 + m_1\omega_1 + m_2\omega_2,$$

$$\Omega' = m_0\omega_0' + m_1\omega_1' + m_2\omega_2',$$

$$\Omega'' = m_0\omega_0'' + m_1\omega_1'' + m_2\omega_2''$$

must be satisfied by integer values of m_0, m_1, m_2. Replacing Ω, ω_0, ..., by rational approximations to these quantities respectively, and solving, we obtain approximations to the values of m_0, m_1, m_2. In general, if Ω, ω_0, ..., are calculated numerically to i places of decimals the values of m_0, m_1, m_2 will be correct to $i-3$ places of decimals; thus, taking $i=4$, the calculated values of m_0, m_1, m_2 should not, when the corpora are identical, differ from integers by more than about ·1. Before it can be inferred that the corpora are identical, however, it is necessary to apply to the equation $a(\theta)=0$ the transformation

$$\Omega_1 = m_0'\omega_0 + m_1'\omega_1 + m_2'\omega_2,$$

where m_0', m_1', m_2' are the integers nearest to the approximate values just obtained, and to verify that the cubic equation $b(\Omega_1)=0$ satisfied by Ω_1 is identical with the equation $B(\Omega)=0$ satisfied by Ω.

It should be observed, however, that there is no *a priori* reason why a given approximation $p+q\iota$ to the numerical value of Ω' should correspond to θ' rather than to θ''. Thus it cannot be inferred that the corpora are not identical without an examination of the values of m_0, m_1, m_2 given by each of the two sets of equations

$$\left.\begin{aligned} \Omega &= m_0\omega_0 + m_1\omega_1 + m_2\omega_2 \\ p+q\iota &= m_0\omega_0' + m_1\omega_1' + m_2\omega_2' \\ p-q\iota &= m_0\omega_0'' + m_1\omega_1'' + m_2\omega_2'' \end{aligned}\right\}, \qquad \left.\begin{aligned} \Omega &= m_0\omega_0 + m_1\omega_1 + m_2\omega_2 \\ p-q\iota &= m_0\omega_0' + m_1\omega_1' + m_2\omega_2' \\ p+q\iota &= m_0\omega_0'' + m_1\omega_1'' + m_2\omega_2'' \end{aligned}\right\}.$$

By two trials at most it is therefore possible to determine whether or not two corpora defined by given numerical equations are identical, and, when they are identical, to give the rational relation connecting the defining equations.

3. In order to determine all the corpora belonging to a discriminant $-\Delta$ we have recourse to a geometrical theorem due to Minkowski, on which most of the subsequent analysis depends and which we now state and illustrate.

First let the whole (x, y) plane be cut up into unit squares by the lines $x=m$, $y=n$, where m, n take all positive and negative integral values (including zero). Then every oval region (*i.e.*, a figure such that the line joining any two points on its boundary lies entirely inside it or coincides with a part of the boundary), whose area is equal to that of four unit squares, and which has the origin for a centre, must have an integer point (m, n) other than the origin within it or on its boundary.

Minkowski proves this statement by describing with every integer

(m, n) as centre, an oval homothetic with that whose centre is at the origin, but of half its lineal dimensions, and thus equal to one unit square in area. Then a large portion of the plane which includes N integer points has an area approximately equal to N unit squares, and the total area of the ovals inside it, one with each integer point as centre is also nearly equal to that of N unit squares. Hence either the ovals exactly fill up the whole plane or else overlap and leave gaps. In either case, on doubling the lineal dimensions, each new oval must stretch at least as far as one other integer point in addition to its centre. The theorem is therefore proved. In particular, if the ovals are anywhere curved, overlapping must exist, and a second integer point is actually inside an oval of area four units. Moreover, the integer points inside such an oval can be arranged in opposite pairs, for if a point lies inside the oval its image in the centre is also inside it.

As an illustration of this theorem, consider the positive quadratic form $ax^2+2bxy+cy^2$, where

$$ac-b^2 = \Delta > 0.$$

The area of the ellipse

$$ax^2+2bxy+cy^2 = K$$

is $\pi K/\Delta^{\frac{1}{2}}$; hence, taking $K = 4\Delta^{\frac{1}{2}}/\pi$, the area is 4, and it must be possible to choose a pair of integers (x, y) other than (0, 0) such that

$$ax^2+2bxy+cy^2 < 4\Delta^{\frac{1}{2}}/\pi.$$

When the form (a, b, c) is indefinite we can write

$$ax^2+2bxy+cy^2 = a^{\frac{1}{2}}(x-\omega y)\,.\,a^{\frac{1}{2}}(x-\omega' y),$$

where

$$a^2(\omega-\omega')^2 = 4(b^2-ac) = 4D.$$

The area of the parallelogram bounded by

$$a^{\frac{1}{2}}(x-\omega y) = \pm\lambda, \quad a^{\frac{1}{2}}(x-\omega' y) = \pm\mu$$

is now $4\lambda\mu/a(\omega-\omega')$ which $= 2\lambda\mu/D^{\frac{1}{2}}$; hence, taking $\lambda\mu = 2D^{\frac{1}{2}}$, at least one pair of integers (x, y) other than (0, 0) must exist such that

$$\text{mod. } ax^2+2bxy+cy^2 < 2D^{\frac{1}{2}}.$$

It should be observed that the coefficients a, b, c are not restricted to be rational numbers.

The theorem possesses an obvious extension to flat space in any number of dimensions; but for our present purpose it is sufficient to consider three-dimensional space. Instead of a plane oval we now have a region which has the origin for a centre and such that the line joining

every two points on its boundary is entirely inside it or else lies on some part of the bounding surface. Such a region, if its volume is equal to that of eight unit cubes, must have an integer point other than the origin, within it or on its boundary. The proof is practically identical with that given for two dimensions, and, as before, if the surface bounding the region is anywhere curved there must be integer points actually within it. Also, since the integer points within the oval are again arranged in opposite pairs, any plane through the centre divides the region into two regions, each of which includes at least one integer point.

As an illustration we take the positive ternary quadratic form

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy.$$

The volume of the ellipsoid

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy = K,$$

is

$$\frac{4}{3}\frac{\pi K^{\frac{3}{2}}}{\sqrt{(abc+2fgh-af^2-bg^2-ch^2)}} = \frac{4\pi K^{\frac{3}{2}}}{3D^{\frac{1}{2}}}.$$

Hence, if $K^{\frac{3}{2}} = 6\sqrt{D/\pi}$, the volume is 8, and a triad of integers (x, y, z) other than (0, 0, 0) must exist such that

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy < \sqrt[3]{(36D/\pi^2)}.$$

Now let $(\omega_0, \omega_1, \omega_2)$ be a minimalbasis of a cubic corpus $k(\theta)$, and let

$$\omega = \omega_0 x+\omega_1 y+\omega_2 z,$$
$$\omega' = \omega_0' x+\omega_1' y+\omega_2' z,$$
$$\omega'' = \omega_0'' x+\omega_1'' y+\omega_2'' z.$$

The region bounded by the planes $\omega = \pm A$, and the elliptic cylinder $\omega'\omega'' = B^2$ satisfies the conditions of Minkowski's theorem, and its volume is $4AB^2\pi/\Delta^{\frac{1}{2}}$. Hence, if this expression $= 8$, there must, since the bounding surface is partly curved, be at least one triad of integers (x, y, z) other than (0, 0, 0) such that

$$|\omega| < A, \quad |\omega'| = |\omega''| < B.$$

In particular, if $A = 1$, $B^2 = 2\Delta^{\frac{1}{2}}/\pi = M$, say, by suitable choice of (x, y, z) we can make

$$|\omega| < 1, \quad |\omega'| = |\omega''| < M^{\frac{1}{2}},$$

and the first of these inequalities shows that ω is actually an integer of $k(\theta)$, and not a rational integer. Again, since $-\omega$ is an integer when ω is such, the inequalities may be replaced by

$$0 < \omega < 1, \quad |\omega'| = |\omega''| < M^{\frac{1}{2}}.$$

From these we deduce

$$\left.\begin{aligned} -2M^{\frac{1}{2}} < \omega+\omega'+\omega'' = -b_1 < 1+2M^{\frac{1}{2}} \\ -2M^{\frac{1}{2}} < \omega(\omega'+\omega'')+\omega'\omega'' = b_2 < 2M^{\frac{1}{2}}+M \\ 0 < \omega\omega'\omega'' = -b_3 < M \end{aligned}\right\}. \qquad (1)$$

Hence, to determine all the corpora of discriminant $-\Delta$, it is sufficient to examine the corpora defined by the equations

$$b(\theta) \equiv \theta^3+b_1\theta^2+b_2\theta+b_3 = 0,$$

where the coefficients b_1, b_2, b_3 are limited by the inequalities (1).

It may be observed that the numerically smallest value of d is -23, the discriminant of the corpus defined by $\theta^3-\theta-1=0$. A small value of d which belongs to two distinct corpora is -972, the discriminant of $k(\sqrt[3]{6})$ and $k(\sqrt[3]{12})$.

Among the integers in $k(\theta)$ there must be a set θ_1, θ_2, ..., such that

$$\Delta(\theta_1) = \Delta(\theta_2) = \ldots < \Delta(\theta),$$

θ being any other integer not contained in the set. Such integers may be called "integers of smallest discriminant" of the corpus. I have not succeeded in finding inequalities sufficient to isolate one of these integers of smallest discriminant. An upper limit for the value of $\Delta(\theta_1)$ can, however, be deduced from the inequalities (1) above. For we have

$$(\omega-\omega')(\omega-\omega'') = \omega^2-\omega(\omega'+\omega'')+\omega'\omega'' < 1+2M^{\frac{1}{2}}+M,$$

and

$$|\omega'-\omega''| < 2M^{\frac{1}{2}}.$$

Hence

$$\Delta(\omega) < 4(1+M^{\frac{1}{2}})^{\frac{1}{2}}M,$$

i.e.,

$$\Delta(\omega) < \frac{16\Delta}{\pi^2}\left[\frac{2^{\frac{1}{4}}\Delta^{\frac{1}{8}}}{\pi^{\frac{1}{4}}}+\frac{\pi^{\frac{1}{4}}}{2^{\frac{1}{4}}\Delta^{\frac{1}{8}}}\right]^4.$$

It can also be shown that there are corpora which contains no integer Ω such that $\Delta(\Omega) < \delta^2\Delta$, where δ is any assigned number.

Let $p_1, p_2, \ldots, p_s$ be the primes of the form $6i+1$, and $q_1, q_2, \ldots, q_t$ those of the form $6i-1$ which do not exceed δ, itself supposed not less than 3, the exceptional primes 2 and 3 being included among the q's and p's respectively. Also let p_0 be another prime greater than δ, and n an integer containing no squared factor and prime to $p_1p_2\ldots p_sp_0q_1q_2\ldots q_t$, but otherwise not yet determined. Now consider the corpus given by

$$\theta^3-p_1^2p_2^2\ldots p_s^2p_0^3q_1^2q_2^2\ldots q_t^2n = 0.$$

Woronoj's minimalbasis is $(1, \theta, \omega_2 \equiv \theta^2/p_1p_2\ldots p_sp_0q_1q_2\ldots q_t)$,

and

$$\Delta = 27p_1^2p_2^2\ldots p_s^2p_0^2q_1^2q_2^2\ldots q_t^2n^2 = \Delta(\theta)/p_1^2p_2^2\ldots p_s^2p_0^2q_1^2q_2^2\ldots q_t^2.$$

Also, if

$$\Omega = x+y\theta+z\omega_2,$$

$$\Delta(\Omega) = \Delta(\theta)N^2(y-z\theta/p_1p_2\ldots p_sp_0q_1q_2\ldots q_t) = \Delta(p_1p_2\ldots p_sp_0q_1q_2\ldots q_ty^3-nz^3)^2.$$

Now, if $\Delta(\Omega) \leqslant \delta^2\Delta$, we can write

$$\Delta(\Omega) = \Delta \cdot p_1^{2\alpha_1} p_2^{2\alpha_2} \dots p_\sigma^{2\alpha_\sigma} q_1^{2\beta_1} q_2^{2\beta_2} \dots q_\tau^{2\beta_\tau} \quad (0 \leqslant \sigma \leqslant S,\ 0 \leqslant \tau \leqslant t),$$

so that $$p_1 p_2 \dots p_s p_0 q_1 q_2 \dots q_t y^3 - nz^3 = \pm p_1^{\alpha_1} p_2^{\alpha_2} \dots p_\sigma^{\alpha_\sigma} q_1^{\beta_1} q_2^{\beta_2} \dots q_\tau^{\beta_\tau},$$

whence $$p_{\sigma+1} \dots p_s p_0 q_{\tau+1} \dots q_t y^3 \equiv \pm p_\sigma^{\alpha_1 - 1} \dots p_\sigma^{\alpha_\sigma - 1} q_1^{\beta_1 - 1} \dots q_\tau^{\beta_\tau - 1} \pmod{n}. \quad (2)$$

We propose to show that this congruence is insoluble when n is suitably chosen.

Let each of the primes p_i $(i = 1, 2, \dots, S, 0)$ be resolved into two conjugate factors π_i, π_i' by adjunction of the number $\rho\ (\equiv \exp 2\pi\iota/3)$ to the rational corpus R. Each of the rational primes $q_1, q_2, \dots, q_t$ remains prime in the wider corpus. The congruence (3) thus becomes

$$\pi_{\sigma+1} \pi'_{\sigma+1} \dots \pi_s \pi'_s \pi_0 \pi'_0 q_{\tau+1} \dots q_t y^3 \equiv (\pi_i \pi_i')^{\alpha_i - 1} \dots (\pi_\sigma \pi'_\sigma)^{\alpha_\sigma - 1} q_1^{\beta_1 - 1} \dots q_\tau^{\beta_\tau - 1} \pmod{n}. \quad (3)$$

Now, if λ is any prime in $k(\rho)$ the complete set of residues, excluding 0 (mod λ), can be divided into two classes $\mu_1, \mu_2, \dots, \nu_1, \nu_2, \dots$, such that λ is a cubic residue of all primes of the form $\lambda x + \mu_i$ and a cubic non-residue of all primes of the form $\lambda x + \nu_i$. Hence definite forms

$$\pi_1 \pi'_1 \dots \pi_s \pi'_s \pi_0 \pi'_0 q_1 \dots q_t x + z \quad (z = z_1,\ z_2,\ \dots)$$

exist such that z is prime to $\pi_1 \pi'_1 \dots \pi_s \pi'_s \pi_0 \pi'_0 q_1 \dots q_t$ and that $\pi_1, \pi'_1, \dots, \pi_s \pi'_s, \pi_0, q_1, \dots, q_t$ are cubic residues, but π'_0 a cubic non-residue, of every prime in these forms.

Further, by Weber's extension of Dirichlet's theorem on the arithmetical progression, each of these forms contains a prime. Let ν be one of these primes and let $n = \nu\nu'$, a rational prime satisfying the restrictions laid down above. With this choice of ν the congruence (3) is insoluble. Hence (2) is insoluble (mod n) in $k(\rho)$, and therefore also insoluble in the more restricted corpus R, of rational numbers. The theorem stated is therefore proved.

From this result it follows that there is no number δ such that every corpus $k(\theta)$ possesses a minimalbasis of the form $\left(1,\ \theta,\ \frac{p + q\theta + \theta^2}{m}\right)$, where $m < \delta$.

For purposes of classification it is necessary to isolate a minimalbasis $(1, \omega_1, \omega_2)$, to be called the canonical basis of the corpus. Such a canonical basis must be unique and capable of being derived from any minimalbasis by a finite number of operations. The rational cubic equation satisfied by ω_1 can then be called the canonical equation of the corpus and ω_1 the canonical integer. Before taking up the question of canonical representation it is convenient to work out a method of approximation, and to this discussion the next four articles will be given.

4. The theory of continued fractions supplies a solution of the following geometrical problem:—The line $y - mx = 0$ being traced in the (x, y) plane, it is required to find a sequence of integer points $(x_1, y_1), (x_2, y_2), \dots$, each nearer to the line than the preceding point, and also such that (x_i, y_i) is nearer to the line than any other integer point $(x'\ y')$ where

$$0 < my' + x' < my_{i+1} + x_{i+1}.$$

In fact, if m is expanded as a simple continued fraction and p_i/q_i is the

i-th principal convergent, (q_1, p_1), (q_2, p_2), ..., is the sequence of points required.

Passing into three dimensions, the following analogous problem presents itself:—It is required to find a sequence of integer points (x_1, y_1, z_1), $(x_2, y_2\ z_2)$, ..., each nearer to the line $x : y : z = l : m : n$ than its predecessor, and also such that no integer point (x', y', z') where

$$0 < lx'+my'+nz' < lx_{i+1}+my_{i+1}+nz_{i+1}$$

is nearer to the line than (x_i, y_i, z_i). We propose, however, to discuss a more general problem which includes this as a particular case.

Let $$U \equiv \alpha x+\beta y+\gamma z = 0$$

be the equation of a real plane, and

$$\left.\begin{aligned} V &\equiv (\alpha'+\iota\alpha'')\, x+(\beta'+\iota\beta'')\, y+(\gamma'+\iota\gamma'')\, z = 0,\\ W &\equiv (\alpha'-\iota\alpha'')\, x+(\beta'-\iota\beta'')\, y+(\gamma'-\iota\gamma'')\, z = 0, \end{aligned}\right\} \qquad (4)$$

those of two conjugate imaginary ones through whose real line of intersection the plane $U = 0$ does not pass. The equation

$$VW \equiv \Phi = \text{a positive constant}$$

is that of an elliptic cylinder which is cut by the plane $U =$ constant in an ellipse.

In subsequent applications the numbers α, β, γ will form a base of an ideal in a cubic corpus $k(\theta)$ and

$$(\alpha'+\iota\alpha'',\ \beta'+\iota\beta'',\ \gamma'+\iota\gamma''), \quad (\alpha'-\iota\alpha'',\ \beta'-\iota\beta'',\ \gamma'-\iota\gamma''),$$

will be the conjugate ideals in $k(\theta')$, $k(\theta'')$. The equation

$$[U_i \equiv]\ \alpha x_i+\beta y_i+\gamma z_i = \alpha x_j+\beta y_j+\gamma z_j\ [\equiv U_j]$$

thus leads to $$\alpha(x_i-x_j)+\beta(y_i-y_j)+\gamma(z_i-z_j) = 0,$$

which can only be satisfied in rational numbers by $x_i = x_j$, $y_i = y_j$, $z_i = z_j$. Again, from the equation $\Phi_i = \Phi_j$, we have

$$U_i\Phi_i U_j = U_j\Phi_j U_i,$$

and, since $U_i\Phi_i$ is a rational number N_i,

$$N_i U_j = N_j U_i.$$

Hence, as before, $$x_i/x_j = y_i/y_j = z_i/z_j = N_i/N_j,$$

so that $$\Phi_i = (N_j^2/N_i^2)\,\Phi_i.$$

The points $(x_i, y_i, z_i)(x_j, y_j, z_j)$ are therefore either coincident or opposite and in the latter case U_i and U_j have different signs.

Now let (x_a, y_a, z_a) be an integer point such that $U_a \equiv \alpha x_a + \beta y_a + \gamma z_a$ is positive, and consider the region bounded by the cylinder $\Phi = \Phi_a$, and the planes $U = 0$, U_a. The only integer points on the boundary of this region are (x_a, y_a, z_a) and the origin, and, since its volume is finite, there can only be a finite number of integer points within it. Among these points [including (x_a, y_a, z_a) but excluding the origin] there must be at least one point (x_1, y_1, z_1) such that there is no other integer point (x', y', z') satisfying the inequalities

$$0 < U' < U_1, \quad \Phi' < \Phi_1.$$

A point (x_1, y_1, z_1) possessing this property is defined to be a *convergent point* to the axis of the cylinder $\Phi = \text{const.}$ by the plane $U = \text{const.}$ Now choosing a positive quantity K such that the volume of the region bounded by the planes $U = 0$, K and the cylinder $\Phi = \Phi_1$ is four units, there can only be a finite number of integer points within this region and by Minkowski's theorem there must be at least one such. Let (x_2, y_2, z_2) be that point, chosen from among this set, for which U takes its least value. Then (x_2, y_2, z_2) also satisfies the conditions defining a convergent point, and $U_1 < U_2$, $\Phi_1 > \Phi_2$. Continuing this process we arrive at a unique sequence of convergent points, determined by (x_1, y_1, z_1), such that

$$U_1 < U_2 < \ldots < U_i < U_{i+1} < \ldots,$$

$$\Phi_1 > \Phi_2 > \ldots > \Phi_i > \Phi_{i+1} > \ldots,$$

and that there is no integer point (x', y', z') satisfying the inequalities

$$U_i < U' < U_{i+1}, \quad \Phi' < \Phi_i.$$

The conditions defining the sequence of convergent points may thus be now formally stated :

$$\left.\begin{array}{ll} \text{(i)} & U_1 \text{ is positive.} \\ \text{(ii)} & \text{There is no integer point } (x', y', z') \text{ such that} \\ & \qquad 0 < U' < U_1, \quad \Phi' < \Phi_1. \\ \text{(iii)} & \qquad U_i < U_{i+1}, \quad \Phi_i > \Phi_{i+1}. \\ \text{(iv)} & \text{There is no integer point } (x', y', z') \text{ such that} \\ & \qquad U_i < U' < U_{i+1}, \quad \Phi' < \Phi_i. \end{array}\right\} \quad (5)$$

From these conditions it is evident that if the plane $U = \text{const.}$ be supposed to move away from the origin it crosses no integer point on or within the cylinder $\Phi = \Phi_i$ till it reaches (x_i, y_i, z_i), a point on the

cylinder, and the next point within the cylinder it crosses is $(x_{i+1}, y_{i+1}, z_{i+1})$. It should be observed that a change in the shape of the cylinder or in the direction of the plane causes in general a change in the convergent points.

Now let the convergent points in order be T_1, T_2, ..., T_i, ..., and 0 the origin of coordinates. If the axis of the cylinder passes through rational points it must pass through integer points. Let (x_0, y_0, z_0) be that integer point on it which is nearer to the origin than any other for which U is positive Then $\Phi_0 = 0$ and T_0 is the last convergent point. But if the axis passes through no rational point (which is true with the above choice of the coefficients) there is an infinite number of convergent points. For, if T_i be any convergent point, $\Phi_i \neq 0$, and therefore another convergent point T_{i+1} must follow it.

A convergent point T_i can always be found such that $U_i > \delta$, any assigned number. For only a finite number of integer points, and therefore only a finite number of convergent points, lie in the region bounded by the planes $U = 0$, δ and the cylinder $\Phi = \Phi_1$. It is similarly evident that a convergent point T_j can always be found such that $\Phi_j < \eta$, any assigned quantity.

The succession of convergent points T_1, T_2, ..., T_i, ..., is uniquely determined by T_1. Now each of the points T_2, T_3, ..., T_i, ..., satisfies the conditions (5, i, ii) defining T_1, and can thus be taken as the first point of a set. Taking T_i as the first point, the points which follow it are T_{i+1}, T_{i+2}, ..., the same points as succeed T_i in the sequence determined by T_1. Again each point of the sequence is preceded by a definite convergent point. For, if T_i and T_j are each followed by T_k, let $U_i < U_j < U_k$. If then $\Phi_i > \Phi_j > \Phi_k$, T_k cannot be the convergent point succeeding T_i; whereas if $\Phi_j > \Phi_i$, T_j is not a convergent point at all. In virtue of this result T_i is a definite convergent point, whether i is a positive or a negative integer.

Further, there is only one sequence of convergent points. For let another convergent point T_1' give rise to the sequence T_1', T_2', ..., T_j', ... Then by suitable choice of j we can make $U_i \leqslant U_j' < U_{i+1}$; whence, as above, it follows that T_i and T_j' are coincident. The sequence determined by T_1' is therefore identical with that given by T_1. For the determination of T_1 another condition is required, and, in the application to ideals, a particular choice will be made later.

5. Before discussing the arrangement of the convergent points it is necessary to mention some properties of Poincaré's nets (*réseaux*). First let OA, OB be two intersecting lines which can be taken as axes in their plane, and let the coordinates of A and B be (1, 0) and (0, 1). Then the

aggregate of points in the plane with positive or negative integral (including zero) coordinates forms a net which is said to be constructed on the base OA, OB. If A', B' are two points belonging to this net, a second net, all of whose points belong to the former, is determined by the base OA', OB'. Taking A', B' to be the points (α, β), (γ, δ) the general point belonging to this net is $(\alpha m+\gamma n, \beta m+\delta n)$ where m, n are integers. The points A, B will belong to it if m, n, m', n' can be chosen so that

$$\alpha m+\gamma n = 1, \quad \beta m+\delta n = 0,$$

$$\alpha m'+\gamma n' = 0, \quad \beta m'+\delta n' = 1.$$

These equations are soluble in integers if, and only if, $\alpha\delta-\beta\gamma = \pm 1$, and when this condition is satisfied we can write

$$p = \alpha(pm+qm')+\gamma(pn+qn'), \quad q = \beta(pm+qm')+\delta(pn+qn'),$$

so that every point belonging to the first net belongs also to the second. Hence if $\alpha\delta-\beta\gamma = \pm 1$ the two nets are identical, and either OA, OB or OA', OB' can be taken as base. The area of the triangle $OA'B'$ is $\pm \frac{1}{2}(\alpha\delta-\beta\gamma) = \frac{1}{2}$, which is the same as that of OAB. A triangle such as OAB or $OA'B'$ is defined to be a *unit triangle* for the net.

If A' is the point (α, β) and $dv(\alpha, \beta) = 1$ the equation $\alpha y-\beta x = \pm 1$ is soluble in integers, its general solution being $x = \alpha t \pm \gamma$, $y = \beta t \pm \delta$, where (γ, δ) is a particular solution. All such points (x, y) lie on the two lines $\alpha y-\beta x = \pm 1$ which are parallel to OA' and on opposite sides of it. There can be no integer point P between OA' and either of these lines, for if so the area of the triangle $OA'P$, whose vertices are integer points would be less than $\frac{1}{2}(\alpha\delta-\beta\gamma)$ or $\frac{1}{2}$, which is impossible. We may thus say that the lines $\alpha y-\beta x = \pm 1$ are the *integer lines* nearest to OA', and it is now evident that the points B' such that $OA'B'$ is a unit triangle are the aggregate of integer points on the two integer lines nearest to OA'.

A net in three dimensions is determined by three lines OA, OB, OC, which are not coplanar. If the coordinates of A, B, C are taken to be (1, 0, 0), (0, 1, 0), (0, 0, 1), all points (m, n, p) with integral coordinates belong to the net. As before, the lines OA', OB', OC' form a base of the net if the integer coordinates of A', B', C' are $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3 \gamma_3)$, where $|\alpha_1, \beta_2 \gamma_3| = \pm 1$.

A two-dimensional net is formed by the integer points in the plane $OA'B'$, and if OA', OB' form a base of this net a third point C' must exist such that OA', OB', OC' form a base of the whole net, *i.e.*, the equation

$$(\beta_1\gamma_2-\beta_2\gamma_1)x+(\gamma_1\alpha_2-\gamma_2\alpha_1)y+(\alpha_1\beta_2-\alpha_2\beta_1)z = \pm 1$$

must be soluble in integers. Integral solutions of this equation exist if

$$dv(\beta_1\gamma_2-\beta_2\gamma_1,\quad \gamma_1\alpha_2-\gamma_2\alpha_1,\quad \alpha_1\beta_2-\alpha_2\beta_1)=1,$$

and, this condition being satisfied, the general solution is

$$x=\alpha_1 m+\alpha_2 n\pm\alpha_3,\quad y=\beta_1 m+\beta_2 n\pm\beta_3,\quad z=\gamma_1 m+\gamma_2 n\pm\gamma_3,$$

where $(\alpha_3, \beta_3, \gamma_3)$ is a particular solution and m, n are integers. All such points (x, y, z) lie on the two planes

$$(\beta_1\gamma_2-\beta_2\gamma_1)x+(\gamma_1\alpha_2-\gamma_2\alpha_1)y+(\alpha_1\beta_2-\alpha_2\beta_1)z=\pm 1,$$

which are parallel to $OA'B'$, and, as before, can be called the *integer planes* nearest to $OA'B'$. The tetraheda $OA'B'C'$, $OABC$ are equal in volume and defined to be *unit tetrahedra*. Thus, if OA', OB' form a base of the net of integer points in their plane, there are points C' such that $OA'B'C'$ is a unit tetrahedron, and such points C' form the aggregate of integer points in the two integer planes nearest to $OA'B'$.

A unit triangle or tetrahedron has no integer point other than its vertices within it or on its boundary. Conversely, if there is no integer point within or on a side of a triangle with integer vertices, the triangle must be unitary. For let $OA'B'$ be such a triangle and complete the parallelogram $OA'C'B'$. Then the entire strip of the plane between the parallel lines OA', $B'C'$ can be filled up with parallelograms congruent to $OA'C'B'$ and contiguous with each other. Moreover, each of these parallelograms is divided by its diagonal parallel to $A'B'$ into two triangles congruent to $OA'B'$, and none of these triangles can have an integer point within it or on a side if $OA'B'$ possesses this property. Hence there are no integer points between OA' and $B'C'$, and the triangle $OA'B'$ is therefore unitary.

A tetrahedron with no integer point within it or on its boundary need not, however, be unitary; the four points (0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, K) are the vertices of such a tetrahedron if $K>1$.

It is now convenient to make a change in the notation for the convergent points. Since the axis of the cylinder is supposed to pass through no rational point except the origin a rational plane meets the cylinder in an ellipse, and therefore only passes through a finite number of convergent points. Let the first convergent point which does not lie in the plane through the origin and the first two be the $[n(1)+1]$-th. The first $n(1)$ points are to be called

$$T_1,\quad T_1^2,\quad T_1^3,\quad \ldots,\quad T_1^{n(1)-1},\quad T_2.$$

Again let the first point which does not lie in the plane through the origin,

the $n(1)$-th and the $[n(1)+1]$-th be the $[n(1)+n(2)]$-th. The $n(2)-1$ points following T_2 are to be called

$$T_2^2, \quad T_2^3, \quad \ldots, \quad T_2^{n(2)-1}, \quad T_3.$$

Continuing this process, the new notation for the whole system of points becomes

$$T_1, T_1^2, T_1^3, \ldots, T_1^{n(1)-1}, T_2, T_2^2, T_2^3, \ldots, T_2^{n(2)-1}, T_3, T_3^2, \ldots,$$
$$T_\mu, T_\mu^2, \ldots, T_\mu^{n(\mu)-1}, T_{\mu+1}, T_{\mu+1}^2, \ldots,$$

where the $n(\mu)$ points $\quad T_\mu, T_\mu^2, \ldots, T_\mu^{n(\mu)-1}, T_{\mu+1},$

in number at least two, lie in a plane Π_μ through the origin. It will frequently be convenient to denote the point $T_{\mu+1}$ by $T_\mu^{n(\mu)}$ or $T_{\mu+1}^1$, the former or latter notation being used according as it is desired to indicate that the point lies on the plane Π_μ or on $\Pi_{\mu+1}$.

The whole investigation rests on the following theorem:—Any two consecutive convergent points form with the origin a unit triangle for the net of integer points in its plane. For if the triangle $OT_\mu^\nu T_\mu^{\nu+1}$ is not unitary there must be an integer point T' within it or on one of its sides. Since $U_\mu^{\nu+1} > U_\mu^\nu > 0$, the greatest value of U for points on a side of the triangle or within it is $U_\mu^{\nu+1}$. Again, since the plane of the triangle meets the cylinder $\Phi = \Phi_\mu^\nu$ in an ellipse within which the two corners O, $T_\mu^{\nu+1}$ of the triangle lie, the greatest value of Φ for points on a side of the triangle or within it is Φ_μ^ν. Hence

$$0 < U' < U_\mu^{\nu+1}, \quad \Phi' < \Phi_\mu^\nu,$$

contrary to the condition (5, iv) defining the convergent point $T_\mu^{\nu+1}$. The triangle $OT_\mu^\nu T_\mu^{\nu+1}$ is therefore unitary. Similar reasoning shows that $OT_\mu^\nu T_{\mu+1}^2$ is a unit triangle, and that the tetrahedron $OT_\mu^\nu T_\mu^{\nu+1} T_{\mu+1}^2$ has no integer point within it or on one of its faces. The tetrahedron, however, cannot be asserted to be unitary.

Since $OT_\mu^\nu T_\mu^{\nu+1}$ is a unit triangle integer points t exist such that $OT_\mu^\nu T_\mu^{\nu+1} t$ is a unit tetrahedron, and such points t form the aggregate of integer points on the two integer planes nearest to $OT_\mu^\nu T_\mu^{\nu+1}$. It is necessary to pick out one of these to be associated with the triangle $OT_\mu^\nu T_\mu^{\nu+1}$. First when $OT_\mu^\nu T^{\nu+1} T_{\mu+1}^2$ is a unit tetrahedron the point chosen is $T_{\mu+1}^2$, and with reference to its association is denoted by t_μ. But when $OT_\mu^\nu T_\mu^{\nu+1} T_{\mu+1}^2$ is not unitary a different choice must be made. Since the triangles $OT_\mu^\nu T_\mu^{\nu+1}$ $[\nu = 1, 2, \ldots, n(\mu)-1]$ are unit triangles in the same

plane, the same point t_μ can be chosen for association with each of them. But $OT_{\mu+1}T_{\mu+1}^2$ being also unitary a finite number of integer points

$$t_{\mu+1}^1,\ t_{\mu+1}^2,\ \ldots,\ t_{\mu+1}^{m(\mu+1)},$$

must exist such that the tetrahedra

$$OT_\mu^{n(\mu)-1}T_{\mu+1}t_{\mu+1}^1,\ Ot_{\mu+1}^1T_{\mu+1}t_{\mu+1}^2,\ \ldots,\ Ot_{\mu+1}^{m(\mu+1)}T_{\mu+1}T_{\mu+1}^2,$$

are all unitary. The choice of the points $t_{\mu+1}^\nu\ [\nu = 1, 2, \ldots, m(\mu+1)]$ in no way affects the convergent points, and is arbitrary so long as the last condition is uniquely satisfied. A unique method of fixing such a set of points is given in § 7. It should be observed that these points

$$t_{\mu+1}^\nu \quad [\nu = 1, 2, \ldots, m(\mu+1)]$$

are defined only when $OT_\mu^{n(\mu)-1}T_\mu T_{\mu+1}^2$ is not a unit tetrahedron. When they exist they are to be called *subsidiary* points. The point to be associated with each of the triangles $OT_\mu^\nu T_\mu^{\nu+1}\ [\nu = 1, 2, \ldots, n(\mu)-1]$ is now taken to be $t_{\mu+1}^1$ and with reference to its association is again denoted by t_μ.

It will now be convenient to denote the point $T_\mu^{n(\mu)-1}$ when $OT_\mu^{n(\mu)-1}T_\mu T_{\mu+1}^2$ is a unit tetrahedron, and $t_{\mu+1}^{m(\mu+1)}$ when $OT_\mu^{n(\mu)-1}T_\mu T_{\mu+1}^2$ is not a unit tetrahedron by $t^{\mu+1}$. The tetrahedra $Ot^{\mu+1}T_{\mu+1}^\nu T_{\mu+1}^{\nu+1}$ are then always unitary.

It will be noticed that T is always a convergent point with the above notation and that t may or may not be one. The coordinates of T will be taken to be (P, Q, R) and those of t to be (p, q, r).

6. We now investigate the equations which connect the coordinates of successive convergent and subsidiary points.

First consider three consecutive convergent points T_μ^ν, $T_\mu^{\nu+1}$, $T_\mu^{\nu+2}$ which lie on a plane through the origin. Since $OT_\mu^\nu T_\mu^{\nu+1}$ is a unit triangle the point $T_\mu^{\nu+2}$ must belong to the net constructed on the base OT_μ^ν, $OT_\mu^{\nu+1}$. Hence

$$(P, Q, R)_\mu^{\nu+2} - A_\mu^{\nu+2}(P, Q, R)_\mu^{\nu+1} - B_\mu^{\nu+2}(P, Q, R)_\mu^\nu = 0,$$

where $A_\mu^{\nu+2}$, $B_\mu^{\nu+2}$ are integers. Similarly T_μ^ν belongs to the net based on $OT_\mu^{\nu+1}$, $OT_\mu^{\nu+2}$, so that $1/B_\mu^{\nu+2}$, $A_\mu^{\nu+2}/B_\mu^{\nu+2}$ must be integers; therefore

$$(P, Q, R)_\mu^{\nu+2} = A_\mu^{\nu+2}(P, Q, R)_\mu^{\nu+1} + B_\mu^{\nu+2}(P, Q, R)_\mu^\nu, \tag{6}$$

where $A_\mu^{\nu+2}$ is an integer and $B_\mu^{\nu+2} = \pm 1$.

Secondly, suppose that $OT_\mu^{n(\mu)-1}T_{\mu+1}T_{\mu+1}^2$ is a unit tetrahedron. Then

$Ot^{\mu}T_{\mu}^{n(\mu)-1}T_{\mu+1}$ being also unitary the point $T_{\mu+1}^{2}$ must belong to the net based on Ot^{μ}, $OT_{\mu}^{n(\mu)-1}$, $OT_{\mu+1}$, so that

$$(P, Q, R)_{\mu+1}^{2}-A_{\mu+1}^{2}(P, Q, R)_{\mu+1}-B_{\mu+1}^{2}(P, Q, R)_{\mu}^{n(\mu)-1}-C_{\mu+1}^{2}(p, q, r)^{\mu}=0.$$

But $OT_{\mu}^{n(\mu)-1}T_{\mu+1}T_{\mu+1}^{2}$ being unitary, t^{μ} must belong to the net formed on the base $OT_{\mu}^{n(\mu)-1}$, $OT_{\mu+1}$, $OT_{\mu+1}^{2}$, so that $A_{\mu+1}^{2}/C_{\mu+1}^{2}$, $B_{\mu+1}^{2}/C_{\mu+1}^{2}$, $1/C_{\mu+1}^{2}$ are integers; therefore

$$(P, Q, R)_{\mu+1}^{2}=A_{\mu+1}^{2}(P, Q, R)_{\mu+1}+B_{\mu+1}^{2}(P, Q, R)_{\mu}^{n(\mu)-1}+C_{\mu+1}^{2}(p, q, r)^{\mu}, \quad (7)$$

where $A_{\mu+1}^{2}$, $B_{\mu+1}^{2}$ are integers, and $C_{\mu+1}^{2}=\pm 1$.

In the third case when $OT_{\mu}^{n(\mu)-1}T_{\mu+1}T_{\mu+1}^{2}$ is not unitary the subsidiary points $t_{\mu+1}^{\nu}$ $[\nu=1, 2, \ldots, m(\mu+1)]$ have been chosen in such a way that the tetrahedra

$$OT_{\mu}^{n(\mu)-1}T_{\mu+1}t_{\mu+1}^{1},\ Ot_{\mu+1}^{\nu}T_{\mu+1}t_{\mu+1}^{\nu+1}\ [\nu=1, 2, \ldots, m(\mu+1)-1],$$

$$Ot_{\mu+1}^{m(\mu+1)}T_{\mu+1}T_{\mu+1}^{2},$$

are unitary. Hence, as in the last case,

$$\left.\begin{aligned}
(p, q, r)_{\mu+1}^{1} &= a_{\mu+1}^{1}(P, Q, R)_{\mu+1} + b_{\mu+1}^{1}(P,Q,R)_{\mu}^{n(\mu)-1}+c_{\mu+1}^{1}(p, q, r)^{\mu}\\
(p, q, r)_{\mu+1}^{2} &= a_{\mu+1}^{2}(p, q, r)_{\mu+1}^{1} + b_{\mu+1}^{2}(P, Q, R)_{\mu+1}+c_{\mu+1}^{2}(P, Q, R)_{\mu}^{n(\mu)-1}\\
(p, q, r)_{\mu+1}^{\nu+1} &= a_{\mu+1}^{\nu+1}(p, q, r)_{\mu+1}^{\nu} + b_{\mu+1}^{\nu+1}(P, Q, R)_{\mu+1}+c_{\mu+1}^{\nu+1}(p, q, r)_{\mu+1}^{\nu-1}\\
(P,Q,R)_{\mu+1}^{2} &= A_{\mu+1}^{2}(p, q, r)_{\mu+1}^{m(\mu+1)}+B_{\mu+1}^{2}(P,Q,R)_{\mu+1}+C_{\mu+1}^{2}(p, q, r)_{\mu+1}^{m(\mu+1)-1}
\end{aligned}\right\} \quad (8)$$

[the last term of the last equation being $C_{\mu+1}^{2}(P, Q, R)_{\mu}^{n(\mu)-1}$ when there is only one subsidiary point] where all the numbers A, B, a, b are integers, and all the c's and C are positive or negative unity.

If for completeness we introduce a point t^{1} such that $Ot^{1}T_{1}T_{1}^{2}$ is a unit tetrahedron, the equations (6, 7, 8) are sufficient to determine all the convergent and subsidiary points in terms of the first two T_{1}, T_{1}^{2}, the point t_{1}^{1}, and the sets of *partial multipliers* (A, B, C), (a, b, c). The set of multipliers $(A_{2}^{2}, B_{2}^{2}, C_{2}^{2})$ or $(a_{2}^{1}, b_{2}^{1}, c_{2}^{1})$, according as $OT_{1}T_{1}^{2}T_{2}^{2}$ is or is not a unit tetrahedron, is dependent on the choice of t^{1}, being such as to give rise to the subsequent unique convergent and subsidiary points.

The ratios $(P_{\mu}^{\nu} : Q_{\mu}^{\nu} : R_{\mu}^{\nu})$ give a rational approximation to the direction ratios

$$(\beta'\gamma''-\beta''\gamma' : \gamma'\alpha''-\gamma''\alpha' : \alpha'\beta''-\alpha''\beta')$$

of the axis of the cylinder $\Phi = \text{const.}$, and $(P_\mu^{\nu+1} : Q_\mu^{\nu+1} : R_\mu^{\nu+1})$ give a better approximation than $(P_\mu^\nu : Q_\mu^\nu : R_\mu^\nu)$ in that the line joining the origin to $(P_\mu^{\nu+1}, Q_\mu^{\nu+1}, R_\mu^{\nu+1})$ cuts the elliptic disc $\Phi = \Phi_\mu^\nu$, $U = U_\mu^\nu$ at a point inside it. The point $(P_\mu^{\nu+1}, Q_\mu^{\nu+1}, R_\mu^{\nu+1})$ is not necessarily nearer to the axis than $(P_\mu^\nu, Q_\mu^\nu, R_\mu^\nu)$, nor need the angle between the axis and $(P_\mu^{\nu+1} : Q_\mu^{\nu+1} : R_\mu^{\nu+1})$ be less than that between it and $(P_\mu^\nu : Q_\mu^\nu : R_\mu^\nu)$.

7. We next indicate a method of finding two consecutive convergent points T_1, T_1^2 and the subsequent sets of partial multipliers $(A, B, C)_\mu^\nu$, $(a, b, c)_\mu^\nu$.

First choosing two positive quantities U_0, Φ_0 such that the volume of the region bounded by the cylinder $\Phi = \Phi_0$ and the pair of planes $U = 0$, U_0 is four units, there must, by Minkowski's theorem, be at least one integer point and therefore at least one convergent point such that

$$0 < U < U_0, \quad \Phi < \Phi_0. \tag{9}$$

Now

$$\Phi = VW = [(\alpha' + \iota\alpha'')\, x + (\beta' + \iota\beta'')\, y + (\gamma' + \iota\gamma'')z] \times [(\alpha' - \iota\alpha'')\, x + (\beta' - \iota\beta'')y + (\gamma' - \iota\gamma'')z]$$

$$= (\alpha' x + \beta' y + \gamma' z)^2 + (\alpha'' x + \beta'' y + \gamma'' z)^2 = \bar{V}^2 + \bar{W}^2, \text{ say.}$$

Thus all the solutions of (9) are included among those of

$$0 < U < U_0, \ -\sqrt{\Phi_0} < \bar{V} < \sqrt{\Phi_0}, \ -\sqrt{\Phi_0} < \bar{W} < \sqrt{\Phi_0}. \tag{10}$$

Determining then the vertices of the parallelopiped bounded by the six planes

$$U = 0, \ U_0; \quad \bar{V} = \pm\sqrt{\Phi_0}; \quad \bar{W} = \pm\sqrt{\Phi_0},$$

within which all the integer points satisfying (10) lie, upper and lower limits for the coordinates of a convergent point are determined. The volume of this parallelopiped being finite, only a finite number of triads of values of (x, y, z) are possibilities for solutions. By rejecting all the triads which fail to satisfy (10) this finite number is reduced, and then a further reduction is made by throwing out such as fail to satisfy (9). Arranging those that remain in ascending sequence for U there must be one triad of integers (x_1, y_1, z_1), and may be more than one, such that there is no other triad (x', y', z') satisfying

$$0 < U' < U_1, \quad \Phi' < \Phi_1.$$

Such a triad $(x_1, y_1, z_1) \equiv (P_1, Q_1 R_1)$ gives, by definition, a first con-

vergent point T_1. If only one such triad exists we replace U_0 and Φ_0 by $U_0\Phi_0/\Phi_1$ and Φ_1 respectively and proceed as before to find the next one. The inequality for U may now, however, be written

$$U_0 < U < U_0\Phi_0/\Phi,$$

which reduces the number of trial solutions. It may happen that two or more convergent points are found among the first set of trial solutions, and when this is the case a saving of labour in the calculation is effected. When T_1, T_1^2 are found, the point t^1 is determined by the equation $|p^1Q_1R_1^2| = 1$, which must be soluble in integers, since $OT_1T_1^2$ is a unit triangle.

For the determination of the subsequent convergent and subsidiary points it is sufficient to calculate the sets of partial multipliers (A, B, C), (a, b, c). We shall suppose that the approximation has been carried as far as the determination of T_μ^ν, $T_\mu^{\nu+1}$, and show how to pass on to the next stage. Applying the integral unitary substitution

$$\left.\begin{aligned} x &= P_\mu^{\nu+1}x_\mu^{\nu+1} + P_\mu^\nu y_\mu^{\nu+1} + p^\mu z_\mu^{\nu+1} \\ y &= Q_\mu^{\nu+1}x_\mu^{\nu+1} + Q_\mu^\nu y_\mu^{\nu+1} + q^\mu z_\mu^{\nu+1} \\ z &= R_\mu^{\nu+1}x_\mu^{\nu+1} + R_\mu^\nu y_\mu^{\nu+1} + r^\mu z_\mu^{\nu+1} \end{aligned}\right\}, \tag{11}$$

to the equations (4) we have

$$(U, \bar{V}, \bar{W}) = (U, \bar{V}, \bar{W})_\mu^{\nu+1}x_\mu^{\nu+1} + (U, \bar{V}, \bar{W})_\mu^\nu y_\mu^{\nu+1} + (u, \bar{v}, \bar{w})^\mu z_\mu^{\nu+1}. \tag{12}$$

The inequalities to be satisfied by the convergent point succeeding $T_\mu^{\nu+1}$ are

$$U_\mu^{\nu+1} < U < U_0\Phi_0/\Phi_\mu^{\nu+1}, \qquad \Phi < \Phi_\mu^{\nu+1}.$$

Using the linear forms (12) and proceeding as before let the first set of values of $(x_\mu^{\nu+1}, y_\mu^{\nu+1}, z_\mu^{\nu+1})$ after (1, 0, 0) be $(\bar{x}_\mu^{\nu+1}, \bar{y}_\mu^{\nu+1}, \bar{z}_\mu^{\nu+1})$. Three cases arise and require separate discussion.

(i) $\bar{z}_\mu^{\nu+1} = 0$. The next convergent point lies on the plane $OT_\mu^\nu T_\mu^{\nu+1}$, and is therefore $T_\mu^{\nu+2}$. Hence, from (6),

$$(\bar{x}_\mu^{\nu+1}, \bar{y}_\mu^{\nu+1}, \bar{z}_\mu^{\nu+1}) = (A_\mu^{\nu+2}, B_\mu^{\nu+2}, 0),$$

where $B_\mu^{\nu+2} = \pm 1$, a check on the calculation.

The substitution to be applied for the next stage in the approximation is

$$\left.\begin{aligned} x &= P_\mu^{\nu+2}x_\mu^{\nu+2} + P_\mu^{\nu+1}y_\mu^{\nu+2} + p^\mu z_\mu^{\nu+2} \\ y &= Q_\mu^{\nu+2}x_\mu^{\nu+2} + Q_\mu^{\nu+1}y_\mu^{\nu+2} + q^\mu z_\mu^{\nu+2} \\ z &= R_\mu^{\nu+2}x_\mu^{\nu+2} + R_\mu^{\nu+1}y_\mu^{\nu+2} + r^\mu z_\mu^{\nu+2} \end{aligned}\right\},$$

which is compounded of (11) and the substitution

$$\left.\begin{aligned} x_\mu^{\nu+1} &= A_\mu^{\nu+2}x_\mu^{\nu+2}+y_\mu^{\nu+2} \\ y_\mu^{\nu+1} &= B_\mu^{\nu+2}x_\mu^{\nu+2} \\ z_\mu^{\nu+1} &= z_\mu^{\nu+2} \end{aligned}\right\}.$$

(ii) $\bar{z}_\mu^{\nu+1} = \pm 1$. In this, the normal, case $\nu+1 = n(\mu)$, so that

$$T_\mu^\nu = T_\mu^{n(\mu)-1}, \quad T_\mu^{\nu+1} = T_{\mu+1}, \quad t^{\mu+1} = T_\mu^{n(\mu)-1}.$$

We thus have, from (7),

$$(\bar{x}_\mu^{\nu+1}, \bar{y}_\mu^{\nu+1}, \bar{z}_\mu^{\nu+1}) = (A_{\mu+1}^2, B_{\mu+1}^2, C_{\mu+1}^2),$$

and the substitution to be applied for the next approximation is

$$\left.\begin{aligned} x &= P_{\mu+1}^2x_{\mu+1}^2+P_{\mu+1}y_{\mu+1}^2+p^{\mu+1}z_{\mu+1}^2 \\ y &= Q_{\mu+1}^2x_{\mu+1}^2+Q_{\mu+1}y_{\mu+1}^2+q^{\mu+1}z_{\mu+1}^2 \\ z &= R_{\mu+1}^2x_{\mu+1}^2+R_{\mu+1}y_{\mu+1}^2+r^{\mu+1}z_{\mu+1}^2 \end{aligned}\right\},$$

which is compounded of (11) and the substitution

$$\left.\begin{aligned} x_\mu^{\nu+1} &= A_{\mu+1}^2x_{\mu+1}^2+y_{\mu+1}^2 \\ y_\mu^{\nu+1} &= B_{\mu+1}^2x_{\mu+1}^2 \qquad +z_{\mu+1}^2 \\ z_\mu^{\nu+1} &= C_{\mu+1}^2x_{\mu+1}^2 \end{aligned}\right\}.$$

(iii) $\bar{z}_\mu^{\nu+1} \neq 0, \pm 1$. Again

$$\nu+1 = n(\mu), \quad T_\mu^\nu = T_\mu^{n(\mu)-1}, \quad T_\mu^{\nu+1} = T_{\mu+1}.$$

The tetrahedron $OT_\mu^{n(\mu)-1}T_{\mu+1}T_{\mu+1}^2$ is not unitary, however, and it is necessary to insert a set of subsidiary points

$$t_{\mu+1}^\nu \quad [\nu = 1, 2, \ldots, m(\mu+1)],$$

such that $OT_\mu^{n(\mu)-1}T_{\mu+1}t_{\mu+1}^1$, $Ot_{\mu+1}^\nu T_{\mu+1}t_{\mu+1}^{\nu+1}$ $[\nu = 1, 2, \ldots, m(\mu+1)-1]$, $Ot_{\mu+1}^{m(\mu+1)}T_{\mu+1}T_{\mu+1}^2$ are unit tetrahedra. We have already noted that the integer points T, such that $OT_\mu^{n(\mu)-1}T_{\mu+1}T$ is unitary, form the aggregate of integer points on the two integer planes nearest to $OT_\mu^{n(\mu)-1}T_{\mu+1}$ and on opposite sides of it. The point $t_{\mu+1}^1$ is chosen to lie on that plane which is on the same side of $OT_\mu^{n(\mu)-1}T_{\mu+1}$ as $T_{\mu+1}^2$. Of the integer points in this plane those which lie on the same side of $OT_\mu T_{\mu+1}^2$ as $T_\mu^{n(\mu)-1}$, and are nearer to this plane than any other, form the aggregate of integer points

on an integer line. The point $t^1_{\mu+1}$ is chosen to lie on this line and to be that point on it for which Φ is least.

With this choice of $t^1_{\mu+1}$ its coordinates are

$$(x^{\nu+1}_\mu,\ y^{\nu+1}_\mu,\ z^{\nu+1}_\mu) = (a^1_{\mu+1},\ b^1_{\mu+1},\ c^1_{\mu+1}),$$

where $c^1_{\mu+1} = +1$ or -1, according as $\bar{z}^{\nu+1}_\mu$ is positive or negative. The distance of this point from the plane $OT_{\mu+1}T^2_{\mu+1}$ whose equation is

$$y^{\nu+1}_\mu \bar{z}^{\nu+1}_\mu - z^{\nu+1}_\mu \bar{y}^{\nu+1}_\mu = 0$$

is proportional to $b^1_{\mu+1}\bar{z}^{\nu+1}_\mu - c^1_{\mu+1}\bar{y}^{\nu+1}_\mu$ and that of $T^{n(\mu)-1}_\mu$ $(0, 1, 0)$, from it to $\bar{z}^{\nu+1}_\mu$. Choosing $b^1_{\mu+1}$ in such a way that $b^1_{\mu+1}\bar{z}^{\nu+1}_\mu - c^1_{\mu+1}\bar{y}^{\nu+1}_\mu$ lies between O and $\bar{z}^{\nu+1}_\mu$, and then $a^1_{\mu+1}$ such that $\bar{V}^2 + \bar{W}^2$ is least, the point $t^1_{\mu+1}$ is determined.

Now, if $Ot^1_{\mu+1}T_{\mu+1}T^2_{\mu+1}$ is a unit tetrahedron, $t^1_{\mu+1}$ is the only subsidiary point to be inserted at this stage, but, if not, there must be at least one more. In general we define $t^{\nu+1}_{\mu+1}$ to be derived from $Ot^\nu_{\mu+1}T_{\mu+1}T^2_{\mu+1}$ in just the same way as $t^1_{\mu+1}$ was derived from $OT^{n(\mu)-1}_\mu T_{\mu+1}T^2_{\mu+1}$. The tetrahedra $OT^{n(\mu)-1}_\mu T_{\mu+1}T^2_{\mu+1}$, $Ot^1_{\mu+1}T_{\mu+1}T^2_{\mu+1}$, $Ot^2_{\mu+1}T_{\mu+1}T^2_{\mu+1}$, ..., are then all integral multiples of the unit tetrahedron in volume, and each is less than its predecessor. Hence a subsidiary point $t^{m(\mu+1)}_{\mu+1}$ must eventually be arrived at such that $Ot^{m(\mu+1)}_{\mu+1}T_{\mu+1}T^2_{\mu+1}$ is unitary.

The relations connecting the coordinates of the successive subsidiary points are given by the equations (8) above, and in this case

$$t_\mu = t^1_{\mu+1}, \quad t^{\mu+1} = t^{m(\mu+1)}_{\mu+1}.$$

The substitution to be applied for the next stage in the approximation is

$$\left.\begin{aligned} x &= P^2_{\mu+1}x^2_{\mu+1} + P_{\mu+1}y^2_{\mu+1} + p^{\mu+1}z^2_{\mu+1} \\ y &= Q^2_{\mu+1}x^2_{\mu+1} + Q_{\mu+1}y^2_{\mu+1} + q^{\mu+1}z^2_{\mu+1} \\ z &= R^2_{\mu+1}x^2_{\mu+1} + R_{\mu+1}y^2_{\mu+1} + r^{\mu+1}z^2_{\mu+1} \end{aligned}\right\},$$

which is compounded of (11) and the substitutions

$$\left.\begin{aligned} x^{\nu+1}_\mu &= a^1_{\mu+1}\xi^1_{\mu+1} + \eta^1_{\mu+1} \\ y^{\nu+1}_\mu &= b^1_{\mu+1}\xi^1_{\mu+1} + \xi^1_{\mu+1} \\ z^{\nu+1}_\mu &= c^1_{\mu+1}\xi^1_{\mu+1} \end{aligned}\right\},$$

$$\left.\begin{aligned} \xi^1_{\mu+1} &= a^2_{\mu+1}\xi^2_{\mu+1} &&+\xi^2_{\mu+1} \\ \eta^1_{\mu+1} &= b^2_{\mu+1}\xi^2_{\mu+1}+\eta^2_{\mu+1} \\ \zeta^1_{\mu+1} &= c^2_{\mu+1}\xi^2_{\mu+1} \end{aligned}\right\},$$

$$\left.\begin{aligned} \xi^\nu_{\mu+1} &= a^{\nu+1}_{\mu+1}\xi^{\nu+1}_{\mu+1} &&+\xi^{\nu+1}_{\mu+1} \\ \eta^\nu_{\mu+1} &= b^{\nu+1}_{\mu+1}\xi^{\nu+1}_{\mu+1}+\eta^{\nu+1}_{\mu+1} \\ \zeta^\nu_{\mu+1} &= c^{\nu+1}_{\mu+1}\xi^{\nu+1}_{\mu+1} \end{aligned}\right\},$$

$$\left.\begin{aligned} \xi^{m(\mu+1)}_{\mu+1} &= A^2_{\mu+1}x^2_{\mu+1} &&+z^2_{\mu+1} \\ \eta^{m(\mu+1)}_{\mu+1} &= B^2_{\mu+1}x^2_{\mu+1}+y^2_{\mu+1} \\ \zeta^{m(\mu+1)}_{\mu+1} &= C^2_{\mu+1}x^2_{\mu+1} \end{aligned}\right\}.$$

8. Some simplification can be introduced into the calculation when the numbers α, β, γ form a base of an ideal in $k(\theta)$ and $(\alpha'+\iota\alpha'', \beta'+\iota\beta'', \gamma'+\iota\gamma'')$, $(\alpha'-\iota\alpha'', \beta'-\iota\beta'', \gamma'-\iota\gamma'')$ are the conjugate ideals in $k(\theta')$, $k(\theta'')$. In the first place no generality is lost in supposing α is rational and thus identical with $\alpha'+\iota\alpha''$ and $\alpha'-\iota\alpha''$. For the sequence of convergent points to the axis of $\Phi = \text{const.}$ by the plane $U = \text{const.}$ is exactly the same as that to the axis of $\lambda\Phi = \text{const.}$ by the plane $\kappa U = \text{const.}$ Taking

$$\kappa = (\alpha'+\iota\alpha'')(\alpha'-\iota\alpha''), \quad \lambda = \alpha(\alpha'-\iota\alpha'')\,\alpha(\alpha'+\iota\alpha''),$$

the leading coefficient of each of the forms U, V, W becomes $N(\alpha)$. We can then take, with a slight change in notation,

$$\begin{aligned} \kappa U &= \alpha x+\beta y+\gamma z, \\ \lambda\Phi &= (\alpha x+\beta' y+\gamma' z)^2+(\beta'' y+\gamma'' z)^2 \\ &= \bar{V}^2+\bar{W}^2, \end{aligned}$$

where

$$\begin{aligned} \bar{V} &= \alpha x+\beta' y+\gamma' z, \\ \bar{W} &= \sqrt{(\beta''^2)}\,[y+(\gamma''/\beta'')\,z]. \end{aligned}$$

In this form each of the coefficients β', γ', β''^2, γ''/β'' is rational in $k(\theta)$, so that it is unnecessary to calculate approximations to the numbers θ', θ'', a considerable saving of labour.

The volume of the region bounded by the planes $U = 0$, U_0 and the cylinder $\Phi = \Phi_0$ is found by direct integration to be

$$2\pi U_0 \Phi_0 / N(\alpha, \beta, \gamma)\,\Delta^{\frac{1}{2}}. \tag{13}$$

It may be added that in the numerical examples worked by the writer the number of consecutive convergent points coplanar with the origin was found to be usually two only and seldom more than four. There is no difficulty, however, in constructing a numerical case in which any assigned number of successive convergent points lie on a plane through the origin. Again the tetrahedron $OT_\mu^\nu T_\mu^{\nu+1} T_{\mu+1}^2$ is nearly always unitary and, when it is not, one subsidiary point was usually found sufficient to complete the unitary system. As before, however, a numerical case can be constructed in which any assigned number of subsidiary points must be inserted to obtain the set of unitary substitutions. These remarks are merely added to show what may be expected in a numerical application to the ideal theory.

9. After discussing these preliminary details, the actual solution of the problem which suggested the title of the present memoir presents but little difficulty.

An ideal $\mathfrak{b} \equiv (\alpha, \beta, \gamma)$ possesses two rational invariants, its norm $N(\mathfrak{b})$, and the smallest positive rational integer contained in it, which we shall denote by $L(\mathfrak{b})$. It is well known that the ideal must possess a base of the form $[L(\mathfrak{b}), \beta_0, \gamma_0]$. We define the ideal to be *primary* when it is not divisible by any rational integer. No generality is lost in supposing the ideals considered to be primary, for rational factors, when present, can be removed. Again, when the ideal is primary, we see at once, by resolving it into prime factors, that

$$L(\mathfrak{b}) \leqslant N(\mathfrak{b}) \leqslant L^2(\mathfrak{b}). \tag{14}$$

If the complex integers α, β, γ form a base of the ideal, the most general triad of numbers forming a base is

$$p_1\alpha+q_1\beta+r_1\gamma, \quad p_2\alpha+q_2\beta+r_2\gamma, \quad p_3\alpha+q_3\beta+r_3\gamma,$$

where $|\, p_1\, q_2\, r_3\,|$ is an integral unit determinant. We adopt a unique set of base integers in the following way.

Let the convergent points to the axis of the cylinder

$$[(\alpha'+\iota\alpha'')\,x+(\beta'+\iota\beta'')\,y+(\gamma'+\iota\gamma'')\,z]$$
$$\times[(\alpha'-\iota\alpha'')\,x+(\beta'-\iota\beta'')\,y+(\gamma'-\iota\gamma'')\,z] = \text{const.},$$

by the plane $$\alpha x+\beta y+\gamma z = \text{const.}$$

be found, and let T_1 be the unique convergent point which satisfies the inequalities

$$U_1 \leqslant L(\alpha, \beta, \gamma), \qquad U_1^2 > L(\alpha, \beta, \gamma).$$

Now applying the unitary substitution

$$S_1 \equiv \begin{pmatrix} P_1 & Q_1 & R_1 \\ P_1^2 & Q_1^2 & R_1^2 \\ p_1 & q_1 & r_1 \end{pmatrix},$$

we have

$$\mathfrak{b} = (P_1\alpha+Q_1\beta+R_1\gamma, \quad P_1^2\alpha+Q_1^2\beta+R_1^2\gamma, \quad p_1\alpha+q_1\beta+r_1\gamma)$$
$$= (U_1, U_1^2, u_1).$$

This expression (U_1, U_1^2, u_1), is to be the *standard form* of the ideal. It should be observed that U_1, U_1^2, u_1 are definite integers of $k(\theta)$ and in no way dependent on the choice of minimalbasis.

The standard form of the ideal formed by the modulus of integers of the corpus is $(1, \Omega_1, \Omega_2)$ and is defined to be the *canonical base.*

The integer Ω_1 is defined to be the *canonical integer*, and the rational cubic equation satisfied by it the *canonical equation* of the corpus.

The *canonical form* of a number in $k(\theta)$ is defined to be $p+q\Omega_1+r\Omega_2$, where p, q, r are rational.

The ideal $\mathfrak{b}$ is said to be expressed in its *canonical standard form* when U_1, U_1^2, u_1 are expressed in canonical form.

Continuing the approximation which gives rise to the standard form of $\mathfrak{b}$, we have the infinite sequence of convergent points

$$T_1, \ T_1^2, \ \ldots, \ T_\mu^\nu, \ T_\mu^{\nu+1}, \ \ldots,$$

and the corresponding set of unitary substitutions,

$$S_1, \ S_1^2, \ \ldots \ S_\mu^\nu \equiv \begin{pmatrix} P_\mu^\nu, & Q_\mu^\nu, & R_\mu^\nu \\ P_\mu^{\nu+1}, & Q_\mu^{\nu+1}, & R_\mu^{\nu+1} \\ p_\mu, & q_\mu, & r_\mu \end{pmatrix}, \ S_\mu^{\nu+1}, \ \ldots .$$

Applying S_μ^ν, we have

$$\mathfrak{b} = S_\mu^\nu \mathfrak{b} = (U_\mu^\nu, U_\mu^{\nu+1}, u_\mu).$$

Now, if F_μ^ν is the greatest rational integer which divides $U_\mu'^\nu U_\mu''^\nu U_\mu^\nu$, $U_\mu'^\nu$, $U_\mu''^\nu U_\mu^{\nu+1}$, $U_\mu'^\nu U_\mu''^\nu u_\mu$, the quotients α_μ^ν, β_μ^ν, γ_μ^ν must form a base of a primary ideal. We shall denote this ideal by $G(U_\mu^\nu, U_\mu^{\nu+1}, u_\mu)$, so that

$$G(U_\mu^\nu, U_\mu^{\nu+1}, u_\mu) = \frac{U_\mu'^\nu U_\mu''^\nu}{F_\mu^\nu} \mathfrak{b} = (\alpha_\mu^\nu, \beta_\mu^\nu, \gamma_\mu^\nu) = \mathfrak{b}_\mu^\nu. \qquad (15)$$

Now since $U'^{\nu}_{\mu} U_{\mu}{}^{\nu} U^{\nu}_{\mu}$ and F^{ν}_{μ} are rational, α^{ν}_{μ} is rational, and α^{ν}_{μ}, β^{ν}_{μ}, γ^{ν}_{μ} being a set of base integers of $\mathfrak{b}^{\nu}_{\mu}$, it follows that

$$\alpha^{\nu}_{\mu} = L(\mathfrak{b}^{\nu}_{\mu}).$$

Again, the points (0, 1, 0)(1, 0, 0) are consecutive convergent points to the axis of the cylinder

$$[\alpha^{\nu}_{\mu} x + (\beta'^{\nu}_{\mu} + \iota\beta''^{\nu}_{\mu}) y + (\gamma'^{\nu}_{\mu} + \iota\gamma''^{\nu}_{\mu}) z][\alpha^{\nu}_{\mu} x + (\beta'^{\nu}_{\mu} - \iota\beta''^{\nu}_{\mu}) y + (\gamma'^{\nu}_{\mu} - \iota\gamma''^{\nu}_{\mu}) z] = \text{const.},$$

by the plane $\alpha^{\nu}_{\mu} x + \beta^{\nu}_{\mu} y + \gamma^{\nu}_{\mu} z = \text{const.},$

and (0, 0, 1) their associate, since T^{ν}_{μ}, $T^{\nu+1}_{\mu}$ are consecutive convergent points to the axis of $\Phi = \text{const.}$ by the plane $U = \text{const.}$ and t_{μ} their associate. The ideal $(\alpha^{\nu}_{\mu}, \beta^{\nu}_{\mu}, \gamma^{\nu}_{\mu})$ is therefore expressed in its standard form. Hence the region bounded by the cylinder

$$[\alpha^{\nu}_{\mu} x + (\beta'^{\nu}_{\mu} + \iota\beta''^{\nu}_{\mu}) y + (\gamma'^{\nu}_{\mu} + \iota\gamma''^{\nu}_{\mu}) z][\alpha^{\nu}_{\mu} x + (\beta'^{\nu}_{\mu} - \iota\beta''^{\nu}_{\mu}) y + (\gamma'^{\nu}_{\mu} - \iota\gamma''^{\nu}_{\mu}) z] = (\alpha^{\nu}_{\mu})^2,$$

and the planes $\alpha^{\nu}_{\mu} x + \beta^{\nu}_{\mu} y + \gamma^{\nu}_{\mu} z = 0,\ \alpha^{\nu}_{\mu},$

having no integer point within it, must be less than four units in volume, so that, from (13),

$$2\pi (\alpha^{\nu}_{\mu})^3 / N(\alpha^{\nu}_{\mu}, \beta^{\nu}_{\mu}, \gamma^{\nu}_{\mu}) \sqrt{\Delta} < 4,$$

or $$\frac{L^2(\mathfrak{b}^{\nu}_{\mu})}{N(\mathfrak{b}^{\nu}_{\mu})} L(\mathfrak{b}^{\nu}_{\mu}) < \frac{2}{\pi} \sqrt{\Delta} \text{ which } = M.$$

But since, from (14), $$N(\mathfrak{b}^{\nu}_{\mu}) \leqslant L^2(\mathfrak{b}^{\nu}_{\mu}),$$

we have $$L(\mathfrak{b}^{\nu}_{\mu}) < M. \tag{16}$$

Now there is only a finite number of ideals $\mathfrak{q}$ such that $L(\mathfrak{q}) < M$, hence the set of ideals $\mathfrak{b}_1, \mathfrak{b}_1^2, \ldots, \mathfrak{b}^{\nu}_{\mu}$ must become periodic. We thus obtain the chain of equivalent ideals

$$[\mathfrak{b} \sim] \mathfrak{b}_1 \sim \mathfrak{b}_1^2 \sim \ldots \sim \mathfrak{b}^{\nu}_{\mu} \sim \mathfrak{b}^{\nu+1}_{\mu} \sim \ldots \sim \mathfrak{b}^{\nu'}_{\mu'} \sim \mathfrak{b}^{\nu+1}_{\mu} \sim \ldots,$$

and since every convergent point in the approximation succeeds a definite convergent point it follows that $\mathfrak{b}^{\nu}_{\mu} = \mathfrak{b}^{\nu'}_{\mu'}$. The cycle of ideals

$$\mathfrak{b}^{\nu+1}_{\mu} \sim \ldots \sim \mathfrak{b}^{\nu}_{\mu}$$

is therefore closed. An ideal which belongs to this closed cycle is defined

to be *reduced*. We are thus able to enunciate the theorem:

Every reduced ideal is primary, and the necessary and sufficient condition for a primary ideal to be reduced is that the leading integer α in its standard form (α, β, γ) shall be a rational number.

The relation connecting two equivalent reduced ideals is found from the equation (15). For since

$$\mathfrak{b}_\mu^\nu = \frac{U_\mu'^\nu U_\mu''^\nu}{F_\mu^\nu}\mathfrak{b}, \qquad \mathfrak{b}_{\mu'}^{\nu'} = \frac{U_{\mu'}'^{\nu'} U_{\mu'}''^{\nu'}}{F_{\mu'}^{\nu'}},$$

we have

$$\frac{\mathfrak{b}_\mu^\nu}{\mathfrak{b}_{\mu'}^{\nu'}} = \frac{U_\mu'^\nu U_\mu''^\nu F_{\mu'}^{\nu'}}{U_{\mu'}'^{\nu'} U_{\mu'}''^{\nu'} F_\mu^\nu} = \frac{F_{\mu'}^{\nu'} N(U_\mu^\nu)}{F_\mu^\nu N(U_{\mu'}^{\nu'})} \frac{U_{\mu'}^{\nu'}}{U_\mu^\nu}. \tag{17}$$

In particular suppose that $\mathfrak{b}_\mu^\nu = \mathfrak{b}_{\mu'}^{\nu'}$. Then, from (15),

$$\frac{U_\mu^{\nu'}}{U_\mu^\nu} = \frac{U_{\mu'}^{\nu'+1}}{U_\mu^{\nu+1}} = \frac{u_{\mu'}}{u_\mu} = \eta, \text{ say,}$$

so that

$$(U_{\mu'}^{\nu'}, U_{\mu'}^{\nu'+1}, u_{\mu'}) = \eta (U_\mu^\nu, U_\mu^{\nu+1}, u_\mu),$$

and, since the two ideals $(U_{\mu'}^{\nu'}, U_{\mu'}^{\nu'+1}, u_{\mu'})$, $(U_\mu^\nu, U_\mu^{\nu+1}, u_\mu)$ are identical, η must be a unit of the corpus. Further, if the ideals $\mathfrak{b}_\mu^{\nu+1}, \ldots, \mathfrak{b}_\mu^\nu$ form one complete cycle, this unit is fundamental; for if ϵ be the fundamental unit, (P, Q, R) a convergent point, and

$$\epsilon(P\alpha+Q\beta+R\gamma) = P'\alpha+Q'\beta+R'\gamma,$$

the point (P', Q', R') must also be a convergent point.

Again, since from (15),

$$\beta_\mu^\nu/\alpha_\mu^\nu = U_\mu^{\nu+1}/U_\mu^\nu,$$

we have

$$\epsilon = \frac{\beta_\mu^{\nu+1} \ldots \beta_\mu^\nu}{\alpha_\mu^{\nu+1} \ldots \alpha_\mu^\nu}. \tag{18}$$

It was found most convenient in numerical applications to apply the operations G before the calculation of the sets of multipliers. From the equations (12) above it is seen that the integers U_μ^ν, $U_\mu^{\nu+1}$, u^μ form a base of the ideal $(U_\mu^\nu, U_\mu^{\nu+1}, u_\mu)$. Now writing

$$\mathfrak{u}_\mu^\nu \equiv \frac{U_\mu'^\nu U_\mu''^\nu}{F_\mu^\nu}(U_\mu^\nu x + U_\mu^{\nu-1} y + u^\mu z)$$

in place of U with corresponding expressions for the conjugate forms V, W, the leading coefficients of the forms become rational. If the convergent point succeeding T_μ^ν [(1, 0, 0) in the new system of coordinates] is (p, q, r), and their associate is (p', q', r'), the ideal $\mathfrak{b}_\mu^\nu$ is expressed in its standard form by applying the substitution $\begin{pmatrix} 1, & 0, & 0 \\ p, & q, & r \\ p', & q', & r' \end{pmatrix}$ to the coefficients of u_μ^ν.

The class-symbols of the corpus are commutative and define an Abelian group. For the formation of the cycles of reduced ideals the following method of procedure is adopted. First taking any ideal $\mathfrak{b}$ such that $L(\mathfrak{b}) < M$, the chain C of reduced ideals equivalent to it is calculated. The chains C^2, C^3, ... equivalent to $\mathfrak{b}^2$, $\mathfrak{b}^3$, ... are then formed in succession, and the number of reduced ideals being finite, the chain C^m of principal ideals must be arrived at in a finite number of steps. Next taking a reduced ideal $\mathfrak{d}$ not contained in the chains already formed, cycles equivalent to

$$\mathfrak{b}^i\mathfrak{d},\ \mathfrak{b}^i\mathfrak{d}^2,\ \ldots,\ \mathfrak{b}^i\mathfrak{d}^{n-1} \quad (i = 0, 1, \ldots, m-1)$$

are constructed, and, if $\mathfrak{d}^n$ is the lowest power of $\mathfrak{d}$ equivalent to a power of $\mathfrak{b}$, all these are distinct. The process is now continued in the same way by taking another reduced ideal and combining it and its powers with ideals in all the cycles previously calculated, and, the number of reduced ideals being finite, all the cycles must be found in a finite number of operations.

10. To complete the investigation it remains to be seen that two equivalent reduced ideals must belong to the same cycle.

Let the two reduced ideals, (α, β, γ), $(\alpha_0, \beta_0, \gamma_0)$, expressed in their standard form, be connected by the equation

$$(h\alpha, h\beta, h\gamma) = \begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{pmatrix} (h_0\alpha_0, h_0\beta_0, h_0\gamma_0),$$

where $|p_1 q_2 r_3|$ is an integral unit determinant, and h, h_0 are integers in $k(\theta)$. Then since $(\alpha_0, \beta_0, \gamma_0)$ is reduced, the points (p_1, q_1, r_1), (p_2, q_2, r_2) must be consecutive convergent points to the axis of $\Phi = \text{const.}$ by the plane $U = \text{const.}$ and (p_3, q_3, r_3) their associate. Hence the result follows.

11. We conclude with the discussion of some numerical examples.

(i) $$\theta^3+\theta-3=0,$$

$$\theta = 1{\cdot}21341, \qquad \Delta = -d = -d(\theta) = 247,$$

$$\theta^2 = 1{\cdot}47237, \qquad M = \frac{2}{\pi}\sqrt{\Delta} = 10{\cdot}0051.$$

Woronoj's minimalbasis is $(1, \theta, \theta^2)$.

Putting $$U = x+y\theta+z\theta^2,$$

we have $$\Phi = (x+y\theta'+z\theta'^2)(x+y\theta''+z\theta''^2)$$

$$= [x-\tfrac{1}{2}\theta y-\tfrac{1}{2}(2+\theta^2)z]^2+(3\theta^2+4)(\tfrac{1}{2}y-\tfrac{1}{2}\theta z)^2 = \bar{V}^2+\bar{W}^2.$$

The first convergent point is (1, 0, 0) and the inequalities satisfied by the next are

$$1 < U < M = 10{\cdot}0051, \quad \Phi < 1.$$

Replacing the second of these by

$$-1 < \bar{V} < 1, \quad -1 < \bar{W} < 1,$$

we find
$$1 < x+1{\cdot}21341y+1{\cdot}47237z < 10{\cdot}0051,$$
$$-1 < x- \ {\cdot}60670y-1{\cdot}73618z < 1,$$
$$-1 < \quad 1{\cdot}45139y-1{\cdot}76019z < 1.$$

Solving for the corners of the parallelopiped, we find

$$0 \leqslant x \leqslant 5, \quad 0 \leqslant y \leqslant 2, \quad 0 \leqslant z \leqslant 2,$$

and the only solution within these limits is seen without difficulty to be (2, 1, 1). From the value of Φ above, we find

$$\Phi(2, 1, 1) = -1+\theta,$$

whence $$N(2+\theta+\theta^2) = (2+\theta+\theta^2)(-1+\theta) = 1.$$

The primitive unit of the corpus is thus $2+\theta+\theta^2$, and the next convergent point is (P_2, Q_2, R_2), where

$$P_2+Q_2\theta+R_2\theta^2 = (2+\theta+\theta^2)^2 = 10+5\theta+4\theta^2.$$

Now, since
$$\begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 10 & 5 & 4 \end{vmatrix} = -1,$$

the standard form of the ideal determined by the modulus of integers of the corpus $(1, \theta, \theta^2)$ is $(1, 2+\theta+\theta^2, 10+5\theta+4\theta^2)$.

Again the i-th convergent point is given by

$$P_i+Q_i\theta+R_i\theta^2 = (2+\theta+\theta^2)^i.$$

The canonical integer of the corpus is

$$\Omega \equiv 2+\theta+\theta^2;$$

its canonical equation $\quad \Omega^3-4\Omega^2-3\Omega-1 = 0;$

and the standard canonical form of the above reduced ideal $(1, \Omega, \Omega^2)$.

The only prime ideals $\mathfrak{b}$ such that $L(\mathfrak{b}) < 11$, are

$$(3, \theta, \theta^2), \quad (3, 3\theta, 1+\theta^2), \quad (5, 1+\theta, 4+\theta^2),$$

$$(5, 5\theta, 2-\theta+\theta^2), \quad (7, 5+\theta, 3+\theta^2), \quad (7, 7\theta, 5+2\theta+\theta^2).$$

Without undergoing the labour of reducing them, however, it is easily seen that all are principal. For, since $N(\theta) = 3$, the ideal $(3, \theta, \theta^2)$ is principal, and therefore $(3, 3\theta, 1+\theta^2)$ is also principal. Similarly, since $N(1+\theta) = 5$, $N(2-\theta) = 7$, the remaining four ideals are principal. Hence there is only one cycle of reduced ideals and the class number is unity.

(ii) $$\theta^3-23 = 0,$$

$$\theta = 2{\cdot}84387, \qquad \Delta = -d = 27 \,.\, 23^2,$$

$$\theta^2 = 8{\cdot}08758, \qquad M = \frac{2}{\pi}\, 3\sqrt{3} \,.\, 23 = 76{\cdot}08339.$$

In this example we shall adopt the minimalbasis $(1, \theta, \theta^2)$, which is not canonical, thereby reducing the calculation to a minimum.

The following table gives the successive u's and the partial multipliers derived from them :—

$$\begin{aligned}
\mathfrak{u}_1 &= x + \theta y + \theta^2 z && (8, 3, 1),\\
\mathfrak{u}_2 &= 6x+(-5-\theta+\theta^2)y+(23-5\theta-\theta^2)z && (6, 0, -1),\\
\mathfrak{u}_3 &= 31x+(9-7\theta+2\theta^2)y+(-42+12\theta+\theta^2)z && (1, 1, 1),\\
\mathfrak{u}_4 &= 20x+(-11+7\theta+\theta^2)y+(2+6\theta-2\theta^2)z && (1, 1, -1),\\
\mathfrak{u}_5 &= 33x+(-1+10\theta-\theta^2)y+(4-7\theta+4\theta^2)z && (0, 1, 1),\\
\mathfrak{u}_6 &= 6x+(-3+3\theta)y+(-2-\theta+\theta^2)z && (2, 1, 1),\\
\mathfrak{u}_7 &= 5x+(1+3\theta-\theta^2)y+(-12-\theta+2\theta^2)z && (3, 1, 1),\\
\mathfrak{u}_7^2 &= 3x+(-2+\theta)y+(-5-\theta+\theta^2)z && (1, 1, 0),
\end{aligned}$$

$$
\begin{aligned}
u_8 &= 8x+(\ \ 1-\ \ \theta+\ \theta^2)y+(-17+\ 9\theta-\ \theta^2)z && (3,\ 3,\ 1),\\
u_8^2 &= 30x+(-22+\ 4\theta+2\theta^2)y+(\ \ 3+\ 9\theta-3\theta^2)z && (1,\ 1,\ 0),\\
u_9 &= 2x+(-\ 4+\ 2\theta\ \ \ \ )y+(-\ 5-\ \ \theta+\ \theta^2)z && (10,\ 2,\ 1),\\
u_9^2 &= 11x+(-10+\ \ \theta+\ \theta^2)y+(\ 43-12\theta-\ \theta^2)z && (1,\ 1,\ 0),\\
u_{10} &= 2x+(-\ 1+\ \ \theta\ \ \ \ )y+(-\ 6+\ 5\theta-\ \theta^2)z && (5,\ 8,\ -1).\\
u_{11} &= 3x+(-\ 5-\ \ \theta+\ \theta^2)y+(\ 14-\ 2\theta-\ \theta^2)z && (7,\ 0,\ -1),\\
u_{12} &= 10x+(\ \ 1+\ 3\theta-\ \theta^2)y+(\ 29-13\theta+\ \theta^2)z && (4,\ -3,\ -1),\\
u_{12}^2 &= 6x+(-\ 4+\ 2\theta\ \ \ \ )y+(-\ 5-\ \ \theta+\ \theta^2)z && (1,\ 1,\ 0),\\
u_{13} &= 16x+(\ \ 2-\ 2\theta+2\theta^2)y+(-17+\ 9\theta-\ \theta^2)z && (2,\ 1,\ 1),\\
u_{14} &= 10x+(\ \ 4-\ 3\theta+\ \theta^2)y+(-11+\ 2\theta+\ \theta^2)z && (1,\ 0,\ 1),\\
u_{14}^2 &= 17x+(-\ 9+\ 5\theta+\ \theta^2)y+(\ \ 1+\ 7\theta-2\theta^2)z && (1,\ 1,\ 0),\\
u_{15} &= 4x+(-\ 3-\ \ \theta+\ \theta^2)y+(\ 12-\ 4\theta\ \ \ \ )z && (5,\ 1,\ -1),\\
u_{15}^2 &= 15x+(-11+\ 2\theta+\ \theta^2)y+(\ 14+\ 7\theta-4\theta^2)z && (1,\ 1,\ 0),\\
u_{16} &= \ \ x+(-\ 2+\ \ \theta\ \ \ \ )y+(-\ 8+\ \ \ \ \ \theta^2)z, &&
\end{aligned}
$$

after which the expansion becomes periodic.

It will be observed that in this example no subsidiary points are required, and that in only six cases do three consecutive convergent points lie in a plane through the origin. Using the values of the partial multipliers as calculated the coordinates of the convergent points are found to be as follow :—

	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_7^2
P	1	8	48	57	97	105	364	1294
Q	0	3	17	20	34	37	128	455
R	0	1	6	7	12	13	45	160

	T_8	T_8^2	T_9	T_9^2	T_{10}
P	1658	8961	10619	125406	136025
Q	583	3151	3734	44097	47831
R	205	1108	1313	15506	16819

	T_{11}	T_{12}	T_{12}^2
P	1674412	11595478	41222651
Q	588780	4077363	14495281
R	207035	1433739	5097032

	T_{13}	T_{14}	T_{14}^2
P	52818129	148533321	189755972
Q	18572644	52229349	66724630
R	6530771	18365609	23462641

	T_{15}	T_{15}^2	T_{16}
P	338289293	1828384308	2166673601
Q	118953979	642921881	761875860
R	41828250	226073120	267901370.

Accordingly the primitive unit of the corpus is

$$\epsilon \equiv 2166673601+761875860\theta+267901370\theta^2,$$

the reciprocal of which is found to be

$$\epsilon^{-1} \equiv -41399-3160\theta+6230\theta^2.$$

The cycle of principal reduced ideals is obtained by applying to $(1, \theta, \theta^2)$ the operations

$$G\begin{Bmatrix} 1, & 0, & 0 \\ 8, & 3, & 1 \\ 48, & 17, & 6 \end{Bmatrix}, \qquad G\begin{Bmatrix} 0, & 1, & 0 \\ 1, & 0, & 0 \\ 1, & 1, & 1 \end{Bmatrix}\begin{Bmatrix} 1, & 0, & 0 \\ 8, & 3, & 1 \\ 48, & 17, & 6 \end{Bmatrix},$$

$$G\begin{Bmatrix} 0, & 1, & 0 \\ 1, & 0, & 0 \\ 1, & 1, & -1 \end{Bmatrix}\begin{Bmatrix} 0, & 1, & 0 \\ 1, & 0, & 0 \\ 1, & 1, & 1 \end{Bmatrix}\begin{Bmatrix} 1, & 0, & 0 \\ 8, & 3, & 1 \\ 48, & 17, & 6 \end{Bmatrix},$$

and so on in succession, or more simply by applying to the coefficients of $u_1, u_2, \ldots, u_{15}^2$ the substitutions

$$\begin{Bmatrix} 1, & 0, & 0 \\ 8, & 3, & 1 \\ 48, & 17, & 6 \end{Bmatrix}, \quad \begin{Bmatrix} 1, & 0, & 0 \\ 6, & 0, & -1 \\ 7, & 1, & -1 \end{Bmatrix}, \quad \begin{Bmatrix} 1, & 0, & 0 \\ 1, & 1, & 1 \\ 2, & 0, & 1 \end{Bmatrix}, \quad \begin{Bmatrix} 1, & 0, & 0 \\ 1, & 1, & -1 \\ 1, & 1, & 0 \end{Bmatrix},$$

$$\begin{Bmatrix} 1, & 0, & 0 \\ 0, & 1, & 1 \\ 1, & 3, & 2 \end{Bmatrix}, \quad \begin{Bmatrix} 1, & 0, & 0 \\ 2, & 1, & 1 \\ 7, & 4, & 3 \end{Bmatrix}, \quad \begin{Bmatrix} 1, & 0, & 0 \\ 3, & 1, & 1 \\ 21, & 7, & 6 \end{Bmatrix}, \quad \begin{Bmatrix} 1, & 0, & 0 \\ 1, & 1, & 0 \\ 6, & 3, & 1 \end{Bmatrix},$$

$$\left\{\begin{matrix}1, & 0, & 0\\ 3, & 3, & 1\\ 46, & 37, & 12\end{matrix}\right\},\quad \left\{\begin{matrix}1, & 0, & 0\\ 1, & 1, & 0\\ 12, & 10, & 1\end{matrix}\right\},\quad \left\{\begin{matrix}1, & 0, & 0\\ 10, & 2, & 1\\ 135, & 25, & 13\end{matrix}\right\},\quad \left\{\begin{matrix}1, & 0, & 0\\ 1, & 1, & 0\\ 13, & 5, & -1\end{matrix}\right\},$$

$$\left\{\begin{matrix}1, & 0, & 0\\ 5, & 8, & -1\\ 35, & 55, & -7\end{matrix}\right\},\quad \left\{\begin{matrix}1, & 0, & 0\\ 7, & 0, & -1\\ 25, & -1, & -4\end{matrix}\right\},\quad \left\{\begin{matrix}1, & 0, & 0\\ 4, & -3, & -1\\ 14, & -8, & -3\end{matrix}\right\},\quad \left\{\begin{matrix}1, & 0, & 0\\ 1, & 1, & 0\\ 3, & 2, & 1\end{matrix}\right\},$$

$$\left\{\begin{matrix}1, & 0, & 0\\ 2, & 1, & 1\\ 2, & 2, & 1\end{matrix}\right\},\quad \left\{\begin{matrix}1, & 0, & 0\\ 1, & 0, & 1\\ 11, & -1, & 6\end{matrix}\right\},\quad \left\{\begin{matrix}1, & 0, & 0\\ 1, & 1, & 0\\ 6, & 5, & -1\end{matrix}\right\},$$

$$\left\{\begin{matrix}1, & 0, & 0\\ 5, & 1, & -1\\ 147, & 26, & -25\end{matrix}\right\},\quad \left\{\begin{matrix}1, & 0, & 0\\ 1, & 1, & 0\\ 25, & 22, & 1\end{matrix}\right\},$$

respectively, and is

$$(1, 8+3\theta+\theta^2, 48+17\theta+6\theta^2) \sim (6, 13+5\theta+\theta^2, 14+4\theta+2\theta^2)$$
$$\sim (31, -2+5\theta+3\theta^2, 20+12\theta+\theta^2) \sim (20, 7+\theta+3\theta^2, 9+7\theta+\theta^2)$$
$$\sim (33, 3+3\theta+3\theta^2, 38+16\theta+5\theta^2) \sim (6, 7+2\theta+\theta^2, 24+9\theta+3\theta^2)$$
$$\sim (5, 4+2\theta+\theta^2, 40+15\theta+5\theta^2) \sim (3, 1+\theta, 7+2\theta+\theta^2)$$
$$\sim (8, 10+6\theta+2\theta^2, 201+71\theta+25\theta^2) \sim (30, 8+4\theta+2\theta^2, 143+49\theta+17\theta^2)$$
$$\sim (2, 7+3\theta+2\theta^2, 105+37\theta+13\theta^2) \sim (11, 1+\theta+\theta^2, 50+17\theta+6\theta^2)$$
$$\sim (2, 8+3\theta+\theta^2, 57+20\theta+7\theta^2) \sim (3, 7+2\theta+\theta^2, 24+9\theta+3\theta^2)$$
$$\sim (10, 8+4\theta+2\theta^2, 45+15\theta+5\theta^2) \sim (6, 2+2\theta, 5+3\theta+\theta^2)$$
$$\sim (16, 17+7\theta+\theta^2, 19+5\theta+3\theta^2) \sim (10, -1+2\theta+\theta^2, 40+15\theta+5\theta^2)$$
$$\sim (17, 8+5\theta+\theta^2, 56+18\theta+7\theta^2) \sim (4, 5+3\theta+\theta^2, 210+74\theta+26\theta^2)$$
$$\sim (15, 4+2\theta+\theta^2, 147+51\theta+18\theta^2).$$

Proceeding as in the former example there is no difficulty in seeing that every ideal $\mathfrak{b}$ such that $L(\mathfrak{b}) < 77$ is equivalent to one of the above reduced ideals. Hence there is only one cycle of reduced ideals and the class number of the corpus is unity.

The relation connecting two reduced ideals is given by equation (17)

As a numerical example we take the ideal

$$(3,\ 7+2\theta+\theta^2,\ 24+9\theta+3\theta^2),$$

which

$$= G(U_{11},\ U_{12},\ U_{12}^2)$$

$$= \frac{N(U_{11})}{U_{11}}(1,\ \theta,\ \theta^2).$$

On equating the norms, we have

$$9 = \frac{N^3(U_{11})}{N(U_{11})},$$

i.e.,

$$3 = N(U_{11}) = N(1674412+588780\theta+207035\theta^2).$$

(iii)

$$\theta^3-65 = 0,$$

$$\theta = 4{\cdot}02072, \qquad \Delta = -d = -d(\theta) = 27\,.\,65^2,$$

$$\theta^2 = 16{\cdot}16623, \qquad M = \frac{2}{\pi}\sqrt{\Delta} = 215{\cdot}0183.$$

The minimalbasis $(1,\ \theta,\ \theta^2)$ which is not canonical will again be adopted to gain simplicity in the calculations. We first consider the ideal

$$(17,\ 2+7\theta-\theta^2,\ -68+17\theta).$$

Writing

$$U = 17x+(2+7\theta-\theta^2)\,y+(-68+17\theta)\,z,$$

$$\bar{V} = 17x+\tfrac{1}{2}(4-7\theta+\theta^2)y+\tfrac{1}{2}(-136-17\theta)\,z,$$

$$\bar{W} = \tfrac{1}{2}\sqrt{3}\,(7\theta+\theta^2)\,y+\tfrac{1}{2}\sqrt{3}\,\theta z,$$

three consecutive convergent points T_0, T_1, T_1^2 are found to be

$$(0,\ 1,\ 0)(1;\ 0,\ 0)(11,\ -3,\ 2).$$

The tetrahedron $OT_0T_1T_1^2$ not being unitary, one or more subsidiary points must be inserted. The coordinates of t_1^1 must be of the form $(h,\ k,\ 1)$, since $OT_0T_1t_1^1$ is unitary, and t_1^1 lies on the same side of OT_0T_1 as T_1^2. Also since t_1^1 lies on the integer plane nearest to $OT_1T_1^2$ on the same side as T_0, its coordinates must be of the form

$$(l+11m,\ 2-3m,\ -1+2m).$$

On comparing these two expressions the point t_1^1 comes to be $(h,\ -1,\ 1)$, and $\Phi(h,\ -1,\ 1)$ takes its least value for integral values of h when $h = 6$.

The tetrahedron $Ot_1^1 T_1 T_1^2$ is unitary, hence no other subsidiary point is required at this stage.

Accordingly a reduced ideal is

$$G\begin{Bmatrix} 0, & 1, & 0 \\ 1, & 0, & 0 \\ 6, & -1, & 1 \end{Bmatrix} (17,\ 2+7\theta-\theta^2,\ -68+17\theta) \equiv (24,\ 9+\theta+\theta^2,\ 188+4\theta-4\theta^2).$$

This gives rise to the cycle

$$\mathfrak{b} \sim (24,\ 9+\theta+\theta^2, 188+4\theta-4\theta^2) \sim (17,\ 45+13\theta+3\theta^2,\ 227+55\theta+14\theta^2)$$
$$\sim (10,\ 10+5\theta+\theta^2,\ 35+30\theta+7\theta^2\ .$$

Forming the cycles equivalent to $\mathfrak{b}^2$, $\mathfrak{b}^3$, ..., I find

$$\mathfrak{b}^2 \sim (3,\ 48+12\theta+3\theta^2,\ 19+5\theta+\theta^2),$$

$$\mathfrak{b}^3 \sim (8,\ 34+10\theta+2\theta^2, 43+11\theta+3\theta^2) \sim (86,\ 17+19\theta+\theta^2, 102+28\theta+6\theta^2)$$
$$\sim (53,\ 58+11\theta+3\theta^2,\ 669+168\theta+41\theta^2),$$

$$\mathfrak{b}^4 \sim (19, 4+2\theta+\theta^2, 150+37\theta+9\theta^2) \sim (3,\ 15+4\theta+\theta^2,\ 760+189\theta+47\theta^2)$$
$$\sim (20,\ 5+5\theta+\theta^2,\ 320+80\theta+20\theta^2),$$

$$\mathfrak{b}^5 \sim (24,\ 26+10\theta+2\theta^2,\ 179+43\theta+11\theta^2)$$
$$\sim (26,\ 39+13\theta+3\theta^2,\ 416+104\theta+26\theta^2)$$
$$\sim (29,\ 19+3\theta+2\theta^2,\ 91+22\theta+5\theta^2),$$

$$\mathfrak{b}^6 \sim (1,\ 16+4\theta+\theta^2,\ 776+193\theta+48\theta^2).$$

An ideal which has not yet occurred is

$$\mathfrak{d} \equiv (4,\ -1+\theta,\ -1+\theta^2).$$

Forming the cycles for $\mathfrak{b}^m \mathfrak{d}^n$, I find

$$\mathfrak{d} \sim (4,\ 3+\theta,\ 15+4\theta+\theta^2) \sim (23,\ 50+14\theta+3\theta^2,\ 159+39\theta+10\theta^2)$$
$$\sim (15,\ 10+5\theta+\theta^2,\ 240+60\theta+15\theta^2) \sim (16,\ 1+\theta+\theta^2,\ 85+21\theta+5\theta^2),$$

$$\mathfrak{b}\mathfrak{d} \sim (6,\ 11+3\theta+\theta^2,\ 19+5\theta+\theta^2) \sim (73,\ 36+16\theta-\theta^2,\ 2+9\theta+4\theta^2)$$
$$\sim (79,\ 8+18\theta+\theta^2,\ 40+11\theta+5\theta^2) \sim (57,\ 25+14\theta+\theta^2,\ 68+13\theta+5\theta^2)$$
$$\sim (40,\ 10+10\theta+2\theta^2,\ 265+65\theta+17\theta^2)$$
$$\sim (38,\ 8+4\theta+2\theta^2,\ 123+33\theta+7\theta^2),$$

$\mathfrak{b}^2\mathfrak{d} \sim (12, 9+3\theta, 13+2\theta+\theta^2) \sim (69, 13+11\theta+4\theta^2, 150+42\theta+9\theta^2)$

$\sim (52, 13+13\theta+\theta^2, 65+13\theta+5\theta^2)$

$\sim (43, 20+13\theta+2\theta^2, 113+24\theta+7\theta^2)$

$\sim (15, 5+\theta^2, 10+5\theta+\theta^2) \sim (58, 38+6\theta+4\theta^2, 43+19\theta+3\theta^2)$

$\sim (48, 3+3\theta+3\theta^2, 52+20\theta+4\theta^2)$,

$\mathfrak{b}^3\mathfrak{d} \sim (2, 32+8\theta+2\theta^2, 169+29\theta+7\theta^2)$,

$\mathfrak{b}^4\mathfrak{d} \sim (5, 15+5\theta+\theta^2, 80+20\theta+5\theta^2) \sim (22, 12-\theta+\theta^2, 35+9\theta+2\theta^2)$

$\sim (34, 45+13\theta+3\theta^2, 499+123\theta+31\theta^2)$,

$\mathfrak{b}^5\mathfrak{d} \sim (6, 19+5\theta+\theta^2, 96+24\theta+6\theta^2) \sim (19, 31+9\theta+2\theta^2, 304+76\theta+19\theta^2)$

$\mathfrak{d}^2 \sim (4, 17+5\theta+\theta^2, 64+16\theta+4\theta^2) \sim (43, 51+14\theta+3\theta^2, 391+93\theta+23\theta^2)$

$\mathfrak{b}\mathfrak{d}^2 \sim (11, 24+9\theta+2\theta^2, 176+44\theta+11\theta^2)$

$\sim (89, 35+4\theta+3\theta^2, 176+43\theta+10\theta^2)$

$\sim (10, 15+5\theta+\theta^2, 145+35\theta+9\theta^2)$,

$\mathfrak{b}^2\mathfrak{d}^2 \sim (13, 39+13\theta+3\theta^2, 208+52\theta+13\theta^2)$

$\sim (58, 19+3\theta+2\theta^2, 91+22\theta+5\theta^2)$

$\sim (12, 13+5\theta+\theta^2, 179+43\theta+11\theta^2)$,

$\mathfrak{b}^3\mathfrak{d}^2 \sim (2, 15+4\theta+\theta^2, 760+189\theta+47\theta^2)$

$\sim (30, -5+5\theta+\theta^2, 480+120\theta+30\theta^2)$

$\sim (19, 9+3\theta+\theta^2, 295+73\theta+18\theta^2)$,

$\mathfrak{b}^4\mathfrak{d}^2 \sim (12, 9+\theta+\theta^2, 13+5\theta+\theta^2) \sim (34, 13+3\theta+2\theta^2, 62+13\theta+3\theta^2)$

$\sim (39, 26+13\theta+2\theta^2, 91+26\theta+7\theta^2)$

$\sim (41, 46+13\theta+\theta^2, 61+11\theta+4\theta^2)$

$\sim (76, 16+8\theta+4\theta^2, 77+29\theta+5\theta^2)$,

$\mathfrak{b}^5\mathfrak{d}^2 \sim (6, 16+5\theta+\theta^2, 29+7\theta+2\theta^2) \sim (47, 8+14\theta+\theta^2, 32+9\theta+4\theta^2)$

$\sim (73, 55-\theta+4\theta^2, 32+18\theta+\theta^2) \sim (30, -5+5\theta+\theta^2, 175+45\theta+11\theta^2)$

$\sim (57, 27+9\theta+3\theta^2, 332+79\theta+20\theta^2)$.

$\mathfrak{d}^3 \sim (1)$, a previous class.

There is now no difficulty in seeing that every ideal $\mathfrak{q}$ such that $L(\mathfrak{q}) < 216$ belongs to one of these eighteen classes. The primitive unit is derived in the usual way [equation (18)] from any of the cycles and is

$$\epsilon \equiv 16 + 4\theta + \theta^2.$$

For an introduction to the more geometrical aspect of this theory and much encouragement received while writing, I acknowledge great indebtedness to Mr. G. B. Mathews. I am also under a debt of gratitude to Mr. Mathews and Mr. Grace for reading the paper in manuscript and making suggestions for its improvement.

THE DIFFRACTION OF LIGHT BY AN OPAQUE PRISM

By H. M. MACDONALD.

[Received February 13th, 1913.—Read March 13th, 1913.]

THE diffraction of light by an opaque prism has been discussed by Jackson,* who compares a perfectly conducting prism with a perfectly absorbing prism. The solution for the perfectly absorbing prism is the usual one which is obtained on the assumption that the electric and magnetic forces vanish over the part of the surface of the absorbing body which is in the shadow, and that the electric and magnetic forces have the same values just outside the part of the surface not in the shadow as in the incident waves. This assumption neglects the creeping effect at the edge of the shadow. The object of the following note is to obtain the exact solution for a perfectly absorbing prism.

It has been proved† that the solution for a perfectly absorbing body can be made to depend on the solution of two allied problems. The argument may be restated as follows. A perfectly absorbing body may be regarded as a body which is incapable of supporting either electric or magnetic force; hence, if M is the electric current distribution on the surface of the body when it is supposed to be perfectly conducting, and E is the magnetic current distribution on the surface of the body when it is supposed to be incapable of supporting magnetic force, the superposition of these two distributions gives the electric and magnetic current distributions on the surface of the body when it is perfectly absorbing and the amplitude of the incident waves is doubled. Therefore, when the solutions for the perfectly conducting body and for the body incapable of supporting magnetic force are both known, the solution for the perfectly absorbing body can be obtained.

For a perfectly conducting prism of angle $2\pi-\alpha$, when the electric force in the incident waves is $e^{\iota\kappa[Vt+r\cos(\theta-\theta_1)]}$ parallel to the edge of the

* *Proc. London Math. Soc.*, Ser. 2, Vol. 1, p. 393, 1904.

† *Phil. Trans.*, Vol. 212, A, p. 337, 1912.

prism, the electric force Z is given by

$$Z = \frac{\iota}{2a} e^{\iota\kappa V t} \int_{\infty\iota+\gamma}^{\infty\iota+\gamma'} e^{\iota\kappa r \cos\zeta} \times \left\{ \frac{\sin\frac{\pi\zeta}{a}}{\cos\frac{\pi\zeta}{a} - \cos\frac{\pi}{a}(\theta-\theta_1)} - \frac{\sin\frac{\pi\zeta}{a}}{\cos\frac{\pi\zeta}{a} - \cos\frac{\pi}{a}(\theta+\theta_1)} \right\} d\zeta,$$

where $$2\pi > \gamma > \pi, \quad 0 > \gamma' > -\pi.^* \tag{1}$$

When the prism is supposed to be incapable of supporting magnetic force, the condition to be satisfied at its surface is that the tangential magnetic force vanishes, that is $\partial Z/\partial\theta = 0$, and the value of Z is now given by

$$Z = \frac{\iota}{2a} e^{\iota\kappa V t} \int_{\infty\iota+\gamma}^{\infty\iota+\gamma'} e^{\iota\kappa r \cos\zeta} \times \left\{ \frac{\sin\frac{\pi\zeta}{a}}{\cos\frac{\pi\zeta}{a} - \cos\frac{\pi}{a}(\theta-\theta_1)} + \frac{\sin\frac{\pi\zeta}{a}}{\cos\frac{\pi\zeta}{a} - \cos\frac{\pi}{a}(\theta+\theta_1)} \right\} d\zeta. \tag{2}$$

Hence for a perfectly absorbing prism the solution is

$$Z = \frac{\iota}{2a} e^{\iota\kappa V t} \int_{\infty\iota+\gamma}^{\infty\iota+\gamma'} e^{\iota\kappa r \cos\zeta} \frac{\sin\frac{\pi\zeta}{a}}{\cos\frac{\pi\zeta}{a} - \cos\frac{\pi}{a}(\theta-\theta_1)} d\zeta. \tag{3}$$

When the magnetic force in the incident waves is $e^{\iota\kappa[Vt + r\cos(\theta-\theta_1)]}$ parallel to the edge of the prism, the magnetic force γ for a perfectly conducting prism is given by

$$\gamma = \frac{\iota}{2a} e^{\iota\kappa V t} \int_{\infty\iota+\gamma}^{\infty\iota+\gamma'} e^{\iota\kappa r \cos\zeta} \times \left\{ \frac{\sin\frac{\pi\zeta}{a}}{\cos\frac{\pi\zeta}{a} - \cos\frac{\pi}{a}(\theta-\theta_1)} + \frac{\sin\frac{\pi\zeta}{a}}{\cos\frac{\pi\zeta}{a} - \cos\frac{\pi}{a}(\theta+\theta_1)} \right\} d\zeta, \tag{4}$$

and for a prism incapable of supporting magnetic force, since the surface

* Macdonald, *Electric Waves*, p. 192, 1902.

condition is now $\gamma = 0$, the solution is

$$\gamma = \frac{\iota}{2a} e^{\iota\kappa Vt} \int_{\infty \iota+\gamma}^{\infty \iota+\gamma'} e^{\iota\kappa r \cos\zeta} \times \left\{ \frac{\sin\frac{\pi\xi}{a}}{\cos\frac{\pi\xi}{a} - \cos\frac{\pi}{a}(\theta-\theta_1)} - \frac{\sin\frac{\pi\xi}{a}}{\cos\frac{\pi\xi}{a} - \cos\frac{\pi}{a}(\theta+\theta_1)} \right\} d\xi. \quad (5)$$

Therefore for a perfectly absorbing prism the solution is

$$\gamma = \frac{\iota}{2a} e^{\iota\kappa Vt} \int_{\infty \iota+\gamma}^{\infty \iota+\gamma'} e^{\iota\kappa r \cos\zeta} \frac{\sin\frac{\pi\xi}{a}}{\cos\frac{\pi\xi}{a} - \cos\frac{\pi}{a}(\theta-\theta_1)} d\xi. \quad (6)$$

At points just outside the shadow which are at a distance from the edge of the prism sufficiently great to secure that the quantities M' and N' of Jackson's paper* are negligible, the integral

$$\int_{\infty \iota+\gamma}^{\infty \iota+\gamma'} e^{\iota\kappa r \cos\zeta} \frac{\sin\frac{\pi\xi}{a}}{\cos\frac{\pi\xi}{a} - \cos\frac{\pi}{a}(\theta+\theta_1)} d\xi$$

is negligible, and the electric and magnetic forces become the same for the perfectly absorbing prism as for the perfectly conducting prism; therefore the position of the diffraction bands is the same for a perfectly absorbing prism as for a perfectly conducting prism. In the above the case of plane waves with their direction perpendicular to the edge of the prism has been treated, but the results are the same whatever the nature of the incident light is.

---

* *L.c.*, p. 404.

ON THE MODE OF OSCILLATION OF A FOURIER SERIES AND OF ITS ALLIED SERIES

By W. H. YOUNG, ScD., F.R.S.

[Received January 29th, 1913.—Read February 13th, 1913.]

1. In a recent very interesting paper whose results were communicated to me privately by Mr. G. H. Hardy, the latter has shown that, by employing a modification of a method indicated in my note in the *Comptes Rendus* of December 23rd, 1912, we may complete the results that I had there enunciated, extending the theorems of Fatou, Jerosch, Weyl, and others, on usually * convergent series. The modification consists essentially in the use of Lebesgue's theorem that, if

$$\phi(t) = f(x+t) + f(x-t) - 2f(x),$$

then $\frac{1}{\lambda}\int_0^\lambda |\phi(t)|\,dt$ converges usually to zero as $\lambda \to \infty$, instead of the theorem that $\int f(t)\,g(x+t)\,dt$ usually exists, if $f(t)$ and $g(t)$ are both summable.

What is common more particularly to Mr. Hardy's method and my own is the preliminary determination of the order of infinity of the partial summations of the series considered almost everywhere, and the application of an auxiliary series formed by a different process from the given series.†

* The words "usually" and "almost everywhere" are used throughout in the technical sense, *i.e.*, as denoting "at all points except at a set of content zero."

† In Mr. G. H. Hardy's paper Fourier series and not their allied series are considered. It will be easy, however, for the reader with the argument of the present communication in his mind to fill in this omission.

We have, in fact, only to remark that, if

$$\phi(t) = f(x+t) - f(x-t),$$

it is still true that $\frac{1}{\lambda}\int_0^\lambda |\phi(t)|\,dt$ converges usually to zero, to see that Mr. Hardy's theorem as to the usual order of infinity of the n-th partial summation of a Fourier series is true for the allied series. The remainder of the argument required is then precisely analogous to that used below in § 6. The factor $1/l_1(n)\,l_2(n)\,\ldots\,\{l_r(n)\}^{1+k}$ changes both the Fourier series and its allied series into Fourier series, and we may accordingly employ the ordinary artifice

It has occurred to me that the subject would gain in unity if I employed the corresponding process to prove theorems I have already obtained in a different way concerning the convergence everywhere of certain Fourier series and allied series. Moreover, for some time still to come, it will be necessary to lay almost as much stress on method as on results in the Theory of Functions of a Real Variable. It is only in this way that the great difficulties of the subject may be expected to be overcome.

For this purpose it is sufficient to use, instead of one of the two theorems above referred to, either the condition that a function F should be the integral of a function f of given type of summability or certain inequalities to which I have already called attention in connexion with the theory of summable functions in general.*

Not only are we able in this way to re-prove and even extend certain of my conclusions already referred to, we actually obtain, with comparative ease, additional information of an almost continuously varying character as to the mode in which the Fourier seres of the various types of functions and their allied series oscillate, and as to the convergence factors which will transform such series into series which converge everywhere.

I do not, of course, assert that the order of infinity obtained is necessarily anything more than an order which is certainly not exceeded.† It is always possible that new methods may give us further information. Moreover, here, as hitherto, I use the term "degree" or "class" of summability as referring to the extreme case. When I say that the degree of summability is of a certain kind I mean, for example, to include all bounded functions, and, more generally, I include all functions of higher degree or class of summability than that explicitly referred to.

I have used freely the notation due to Landau and employed with success by various writers, in particular by Messrs. Hardy and Littlewood. I have also found the advantage of adopting their plan of deducing the convergence of a series from its convergence $(C1)$ when a certain simple and obvious condition is satisfied. It is a great advantage in dealing with

of Messrs. Hardy and Littlewood, and so obtain the required results. It will be remarked, however, that we cannot employ the factor $1/\log(n)$ for the allied series, so that the convergence factor just written down is the most general given by the theory in the case of the allied series. In fact, if $g(x)$ denote the sum of the sine series whose general coefficient is $1/\log n$, $g(x)$ is not summable, and we are not able to assert that the factor $1/\log n$ changes the allied series into a Fourier series, so that this part of the reasoning breaks down.

* See "On Classes of Summable Functions and their Fourier Series," 1912, *Proc. Roy. Soc.*, A, Vol. 87, pp. 225–229.

† It is in this sense that the mark = 0 in the present communication should be interpreted.

Fourier series that they necessarily converge ($C1$) when the functions to which they belong are continuous, or simply discontinuous, and converge usually ($C1$) whatever these functions may be.

In the more refined applications of the theory, the series whose coefficients belong to what we may call the logarithmic scale, involving as they do the product of logarithms, play a prominent part. I have accordingly worked out the order of summability of the sum-functions of these series in the most general circumstances that can arise.

It is worth remarking that there is a theory of convergence ($C1$) different to that of convergence *per se*, even when we are concerned with such convergence factors as belong to the logarithmic scale. This is due to the fact that, if g be the function associated with a Fourier series belonging to what may be called the logarithmic scale, it may very well happen that $\int f(x+t)g(t)dt$ is a continuous function of x without the coefficients of the Fourier series of $g(t)$ approaching zero sufficiently rapidly to ensure the proper convergence of the Fourier series of $f(x)$, when these coefficients are used as convergence factors. This may be compared with the fact that

$$\underset{E\to 0,\ n\to\infty}{\mathrm{Lt}} \int_E s_n(x)dx = 0,$$

when s_n is the partial summation of a Fourier series belonging to the logarithmic scale down to a certain point in the scale, but not apparently beyond, and the explanation would appear to be an analogous one. Thus, while it is true that when the Fourier successions belonging to complementary classes are multiplied together we get the coefficients of a series converging ($C1$), it is not, it would appear, true that the mere fact that one of the two functions corresponds to a Fourier series in the logarithmic scale is sufficient to ensure the ordinary convergence.

It will also at once be noticed that there is an essential difference between the theory of Fourier series which converge everywhere here presented and that of Fourier series which converge almost everywhere, as far as the latter has yet been developed. Whereas the convergence factors required to change the Fourier series into one which converges everywhere vary continuously as the class to which the function belongs alters, there is no such distinction when we are content to secure convergence almost everywhere. It would almost appear as if this distinction was in the nature of things; and, if this be so, it is clear that a guess which has been hazarded that functions whose squares are summable have Fourier series which converge almost everywhere is an incorrect one. It is not, however, wholly impossible that the application

of some new method may lead to further notable increase of our knowledge of the theory of Fourier series which converge almost everywhere.

2. From the known expressions for the partial summations s_n and σ_n of the Fourier series and allied series it is easily shown that, in the determination of the order of infinity, it will be sufficient in what follows to consider the following integrals :

$$\int_0^c f(x+t)\frac{\sin nt}{t}\,dt, \qquad \int_0^c f(x+t)\frac{1-\cos nt}{t}\,dt.$$

Here c may be taken as small as we please, but it must be a *fixed* quantity. By the theorem of Rieman-Lebesgue, for example, when the limits are fixed, we know that

$$\int \phi(t)\sin nt\,dt \quad \text{and} \quad \int \phi(t)\cos nt\,dt$$

have the unique limit zero, when $n\to\infty$, provided only that $\phi(t)$ is summable between the limits. Thus to a quantity of order $o(1)$, the partial summations will be unaffected by the change in the upper limit in the integrals.

3. Let us now begin with the simplest case, and suppose $f(x)$ bounded. Then

$$s_n = \int_0^c f(x+t)\frac{\sin nt}{t}\,dt + o(1) = O\int_0^c \left|\frac{\sin nt}{t}\right| dt + o(1) = O\log n, \quad (1)$$

by a theorem due to Lebesgue, which we may easily prove. In fact,

$$\int_0^c \left|\frac{\sin nt}{t}\right| dt = \int_0^{1/n} \left|\frac{\sin nt}{t}\right| dt + \int_{1/n}^c \left|\frac{\sin nt}{t}\right| dt$$

$$= O\int_0^1 \left|\frac{\sin t}{t}\right| dt + O\int_{1/n}^c \frac{dt}{t} = O\log n.$$

In precisely similar way it may be seen that

$$\sigma_n = O\log n. \quad (2)$$

4. Let us further suppose that the function $f(x)$ is not only bounded, but possesses, if any, only discontinuities of the first kind. Then there is

at each point an unique limit $f(x+o)$ for $f(x+t)$ as $t \to o$. Thus,

$$s_n = O\int_0^c |f(x+t)-f(x+0)| \left|\frac{\sin nt}{t}\right| dt + O\int_0^c f(x+0)\frac{\sin nt}{t}\,dt + o(1)$$

$$= O\int_0^c |f(x+t)-f(x+o)| \left|\frac{\sin nt}{t}\right| dt + o(1).$$

Now, taking c sufficiently small, we can ensure that the former of the two factors in the integrand is less than a fixed small quantity, say ϵ, independent of x. Hence

$$s_n/\log n = \epsilon\, O(1).$$

Thus, the limits of $s_n/\log n$ are as small as we please and are therefore zero. In other words, with Landau's further notation,

$$s_n = o \log n. \tag{3}$$

In precisely similar way we prove that

$$\sigma_n = o \log n. \tag{4}$$

5. Before proceeding farther we may conveniently apply the results of the preceding article to obtain suitable convergence factors for the Fourier series and allied series of a bounded function.

Consider first the Fourier series, and let us write for brevity*

$$f(x) \sim \sum_{n=2}^{\infty} A_n,$$

where $$A_n = a_n \cos nx + b_n \sin nx.$$

Then since, as is known, $\sum_{n=2}^{\infty} \frac{\cos nx}{\log n}$ is a Fourier series, the series $\sum_{n=2}^{\infty} A_n/\log n$ is the Fourier series of a continuous function. Denote the n-th partial summation of the latter series by S_n, then

$$nS_n - \{S_2+S_3+\ldots+S_n\} = \sum_{r=2}^{n} rA_r/\log r = \sum_{r=2}^{n} r\,(s_r - s_{r-1})/\log r$$

$$= O\sum_{r=2}^{n-1} s_r/\log r + On\, s_n/\log n = O(n). \tag{5}$$

Now the Fourier series of a continuous function necessarily converges

* Without loss of generality we may suppose the terms A_0 and A_1 to be absent.

uniformly when summed ($C1$). Thus, the second term on the left-hand side $= O(n)$. Hence

$$S(n) = O(1).$$

In other words, we have proved that *the Fourier series of any bounded function oscillates boundedly everywhere, when the convergence factor* $1/\log n$ *is employed.*

From this it also follows that *the Fourier series of any bounded function converges uniformly throughout any interval in the interval of periodicity, when the convergence factor* $g_n/\log n$ *is employed, provided only* g_n *converges monotonely to zero as* $n \to \infty$.

6. Next consider the allied series. Then the argument just used will fail to apply, since the series $\Sigma \frac{\sin nx}{\log n}$ is not a Fourier series. If, however, we introduce a suitable factor k_n, such that $\Sigma \frac{k_n}{n \log n}$ converges, the series $\Sigma k_n \sin nx/\log n$ will be a Fourier series. Denoting by B_n the general term $(b_n \cos nx - a_n \sin nx)$ of the allied series of $f(x)$, we may now assert that $\Sigma k_n B_n/\log n$ is a Fourier series, and is moreover the Fourier series of a continuous function.

The whole argument employed for the series $\Sigma A_n/\log n$ will now apply *mutatis mutandis* for the series $\Sigma k_n B_n/\log n$. But we can, however, go farther if we make, as we must in any case, a further assumption with regard to the factor k_r. We suppose that

$$k_r - k_{r+1} = o\left(\frac{1}{r}\right);$$

or, more precisely, we may replace our two hypotheses with regard to k_r by the definite single hypothesis that

$$k_r = 1/l_2(r)\, l_3(r) \ldots l_{p-1}(r)\,[l_p(r)]^{1+\epsilon}.$$

We thus get instead of (5), S'_n referring to the series $\Sigma k_n \sin nx/\log n$,

$$nS'_n - \{S'_2 + S'_3 + \ldots + S'_{n-1}\} = O \sum_{r=2} kr\sigma_r/\log r + Ok_n\sigma_n/\log n = o(n).$$

Now the second term on the left when divided by n gives us the Cesàro partial summation (index unity) of the Fourier series of a continuous function, and therefore converges to that function everywhere.

Hence, finally, we see that the convergence factor

$$1/l_1(n)\, l_2(n) \ldots l_{p-1}(n)\,[l_p(n)]^{1+\epsilon}$$

changes the allied series of the Fourier series of any bounded function into a convergent series.

7. In the two preceding articles we have taken $f(x)$ to be a perfectly general bounded function. If we suppose the function to have but discontinuities of the first kind, we can somewhat extend the results of § 5. This follows from the two facts: first, that $s_n = o \log n$ by (3) instead of $O \log n$, and, secondly, that the Fourier series of a continuous function always converges uniformly when summed ($C1$). We thus obtain at once the result that *the Fourier series of a continuous or simply discontinuous function necessarily converges uniformly, when the convergence factor* $1/\log n$ *is employed.*

8. Let us now suppose that $f(x)$ is a function whose $(1+p)$-th power is summable. We may then conveniently apply the extension of Schwarz's inequality to which F. Riesz has called attention. We have

$$\begin{aligned} |s_n| &= O\left|\int_0^c f(x+t)\frac{\sin nt}{t}\,dt\right| \\ &= O\left\{\int_0^c |f(x+t)|^{1+p}\,dt\right\}^{1/(1+p)}\left\{\int_0^c \left|\frac{\sin nt}{t}\right|^{1+1/p} dt\right\}^{1/(1+1/p)} \\ &= \epsilon O\left\{n^{1/p}\int_0^{nc}\left|\frac{\sin t}{t}\right|^{1+1/p} dt\right\}^{1/(1+1/p)}, \end{aligned}$$

where ϵ is a fixed quantity which, by a proper choice of c is as small as we please. Hence

$$s_n = o(n^{1/(1+p)}). \tag{6}$$

In precisely the same way, we prove that

$$\sigma_n = o(n^{1/(1+p)}). \tag{7}$$

9. Now though both the series $\Sigma n^{-1/(1+p)}\cos nx$ and $\Sigma n^{-1/(1+p)}\sin nx$ are Fourier series, neither of them is contained in the class complementary to that to which $f(x)$ belongs, when $|f(x)|^{1+p}$ is summable. The functions of which they are the Fourier series are of the order $x^{-p/(1+p)}$. Accordingly when the convergence factor $n^{-1/(1+p)}$ is employed, the series obtained from the Fourier series of $f(x)$ and its allied series will not in general be the Fourier series of continuous functions, and therefore the argument used in § 5 will not apply in its entirety. By using slightly stronger convergence factors, however, this end will be obtained. Thus it will be obtained by using the convergence factor n^{-q}, where $q > 1/(1+p)$. It will equally be obtained

by using the convergence factor $n^{1/(1+p)}(\log n)^{-q}$, where $q > p/(1+p)$, as will be seen by considerations to be exposed shortly. In fact the cosine and sine series which have these convergence factors for coefficients are the Fourier series of functions whose $(1+1/p)$-th powers are summable.

We thus obtain in precisely the same manner as that already explained the result that *the use of convergence factors such as these change the Fourier series of a function whose* $(1+p)$*-th power is summable and the allied series into uniformly convergent Fourier series.*

10. Already in the preceding article we have seen the importance of the consideration of the cosine and sine series whose coefficients belong to what may be conveniently called the logarithmic scale. In our further developments such series play a fundamental part. It will be convenient, therefore, to obtain some information as to the class of functions to which the sums of such series belong.

In the succeeding articles we shall therefore take $\phi(n)$ to be a function of the form

$$\{l_1(n)\}^{q_1}\{l_2(n)\}^{q_2}\ldots\ldots\{l_p(n)\}^{q_p}$$

where $q_1, q_2, \ldots, q_p$ are $\geqslant 0$, and are not all zero. Thus, $\phi(n)$ is a monotone increasing function of n and has the property that

$$\operatorname*{Lt}_{n\to\infty}\frac{n\phi'(n)}{\phi(n)} = 0. \tag{8}$$

Hence also $\log\phi(y)/\log y$ and $\phi(y)/y$ both have the unique limit zero. It will be seen that a more general form may be given to $\phi(n)$, if desired, without invalidating the reasoning or the conclusions arrived at.

11. Let us then first write

$$g(x) = \sum_{n=n_0}^{\infty}\sin nx/\phi(n) = \sum_{n=n_0}^{N}\sin nx/\phi(n) + \sum_{n=N+1}^{\infty}\sin nx/\phi(n),$$

where N denotes the greatest integer in $1/x$.

To the second summation on the right we apply Abel's Lemma, which we are able to do since $1/\phi(n)$ traces out a monotone decreasing sequence with zero as limit. Bearing in mind that the sum of any finite number of consecutive terms of the series $\Sigma\sin nx$ is numerically less than a constant multiplier of $1/x$, we see that the second summation is numerically $< B/x\,\phi(N+1)$, where B is a constant.

In the first summation on the right we replace $\sin nx$ by unity. Thus

$$|g(x)| \leqslant \sum_{n=n_0}^{N} 1/\phi(n) + Bx/\phi(N+1)$$

$$\leqslant \int_{n_0-1}^{(1\,x)-1} \frac{dt}{\phi(t)} + \frac{B}{x\phi\left(\frac{1}{x}\right)} = O\int_{n_0}^{1/x} \frac{dt}{\phi(t)} + O\frac{1}{x\phi\left(\frac{1}{x}\right)}.$$

Let us write y for $(1/x)$, and let $x \to 0$, and therefore $y \to \infty$. Then, as already remarked in § 10, $\phi(y)/y$ has the unique limit zero; therefore the second term on the right in the last inequality approaches infinity. If the first term does not also approach infinity, therefore, it is this second term which determines the order of infinity of $g(x)$. In the contrary case we proceed to examine the indeterminate form constituted by the ratio of the two terms. Now, by (8),

$$\operatorname*{Lt}_{y\to\infty} \frac{\frac{y}{\phi(y)}}{\int_{n_0}^{y} \frac{dt}{\phi(t)}} = \operatorname*{Lt}_{y\to\infty} \frac{\frac{1}{\phi(y)}\left\{1 - \frac{y\phi'(y)}{\phi(y)}\right\}}{\frac{1}{\phi(y)}} = \operatorname*{Lt}_{y\to\infty} \left\{1 - \frac{y\phi'(y)}{\phi(y)}\right\} = 1.$$

Hence in the neighbourhood of the origin we have in any case

$$\sum_{n=n_0}^{\infty} \sin nx/\phi(n) = O\left\{1/x\,\phi\left(\frac{1}{x}\right)\right\}. \tag{9}$$

Integrating between the limits x and a, where $a > 0$, we obtain the relation

$$\sum_{n=n_0}^{\infty} \cos nx/n\phi(n) = O\int_a^x \frac{dt}{t\phi(t)}. \tag{10}$$

Cor.—
$$\sum_{n=3}^{\infty} \sin nx/\log n = O\left\{1/x \log \frac{1}{x}\right\}, \tag{11}$$

$$\sum_{n=3}^{\infty} \cos nx/n \log n = O\left\{\log \log \frac{1}{x}\right\}. \tag{12}$$

12. Next let us consider

$$g(x) = \sum_{n=n_0}^{\infty} \sin nx/n\phi(n) = \sum_{n=n_0}^{N} \frac{\sin nx}{nx} \frac{x}{\phi(n)} + \sum_{n=N+1}^{\infty} \frac{\sin nx}{n} \frac{1}{\phi(n)}.$$

Replacing in the first summation $\dfrac{\sin nx}{nx}$ by unity, and, as in the preceding article changing the summation to an integral, while for the evaluation of

the second summation we use Abel's Lemma, we get

$$|g(x)| \leqslant x \int_{n_0}^{1/x} \frac{dt}{\phi(t)} + \frac{B}{\phi\left(\frac{1}{x}\right)}.$$

Hence, as in § 11, $$\sum_{n=n_0}^{\infty} \sin nx/n\phi(n) = O\left\{\frac{1}{\phi\left(\frac{1}{x}\right)}\right\}. \tag{13}$$

COR.— $$\sum_{n=3}^{\infty} \sin nx/n \log n = O\left\{\frac{1}{\log \frac{1}{x}}\right\}, \tag{14}$$

$$\sum_{n=3}^{\infty} \sin nx/n(\log n)^q = O\left\{\frac{1}{\left(\log \frac{1}{x}\right)^q}\right\}. \tag{15}$$

13. We might justify the process of differentiation applied to the relation (13), but it is simpler to consider the series $\Sigma \cos nx/\phi(n)$ directly.

Let $$g(x) = \sum_{n=n_0}^{\infty} \cos nx/\phi(n).$$

Then

$$2 \sin \tfrac{1}{2}x\, g(x) = -\frac{\sin(n_0 - \frac{1}{2})x}{\phi(n_0)} + \sum_{n=n_0}^{\infty} \left\{\frac{1}{\phi(n)} - \frac{1}{\phi(n+1)}\right\} \sin(n+\tfrac{1}{2})x.$$

Let us write $$\frac{d}{dx}\left\{\frac{1}{\phi(x)}\right\} = -\frac{1}{x\phi_1(x)}.$$

Then

$$2 \sin \tfrac{1}{2}x \,.\, g(x) = O\Sigma \sin nx/n\phi_1(n) = O\left\{1/\phi_1\left(\frac{1}{x}\right)\right\} = O\left\{x \frac{d}{dx} \frac{1}{\phi\left(\frac{1}{x}\right)}\right\},$$

by (13), since $1/\phi_1(n)$ is a monotone decreasing function of n with zero as limit, and

$$\operatorname*{Lt}_{n \to \infty} \frac{n\phi_1'(n)}{\phi_1(n)} = 0.$$

Hence $$\sum_{n=n_0}^{\infty} \cos nx/\phi(n) = O\left\{\frac{d}{dx} \frac{1}{\phi\left(\frac{1}{x}\right)}\right\}. \tag{16}$$

Cor.—
$$\sum_{n=3}^{\infty} \cos nx/\log n = O\left\{\frac{1}{x\left(\log\frac{1}{x}\right)^2}\right\}, \tag{17}$$

$$\sum_{n=3}^{\infty} \frac{\cos nx}{(\log n)^p} = O\left\{\frac{1}{x\left(\log\frac{1}{x}\right)^{1+p}}\right\}. \tag{18}$$

14. In the cases we have considered the exponent of $1/n$ is either zero or unity. Let us now consider the cases where it is fractional.

Consider
$$g(x) = \sum_{n=n_0}^{\infty} \sin nx/n^q\phi(n), \quad \ldots \quad (0 < q < 1).$$

We have, as before,

$$g(x) = x^q \sum_{n=n_0}^{N} \frac{\sin nx}{n^q x^q}\frac{1}{\phi(n)} + \sum_{n=N+1}^{\infty} \frac{\sin nx}{n^q\phi(n)}$$

$$= Ox^q \int_{n_0}^{1/x} \frac{dt}{\phi(t)} + O\frac{x^{q-1}}{\phi\left(\frac{1}{x}\right)}.$$

Hence, by the reasoning already exposed,

$$\sum_{n=n_0}^{\infty} \sin nx/n^q\phi(n) = O\frac{1}{x^{1-q}\phi\left(\frac{1}{x}\right)}. \tag{19}$$

Similarly, writing
$$g(x) = \sum_{n=n_0}^{\infty} \cos nx/n^q\phi(n) \quad (0 < q < 1),$$

we have
$$g(x) = O\int_{n_0}^{1/x} \frac{dt}{t^q\phi(t)} + O\frac{x^{q-1}}{\phi\left(\frac{1}{x}\right)},$$

whence since

$$\operatorname*{Lt}_{y\to\infty} \frac{\int_{n_0}^{y} \frac{dt}{t^q\phi(t)}}{\frac{y^{1-q}}{\phi(y)}} = \operatorname*{Lt}_{y\to\infty} \frac{\frac{1}{y^q\phi(y)}}{(1-q)\frac{1}{y^q\phi(y)} - \frac{y^{1-q}\phi'(y)}{\{\phi(y)\}^2}}$$

$$= \operatorname*{Lt}_{y\to\infty} \frac{1}{1-q-\frac{y\phi'(y)}{\phi(y)}} = \frac{1}{1-q},$$

we get
$$\sum_{n=n_0}^{\infty} \cos nx/n^q\phi(n) = O\frac{1}{x^{1-q}\phi\left(\frac{1}{x}\right)}. \tag{20}$$

15. Again $$\sum_{n=n_0}^{\infty} \frac{\phi(n)}{n} \sin nx = Ox \int_{n_0}^{1/x} \phi(t)\,dt + O\left\{\phi\left(\frac{1}{x}\right)\right\}.$$

Hence since $$\operatorname*{Lt}_{y\to\infty} \frac{\int_{n_0}^{y} \phi(t)\,dt}{y\phi(y)} = \operatorname*{Lt}_{y\to\infty} \frac{\phi(y)}{\phi(y)+y\phi'(y)} = 1,$$

$$\Sigma \frac{\phi(n)}{n} \sin nx = O\left\{\phi\left(\frac{1}{x}\right)\right\}. \tag{21}$$

Similarly, $$\sum_{n=n_0}^{\infty} \frac{\phi(n)}{n} \cos nx = O\int_{n_0}^{1/x} \frac{\phi(t)}{t}\,dt + O\left\{\phi\left(\frac{1}{x}\right)\right\}.$$

Hence since $$\operatorname*{Lt}_{y\to\infty} \frac{\int_{n_0}^{y} \frac{\phi(t)}{t}\,dt}{\phi(y)} = \operatorname*{Lt}_{y\to\infty} \frac{\phi(y)}{y\phi'(y)} = \infty,$$

$$\Sigma \frac{\phi(n)}{n} \cos nx = O\int_{a}^{x} \frac{\phi\left(\frac{1}{t}\right)}{t}\,dt. \tag{22}$$

Similarly, $$\sum_{n=n_0}^{\infty} \frac{\phi(n)}{n^q} \sin nx = Ox^{q-1}\phi\left(\frac{1}{x}\right) \quad (0<q<1), \tag{23}$$

$$\sum_{n=n_0}^{\infty} \frac{\phi(n)}{n^q} \cos nx = Ox^{q-1}\phi\left(\frac{1}{x}\right) \quad (0<q<1), \tag{24}$$

since $$\operatorname*{Lt}_{y\to\infty} \frac{\int_{n_0}^{y} \phi(t)\,t^{-q}\,dt}{y^{1-q}\phi(y)} = \operatorname*{Lt}_{y\to\infty} \frac{\phi(y)\,y^{-q}}{(1-q)y^{-q}\phi(y)+y^{1-q}\phi' y} = \frac{1}{1-q}.$$

16. We now refer to the classification of summable functions* into complementary classes. Let $f(x)$ belong to the class of summable functions complementary to that of those functions u which are such that

$$\psi(u) = u\{l_1(u)\}^{q_1}\{l_2(u)\}^{q_2}\ldots\ldots\{l_p(u)\}^{q} = u\chi(u)$$

is a summable function of x.

Then there is a certain function $\phi(z)$ such that $\phi\{f(x)\}$ is summable and which has the property that

$$uv \leqslant \phi(u)+\psi(v).$$

---

* *Loc. cit. supra.*

Let $s_n(x)$ be the sum of the first n terms of the Fourier series of $f(x)$, as before. Then

$$s_n = O\int_0^c f(x+t)\frac{\sin nt}{t}\,dt = O\int_0^{1/n} f(x+t)\frac{\sin nt}{t}\,dt + O\int_{1/n}^c \frac{f(x+t)}{t}\,dt$$

$$= O(1) + O\int_n^{1/n} \psi\left(\frac{\sin nt}{t}\right)dt + O\int_{1/n}^c \psi\left(\frac{1}{t}\right)dt$$

$$= O\frac{\psi(n)}{n} + O\int_{1/n}^c \psi\left(\frac{1}{t}\right)dt.$$

Now

$$\operatorname*{Lt}_{\epsilon\to 0}\frac{\int_\epsilon^c \psi\left(\frac{1}{t}\right)dt}{\chi\left(\frac{1}{\epsilon}\right)} = -\operatorname*{Lt}_{\epsilon\to 0}\frac{\psi\left(\frac{1}{\epsilon}\right)}{\frac{1}{\epsilon^2}\chi\left(\frac{1}{\epsilon}\right)} = \operatorname*{Lt}_{\epsilon\to 0}\frac{\epsilon\chi\left(\frac{1}{\epsilon}\right)}{\chi'\left(\frac{1}{\epsilon}\right)} = \operatorname*{Lt}_{y\to\infty}\frac{\chi(y)}{y\chi'(y)} = \infty.$$

Hence

$$s_n = O\int_{1/n}^c \psi\left(\frac{1}{t}\right)dt = O\int_{1/c}^n \frac{\psi(t)}{t^2}\,dt = O\int_{1/c}^n \frac{\chi(t)}{t}\,dt = O\,\chi(n)\log n$$

$$= O\left\{l_1(n)\right\}^{1+q_1}\left\{l_2(n)\right\}^{q_2}\ldots\left\{l_r(n)\right\}^{q_r}. \qquad (23)$$

17. Now we have seen that

$$\Sigma\cos nx/\left\{l_1(n)\right\}^{p_1}\ldots\left\{l_r(n)\right\}^{p_r} = O\frac{d}{dx}\frac{1}{\left\{l_1\left(\frac{1}{x}\right)\right\}^{p_1}\ldots\left\{l_r\left(\frac{1}{x}\right)\right\}^{p_r}}$$

$$= O\frac{1}{x\left\{l_1\left(\frac{1}{x}\right)\right\}^{1+p_1}\ldots\left\{l_r\left(\frac{1}{x}\right)\right\}^{1+p_r}}.$$

Hence denoting the sum-function by u,

$$\log u = O\left\{l_1\left(\frac{1}{x}\right) - (1+p_1)\,l_2\left(\frac{1}{x}\right) - \ldots - (1+p_r)\,l_{r+1}\left(\frac{1}{x}\right)\right\},$$

whence

$$\left\{l_1(u)\right\}^{p_1} = O\left\{l_1\left(\frac{1}{x}\right)\right\}^{p_1}.$$

Similarly,

$$\left\{l_2(u)\right\}^{p_2} = O\left\{l_2\left(\frac{1}{x}\right)\right\}^{p_2},$$

and so on. Thus $u\{l_1(u)\}^{p_1}\{l_2(u)\}^{p_2}\dots\{l_r(u)\}^{q_r}$

is summable, provided $q_r < p_r$.

Hence, *if $f(x)$ is a function belonging to the class complementary to that which contains the sum-function $g(x)$ of the series*

$$\Sigma \cos nx/\{l_1(n)\}^{q_1}\{l_2(n)\}^{q_2}\dots\{l_{r-1}(n)\}^{q_{r-1}}\{l_r(n)\}^{p_r},$$

where $p_r > q_r$, we have

$$s_n = O\int_{1/c}^{n}\frac{\phi(t)}{t}\,dt = O\phi(n)\log n = O\{l_1(n)\}^{1+q_1}\{l_2(n)\}^{q_2}\dots\{l_r(n)\}^{q_r}.$$

Moreover $\int f(x+t)\,g(t)\,dt$ is a continuous function of x; hence

$$\Sigma A_n/\{l_1(n_1)\}^{q_1}\{l_2(n)\}^{q_2}\dots\{l_r(n)\}^{p_r} \quad (q_r < p_r)$$

converges when summed $(C1)$.

18. Similarly the same is true if we increase the powers in the denominator. It is therefore true of $\Sigma A_n/\lambda_n$, where

$$\lambda_n = \{l_1(n)^{1+q_1}\{l_2(n)\}^{q_2}\dots\{l_{r-1}(n)\}^{q_{r-1}}\{l_r(n)\}^{p_r}.$$

This latter series will therefore converge in the ordinary manner if

$$\Sigma nA_n/\lambda(n) = o(n),$$

that is, if

$$\Sigma\frac{s_n}{\lambda_n}+\frac{ns_n}{\lambda_n} = o(n),$$

and this is true, since $q_r < p_r$.

Hence, finally, *if*

$$f(x) \sim \Sigma(a_n\cos nx + b_n\sin nx),$$

and $f(x)$ belong to the class complementary to that containing the sum-function of the series

$$\Sigma \cos nx/\{l_1(n)\}^{q_1}\{l_2(n)\}^{q_2}\dots\{l_{r-1}(n)\}^{q_{r-1}}\{l_r(n)\}^{q_r+\epsilon},$$

then

$$\Sigma\frac{a_n\cos nx+b_n\sin nx}{\{l_1(n)\}^{1+q_1}\{l_2(n)\}^{q_2}\dots\{l_{r-1}(n)\}^{q_{r-1}}\{l_r(n)\}^{q_r+\epsilon}}$$

converges in the ordinary manner.

This result is equally true if q_1 is zero, or, indeed,

$$q_1 = q_2, \ldots, = q_{r-1} = 0.$$

The corresponding statement for the allied series is, however, not true.

19. To obtain the corresponding results for the allied series we have first to remark that the sine series

$$\Sigma \sin nx / \{l_1(n)\}^{p_1} \ldots \{l_r(n)\}^{p_r} = O\ \frac{1}{x\left[l_1\left(\frac{1}{x}\right)\right]^{p_1}\left[l_2\left(\frac{1}{x}\right)\right]^{p_2}\ldots\left[l_r\left(\frac{1}{x}\right)\right]^{p_r}}.$$

Thus, if u denote the sum of this sine series, we can only assert that

$$u\{l_1(u)\}^{q_1}, \quad \{l_2(u)\}^{q_2}, \quad \ldots, \quad \{l_r(u)\}^{q_r}$$

is summable if

$$p_1 = 1+q_1, \quad p_2 = 1+q_2, \quad \ldots, \quad p_{r-1} = 1+q_{r-1}, \quad p_r > 1+q_r.$$

Thus, the corresponding statement with regard to the allied series is that *the series*

$$\Sigma \frac{b_n \cos nx - a_n \sin nx}{\{l_1(n)\}^{2+q_1}\{l_2(n)\}^{1+q_2}\ldots\{l_{r-1}(n)\}^{1+q_{r-1}}\{l_r(n)\}^{1+q_r+\epsilon}}$$

converges in the ordinary manner under the same circumstances with regard to the function $f(x)$.

20. We now proceed to illustrate a different method by proving certain theorems which are closely connected in the theorem of § 17.

THEOREM.—*If $e^{|f(x)|}$ is summable and $f(x) \sim \Sigma(a_n \cos nx + b_n \sin nx), \ldots,$ then*

$$\Sigma (a_n \cos nx + b_n \sin nx)/(\log n)^2, \ldots,$$

oscillates finitely everywhere and, in fact, boundedly.

Let $F(x)$ denote the indefinite integral of $f(x)$, and suppose, for convenience, $f(x) \geqslant 0$. Then, since $e^{|f(x)|}$ is summable,

$$h\, e^{|F(x+h)-F(x)|/h} < K,$$

where K is a constant, which, however, may be taken as small as we please, provided the range of h is suitably restricted. Hence it is easily seen that

$$|F(x+h)-F(x)| = Oh \log\left|\frac{1}{h}\right| + Oh.$$

Denoting, as usual, by s_n the n-th partial summation of the Fourier series

of $f(x)$, then

$$\begin{aligned} s_n &= O\int_0^c \{f(x+t)+f(x-t)\}\,\frac{\sin nt}{t}\,dt \\ &= On\int_0^{1/n} \{f(x+t)+f(x-t)\}\,dt + O\int_{1/n}^c \frac{f(x+t)+f(x-t)}{t}\,dt \\ &= O\log n + O\int_{1/n}^c \frac{F(x+t)-F(x-t)}{t^2}\,dt \\ &= O\log n + O\int_{1/n}^c t^{-1}\log t^{-1}\,dt \\ &= O(\log n)^2. \end{aligned}$$

Hence writing, as usual, A_n for $a_n\cos nx+b_n\sin nx$, we have

$$\begin{aligned} \sum_{n=2}^{\infty}\frac{nA_n}{(\log n)^2} &= O\sum_{n=2}^{\infty}\left\{\frac{n}{(\log n)^2}-\frac{n+1}{(\log n+1)^2}\right\}s_n + O\frac{ns_n}{(\log n)^2} \\ &= O\sum_{n=2}(\log n)^{-2}s_n + (\log n)^{-2}ns_n = O(n). \end{aligned}$$

Thus, the series $\Sigma A_n(\log n)^{-2}$ will oscillate finitely, and even boundedly in the ordinary way, if it does so in the Cesàro way, index unity.

But this has been proved in § 17 to be the case. For, if

$$g(x)\sim\Sigma(\log n)^{-2}\cos nx,$$

we have seen in § 17 that $g(x)\log|g(x)|$ is summable, whence it follows that $g(x)$ and $f(x)$ belong to complementary classes of summable functions. Thus the theorem is proved.

21. In a similar manner we can prove the following theorem :—

THEOREM.—*If* $e^{|f(x)|}$ *is summable,* $\Sigma(a_n\cos nx+b_n\sin nx)/\log n\log\log n$ *oscillates boundedly everywhere.*

In this case we have

$$h\,e^{|F(x+h)-F(x)|/h} < K,$$

whence

$$F(x+h)-F(x) = Oh\log\log\frac{1}{h}.$$

Just as before we get, *mutatis mutandis*,

$$s_n = O\log\log n + O\int_{1/n}^c t^{-1}\log\log t^{-1}\,dt = O\log n\log\log n.$$

Now writing $$g(x) \sim \sum_{n=3}^{\infty} \cos nx/\log n \log\log n,$$

we have seen in § 17 that $g(x)\log|g(x)|\{\log\log g(x)\}^{1-\epsilon}$ is summable; *a fortiori* this is the case if we omit ϵ. Thus $f(x)$ and $g(x)$ belong to complementary classes of summable functions, and therefore, by § 17, the series under discussion converges ($C1$).

But

$$\sum_{n=3} \frac{nA_n}{\log n \log\log n} = Os_n/\log n \log\log n + On s_n/\log n \log\log n = O(n),$$

whence, as before, the series under discussion oscillates boundedly, when summed in the ordinary way, which proves the theorem.

22. It is clear that the argument used is generalisable. Let us denote e^z by $E_1(z)$, e^{e^z} by $E_2(z)$, and generally write

$$E_p(z) = e^{E_{p-1}(z)};$$

then we have the following theorem.

Theorem.—*If* $E_p\{|f(x)|\}$ *is summable, the series*

$$\Sigma(a_n \cos nx + b_n \sin nx)/l_1(n)\, l_p(n)$$

oscillates boundedly everywhere.

23. Finally, we consider the case when the function $|f(x)|$ is slightly more than summable and show how, in this case, the integrated series of the Fourier series of $f(x)$ and of its allied series will continue to converge when their coefficients are multiplied by suitable monotone increasing constants. It may be remarked that the integrated series of the allied series will not in general converge, but it will do so, as I have already pointed out, if $f(x)\log|f(x)|$ is summable. We shall suppose, then, that $u\phi|u|$ is summable, where

$$\phi(u) = \{l_1(u)\}^{q_1}\{l_2(u)\}^{q_2}\ldots\{l_r(u)\}^{q_r}.$$

We have, as before,

$$s_n = On\int_0^{1/n} f(x+t)\,dt + O\int_{1/n}^{c} \frac{f(x+t)}{t}\,dt.$$

Now, in our case,

$$|F(x+h)-F(x)|\,\phi\left\{\frac{|F(x+h)-F(x)|}{h}\right\} < K,$$

where K may be chosen as small as we please, provided h is also suitably restricted. We thus see that

$$\left|\frac{F(x+h)-F(x)}{h}\right| = o\left\{1/\phi\left(\frac{1}{h}\right)\right\}.$$

Therefore

$$s_n = o\left\{\frac{n}{\phi(n)}\right\} + o\int_{1,n}^{c} \frac{dt}{t^2\phi\left(\frac{1}{t}\right)} = o\left\{\frac{n}{\phi(n)}\right\} + o\int^{n} \frac{dt}{\phi(t)} = o\left\{\frac{n}{\phi(n)}\right\}.$$

Now write

$$\begin{aligned} v = h\left(\frac{1}{x}\right) &= \Sigma n \cos nx / \{l_1(n)\}^{p_1} \dots \{l_r(n)\}^{p_r} \\ &= O\int_x \frac{\left\{l_1\left(\frac{1}{t}\right)\right\}^{p_1} \dots \left\{l_r\left(\frac{1}{t}\right)\right\}^{p_r}}{t}\, dt \\ &= O\left\{l_1\left(\frac{1}{x}\right)\right\}^{1+p_1}\left\{l_2\left(\frac{1}{x}\right)\right\}^{p_2} \dots \left\{l_r\left(\frac{1}{x}\right)\right\}^{p_r} \\ &= O\left\{l_1\left(\frac{1}{\sqrt{x}}\right)\right\}^{1+p_1} \dots \left\{l_2\left(\frac{1}{\sqrt{x}}\right)\right\}^{p_2} \dots \left\{l_r\left(\frac{1}{\sqrt{x}}\right)\right\}^{1+p_r}. \qquad \text{(A)} \end{aligned}$$

Now, if we reflect on the relation between the exponential function and the logarithm function, we shall see that we may say that, if a function is of the order of a logarithm of $\log(1/x)$, it is such that the inverse of the logarithm of the function, that is the exponential function of the function, is summable if the function be multiplied by a suitable constant. We have had already occasion to use this fact in the papers on "Summable Functions and their Fourier Series" and "On a Certain Series of Fourier." In precisely the same way we see that, when we are concerned with a product of logarithms, so that the function whose order we are considering is of the order $\phi(1/x)$, then, by a suitable treatment of the function, say u, it is seen that $\phi^{-1}(u)$ is a summable function of x. Here we make use of this principle when we assert that $1/x = h^{-1}(v)$ gives substantially the function of v which is summable. Also

$$\frac{d}{dv}\{h^{-1}(v)\} = -\frac{1}{x^2}\frac{dx}{dv} = O\frac{1}{x}\left\{l_1\left(\frac{1}{x}\right)\right\}^{-p_1} \dots \left\{l_r\left(\frac{1}{x}\right)\right\}^{-p_r}.$$

Now suppose that $q_1, q_2, \dots, q_r$ are such that the complementary class of functions u is that for which

$$u\{l_1(u)\}^{q_1} \dots \{l_r(u)\}^{q_r}$$

is summable. Then writing, in accordance with the definition of complementary classes of functions,

$$v = \{l_1(u)\}^{q_1} \dots \{l_r(u)\}^{q_r} = \phi(u),$$

we ought to have $$\phi^{-1}(v) = O\frac{d}{dv}\{h^{-1}(v)\},$$

and therefore

$$v = O\phi\left[\frac{d}{dv}\{h^{-1}(v)\}\right] = O\phi\left\{-\frac{1}{x^2}\frac{dx}{dv}\right\}$$

$$= O\phi\left[\frac{1}{x}\left\{l_1\left(\frac{1}{x}\right)\right\}^{-p_1} \dots \left\{l_r\left(\frac{1}{x}\right)\right\}^{-p_r}\right] = \left\{l_1\left(\frac{1}{x}\right)\right\}^{q_1} \dots \left\{l_r\left(\frac{1}{x}\right)\right\}^{q_r}.$$

Comparing this with (A), we get

$$q_1 = 1+p_1,\ q_2 = p_2,\ \dots,\ q_r = p_r.$$

Hence, if u be a function of x, such that

$$u\{l_1(u)\}^{1+p_1}\{l_2(u)\}^{p_2} \dots \{l_r(u)\}^{p_r}$$

is summable, then u and v are functions belonging to complementary classes.

Hence, if $u\phi(u)$ is summable, the series

$$\Sigma\frac{A_n}{nl_1(n)}\phi(n) \equiv \Sigma\frac{a_n\cos nx + b_n\sin nx}{n}\{l_1(n)\}^{q-1}\{l_2(n)\}^{q_2} \dots \{l_r(n)\}^{q_r}$$

converges ($C1$).

It will therefore also converge in the ordinary way, if

$$\Sigma A_n\phi(n)/l_1(n) = o(n).$$

Now the order of infinity of this last series is

$$O\Sigma\frac{s_n\{l_2(n)\}^{q_2} \dots \{l_r(n)\}^{q_r}}{\{l_1(n)\}^{2-q_1}n} + Os_n\{l_1(n)\}^{q_1-1}\{l_2(n)\}^{q_2} \dots \{l_r(n)\}^{q_r}$$

$$= O\Sigma\frac{s_n\phi(n)}{\{l_1(n)\}^2} + O\Sigma\frac{s_n\phi(n)}{l_1(n)} = o\Sigma\frac{1}{\{l_1(n)\}^2} + o\left\{\frac{n}{l_1(n)}\right\} = o(n).$$

Thus the theorem is true, viz., *if* $f(x)\ \phi\{|f(x)|\}$ *is summable, where*

$$\phi(u) = \{l_1(u)\}^{q_1} \dots \{l_r(u)\}^{q_r},$$

the series $\Sigma A_n\phi(n)/nl_1(n)$ *converges everywhere in the ordinary way.*

24. Now consider

$$g(x) = \Sigma \frac{\sin nx}{n} \{l_1(n)\}^{p_1} \dots \{l_r(n)\}^{p_r}$$

$$= O\left[l_1\left(\frac{1}{x}\right)^{p_1} \left\{l_2\left(\frac{1}{x}\right)\right\}^{p_2} \dots \left\{l_r\left(\frac{1}{x}\right)\right\}^{p_r}\right].$$

Then the whole of the preceding argument applies except that it will now be sufficient to take $q_1 = p_1$ instead of $1+p_1$. Thus, we shall find that, *if* $f(x)\phi\{|f(x)|\}$ *is summable,*

$$\Sigma \frac{B_n}{n} \phi(n) \equiv \Sigma \frac{b_n \cos nx - a_n \sin nx}{n} \{l_1(n)\}^{q_1} \{l_2(n)\}^{q_2} \dots \{l_r(n)\}^{q_r}$$

converges everywhere in the ordinary way.

Accordingly we may say that *if* $u\phi(u)$ *is summable its integrated series continues to converge to an integral when the coefficients are increased in the ratio* $\phi(n) : 1$, *while the integrated series of the allied series converges to a function which is at least a continuous function when the coefficients are increased in the ratio* $\phi(n) : \log n$.

25. We may slightly extend the result we have arrived at in regard to the Fourier series of a function whose $(1+p)$-th power is summable, and its allied series. It is, in fact, clear that, instead of the factor n^{-q}, where $q > 1/(1+p)$, we may take the factor

$$1/n^{1/(p+1)} \{l_1(n)\}^{q_1} \{l_2(n)\}^{q_2} \dots \{l_{r-1}(n)\}^{q_{r-1}} l_r(n)^{q_r},$$

provided $$q_r > p/(1+p)$$

and $$q_1 = q_2 = \dots = q_{r-1} = p/(1+p).$$

In fact, denoting either the cosine or the sine series having this last-named typical coefficient by $g(x)$, we see that

$$\{g(x)\}^{1/(1+p)}$$

is summable, whence, by the usual argument the convergence factor in question certainly changes both the Fourier series and the allied series into the Fourier series of continuous functions, and the rest of the reasoning is as before.*

* [*Note added August 9th*, 1913. — Readers may be glad to know that the paper of Mr. G. H. Hardy referred to in the introductory remarks is that just published in these *Proceedings*, pp. 365–373. It may also be well to call attention to the fact that in a paper presented to the Society in May, and shortly to appear in the *Proceedings*, I have shown that Mr. Hardy's extensions of my results may themselves be extended; they are indeed true of series which, without being necessarily Fourier series, are the derived series of the Fourier series of functions of bounded variation.]

ON SOME CASES OF TIDAL MOTION OF ROTATING SHEETS OF WATER

By J. PROUDMAN.

Communicated by Prof. H. LAMB.

[Received February 27th, 1913.—Read March 13th, 1913.]

1. The following investigations arose out of an attempt to obtain an expression for the free tidal motion of a rotating flat sheet of water in the form of a circular sector. This was suggested as a subject for research by Prof. Lamb. No progress, however, was made with the original problem except in the case of the semicircle, and in this case only when the angular velocity of rotation was small compared with the free periods of the relative motion of the water. Even in this restricted case the resulting expression was so complicated as to make interpretation very difficult, if not impossible, and consequently it is not given below.

After a statement of the general conditions (which were first investigated by Lord Kelvin *), and their application to the case of a nearly circular sheet of water, a discussion is given of the limiting forms of forced tides as the period of the disturbing force tends to become infinite. It appears that these limiting forms are, for rotating flat sheets of water of uniform depth, always different from the forms given by the equilibrium theory. The limiting form is calculated for a *rectangular* sheet, and the result may, perhaps, be of interest, as the complete determination of the tides on such a sheet has not yet been obtained.† Some partial solutions are then given, and, finally, the free symmetrical motion is obtained for a particular small circular sea on a sphere, as the next approximation after that which regards the sea as flat. This may, perhaps, give a little help

* "On Gravitational Oscillations of Rotating Water," *Proc. Roy. Soc. Edin.*, Vol. x, p. 92 (1879) [*Phil. Mag.* (5), Vol. x, p. 109 (1880); *Math. and Phys. Papers*, Vol. IV, p. 141].

† Cf. Lord Kelvin, *ibid.*, and Lord Rayleigh, "On the Vibrations of a Rectangular Sheet of Rotating Liquid," *Phil. Mag.* (6), Vol. v, p. 297 (1903).

in forming an estimate as to how far the results obtained for flat sheets of water may be applied to small seas on the earth's surface.

General Equations for Flat Rotating Sheets.

2. Suppose the sheet of water to be rotating with constant angular velocity ω about an axis perpendicular to its plane, and let the depth of the water (as rotating in free relative equilibrium) be uniform and equal to h. Suppose the disturbing forces to have a potential Ω, and let ζ denote the elevation of the free surface at any point. Then for a simple harmonic disturbance, in which the time factor is $e^{i\sigma t}$, we have the equation *

$$\nabla_1^2 \zeta + \frac{\sigma^2 - 4\omega^2}{gh} \zeta = \nabla_1^2 \bar{\zeta}, \tag{1}$$

where $\nabla_1^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ in Cartesian coordinates, and $\bar{\zeta} = -\Omega/g$, g being the acceleration due to gravity.

The relative velocity in any direction is given by

$$\frac{g}{\sigma^2 - 4\omega^2} \left(i\sigma \frac{\partial}{\partial s_1} + 2\omega \frac{\partial}{\partial s_2} \right) (\zeta - \bar{\zeta}), \tag{2}$$

where $\partial/\partial s_1$ denotes differentiation in that direction, and $\partial/\partial s_2$ in the direction which is a right angle in advance.

From this we see that the boundary condition is

$$\left(i\sigma \frac{\partial}{\partial n} + 2\omega \frac{\partial}{\partial s} \right) (\zeta - \bar{\zeta}) = 0, \tag{3}$$

where $\partial/\partial n$ denotes differentiation along the outward drawn normal, and $\partial/\partial s$ along the positive direction of the arc of the boundary.

The above equations do not necessarily hold when $\sigma = 0$, or $\sigma^2 = 4\omega^2$.

If we integrate (3) round any closed boundary line, we obtain

$$i\sigma \int \frac{\partial}{\partial n} (\zeta - \bar{\zeta})\, ds = -2\omega \int \frac{\partial}{\partial s} (\zeta - \bar{\zeta})\, ds. \tag{4}$$

Now ζ is essentially a one-valued function of position, and we assume $\bar{\zeta}$ to be so also. It then follows that the integral on the right-hand side of (4)

* See Lamb, *Hydrodynamics*, 3rd ed., p. 303.

always vanishes, and consequently

$$\int \frac{\partial}{\partial n}(\xi-\bar{\xi})\,ds = 0, \tag{5}$$

from (4) if $\sigma \neq 0$, the integral being taken round any closed boundary line. Now the "relative circulation" round such a boundary line is given by

$$\frac{g}{\sigma^2-4\omega^2}\left[i\sigma\int\frac{\partial}{\partial s}(\xi-\bar{\xi})\,ds-2\omega\int\frac{\partial}{\partial n}(\xi-\bar{\xi})\,ds\right],$$

from (2), and by the above each integral in this is seen to vanish. A more general result, however, is the following: In any case of tidal motion under forces which have a single-valued potential, the circulation round any closed boundary line is constant. This easily follows from the theorem that the circulation in a closed chain of particles is constant.

We shall require equation (1) in polar coordinates. Let

$$\kappa^2 = \frac{\sigma^2-4\omega^2}{gh}. \tag{6}$$

Then (1) becomes in polars

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)(\xi-\bar{\xi})+\kappa^2\xi = 0. \tag{7}$$

For free motion we have $\bar{\xi} = 0$, and then normal solutions of (7) are of the type $(A\cos n\theta+B\sin n\theta)J_n(\kappa r)$, where A, B and n are any constants and $J_n(\kappa r)$ is the Bessel's function of the first kind of order n and argument κr.

For forced motion the forms of $\bar{\xi}$ which are of greatest interest are*

$$\bar{\xi} = C\left(\frac{r}{a}\right)^s e^{i(\sigma t+s\theta)}, \tag{8}$$

and

$$\bar{\xi} = C(\kappa r)^2 e^{i\sigma t}, \tag{9}$$

C being a constant.

Tides on a Nearly Circular Sheet.†

3. Using polar coordinates, let the equation of the boundary be

$$r = a\{1+f(\theta)\}, \tag{10}$$

* See Lamb, *l.c.*, chap. viii, App.; (9) is taken as corresponding (as far as is necessary) for a plane, to equation (13) of Lamb's Appendix, κ being inserted for convenience. The value of C in (9), however, is of a higher order of small quantities than that in (8).

† The following method is similar to that used by Lord Rayleigh for the solution of an analogous problem, *Theory of Sound*, § 209.

where $f(\theta)$ is a differentiable function of θ, periodic in 2π, and so small that $|f(\theta)|^2$ and $|f'(\theta)|^2$ may be neglected.

Then for the boundary we have

$$\frac{\partial}{\partial n} = \frac{\partial}{\partial r} - f'(\theta)\frac{\partial}{a\partial\theta},$$

and

$$\frac{\partial}{\partial s} = \frac{\partial}{r\partial\theta} + f'(\theta)\frac{\partial}{\partial r} = \{1-f(\theta)\}\frac{\partial}{a\partial\theta} + f'(\theta)\frac{\partial}{\partial r},$$

so that the boundary condition (3) becomes

$$\left[i\sigma\left\{\frac{\partial}{\partial r} - f'(\theta)\frac{\partial}{a\partial\theta}\right\} + 2\omega\left\{[1-f(\theta)]\frac{\partial}{a\partial\theta} + f'(\theta)\frac{\partial}{\partial r}\right\}\right](\zeta-\bar{\zeta}) = 0, \quad (11)$$

for

$$r = a\{1+f(\theta)\},$$

and all values of θ.

If we take a to be the radius of the circle of area equal to that of the sheet under consideration, we shall have

$$\int_0^{2\pi} f(\theta)\,d\theta = 0. \quad (12)$$

We may suppose $f(\theta)$ expanded as a Fourier's series, and then the following method is applicable; but, owing to the complication of the results, we shall only consider the cases in which $f(\theta) = \epsilon\cos n\theta$, n being an integer and ϵ^2 negligible. Also we shall only consider the modes of vibration which are nearly symmetrical about the origin, but the others may be calculated with very little more trouble.

4. For the *free oscillations* in the modes which are *nearly symmetrical* about the origin we have to satisfy

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \kappa^2\right)\zeta = 0, \quad (13)$$

together with the condition (11).

We assume

$$\zeta = J_0(\kappa r) + \epsilon(A\cos n\theta + B\sin n\theta)J_n(\kappa r), \quad (14)$$

where A and B are constants, and the time factor $e^{i\sigma t}$ has been omitted for shortness. This satisfies (13), and we shall see that A, B and κ can be chosen so as to make it satisfy (11).

Substituting from (14) in (11) we have, always neglecting ϵ^2,

$$i\sigma\{\kappa J_0'(\kappa r)+\epsilon(A\cos n\theta+B\sin n\theta)\,\kappa J_n'(\kappa r)\}$$

$$+2\omega\left\{\frac{n\epsilon}{a}(-A\sin n\theta+B\cos n\theta)\,J_n(\kappa r)-\epsilon n\sin n\theta\,\kappa J_0'(\kappa r)\right\}=0, \quad (15)$$

for $$r=a(1+\epsilon\cos n\theta),$$

and all values of θ.

Now on the boundary we may put

$$J_0'(\kappa r)=J_0'(\kappa a)+\kappa a\epsilon\cos n\theta\,J_0''(\kappa a),$$

and similarly for $J_n(\kappa r)$ and $J_n'(\kappa r)$, so that we see that in the terms in (15) involving ϵ we may replace r by a. Consequently we obtain, after a little rearrangement,

$$i\sigma(\kappa a)\,J_0'(\kappa a)+\epsilon\cos n\theta\{i\sigma(\kappa a)^2J_0''(\kappa a)+Ai\sigma(\kappa a)J_n'(\kappa a)+B\,2\omega n\,J_n(\kappa a)\}$$

$$+\epsilon\sin n\theta\{Bi\sigma(\kappa a)\,J_n'(\kappa a)-A\,2\omega n\,J_n(\kappa a)-2\omega n(\kappa a)\,J_0'(\kappa a)\}=0, \quad (16)$$

for all values of θ.

To satisfy this we take

$$J_0'(\kappa a)=0, \quad (17)$$

$$Ai\sigma(\kappa a)\,J_n'(\kappa a)+B\,2\omega n\,J_n(\kappa a)+i\sigma(\kappa a)^2J_0''(\kappa a)=0, \quad (18)$$

$$A\,2\omega n J_n(\kappa a)-Bi\sigma(\kappa a)J_n'(\kappa a)=0. \quad (19)$$

Equation (17) gives the possible values of κ, and thence of σ from (6). We notice that the free periods are the same as for a circular sheet of water of the same area.* This remark also holds for the free modes which are not nearly symmetrical.

Solving (18) and (19) for A and B we obtain

$$A=\frac{\sigma^2(\kappa a)^3J_0''(\kappa a)\,J_n'(\kappa a)}{4\omega^2n^2J_n^2(\kappa a)-\sigma^2(\kappa a)^2J_n'^2(\kappa a)}, \quad (20)$$

and

$$B=-i\,\frac{2\omega n\sigma(\kappa a)^2J_0''(\kappa a)\,J_n(\kappa a)}{4\omega^2n^2J_n^2(\kappa a)-\sigma^2(\kappa a)^2J_n'^2(\kappa a)}. \quad (21)$$

Substituting these in (14) we obtain the desired result, viz.,

$$\xi=J_0(\kappa r)+\epsilon\,\frac{\sigma(\kappa a)^2J_0''(\kappa a)}{4\omega^2n^2J_n^2(\kappa a)-\sigma^2(\kappa a)^2J_n'^2(\kappa a)}$$

$$\times\{\sigma(\kappa a)\,J_n'(\kappa a)\cos n\theta-2\omega nJ_n(\kappa a)\,i\sin n\theta\}\,J_n(\kappa r), \quad (22)$$

subject to (17).

* See Lamb, *l.c.*, p. 306.

The above investigation will not be valid if κ and n are such as to make

$$4\omega^2 n^2 J_n^2(\kappa a) - \sigma^2 (\kappa a)^2 J_n'^2(\kappa a)$$

small enough to be comparable with ϵ. The vanishing of this expression would mean that κ was also appropriate to that type of free oscillations of the sheet in which the principal term in ξ is $e^{in\theta} J_n(\kappa r)$ or $e^{-in\theta} J_n(\kappa r)$.

Of course, to apply (22) we must restore the time factor $e^{i\sigma t}$ and then take the real part.

5. For the *forced oscillations* of *nearly symmetrical* type we have to satisfy, from (7) and (9),

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \kappa^2\right) \xi = 4\kappa^2 C. \tag{23}$$

A particular integral of this is $\xi = 4C$, so that we assume

$$\xi = 4C + DJ_0(\kappa r) + \epsilon (A \cos n\theta + B \sin n\theta) J_n(\kappa r), \tag{24}$$

where A, B and D are constants, as this satisfies (23).

Substituting in (11) and proceeding as in the last section, we obtain

$$\begin{aligned} &Di\sigma(\kappa a) J_0'(\kappa a) - 2Ci\sigma(\kappa a)^2 + \epsilon \cos n\theta \\ &\qquad \times \{Di\sigma(\kappa a)^2 J_0''(\kappa a) + Ai\sigma(\kappa a) J_n'(\kappa a) + B\, 2\omega n J_n(\kappa a) - 2Ci\sigma(\kappa a)^2\} \\ &+ \epsilon \sin n\theta \{Bi\sigma(\kappa a) J_n'(\kappa a) - A\, 2\omega n J_n(\kappa a) - D\, 2\omega n (\kappa a) J_0'(\kappa a) \\ &\qquad + 4C\omega n (\kappa a)^2\} = 0, \end{aligned} \tag{25}$$

for all values of θ.

To satisfy this we take

$$DJ_0'(\kappa a) - 2C(\kappa a) = 0, \tag{26}$$

$$Ai\sigma(\kappa a) J_n'(\kappa a) + B\, 2\omega n J_n(\kappa a) + Di\sigma(\kappa a)^2 J_0''(\kappa a) - 2Ci\sigma(\kappa a)^2 = 0, \tag{27}$$

$$A\, 2\omega n J_n(\kappa a) - Bi\sigma(\kappa a) J_n'(\kappa a) = 0. \tag{28}$$

Solving these for D, A and B, we obtain

$$D = \frac{2C(\kappa a)}{J_0'(\kappa a)}, \tag{29}$$

$$A = -\frac{2C\sigma^2(\kappa a)^3 J_n'(\kappa a)\{J_0'(\kappa a) - (\kappa a) J_0''(\kappa a)\}}{J_0'(\kappa a)\{4\omega^2 n^2 J_n^2(\kappa a) - \sigma^2(\kappa a)^2 J_n'^2(\kappa a)\}}, \tag{30}$$

and

$$B = i\frac{4C\omega n\sigma(\kappa a)^2 J_n(\kappa a)\{J_0'(\kappa a) - (\kappa a) J_0''(\kappa a)\}}{J_0'(\kappa a)\{4\omega^2 n^2 J_n^2(\kappa a) - \sigma^2(\kappa a)^2 J_n'^2(\kappa a)\}}, \tag{31}$$

so that on substituting in (24) we obtain the result

$$\xi = 4C + \frac{2C(\kappa a)}{J_0'(\kappa a)}\left[J_0(\kappa r) - \epsilon \frac{\sigma(\kappa a)\{J_0'(\kappa a) - (\kappa a) J_0''(\kappa a)\}}{4\omega^2 n^2 J_n^2(\kappa a) - \sigma^2(\kappa a)^2 J_n'^2(\kappa a)} \times \{\sigma(\kappa a) J_n'(\kappa a)\cos n\theta - 2\omega n J_n(\kappa a)\, i \sin n\theta\} J_n(\kappa r)\right]. \quad (32)$$

The above will not be valid if either or the product of both of $J_0'(\kappa a)$, $4\omega^2 n^2 J_n^2(\kappa a) - \sigma^2(\kappa a)^2 J_n'^2(\kappa a)$ is small enough to be comparable with ϵ. The vanishing of the first of these would mean that the period of the disturbing force was equal to one of the free periods investigated in the last section, while the significance of the vanishing of the second has already been given.

6. It is interesting to obtain the *limiting form* of (32) as $\sigma \to 0$, and to compare it with the value obtained by the equilibrium theory.

On making $\sigma \to 0$, since $n \neq 0$, (32) takes the form

$$\xi = 4C + \frac{2C(\kappa a)}{J_0'(\kappa a)} J_0(\kappa r), \quad (33)$$

where κ has its limiting value, which, however, is imaginary, since $\kappa^2 \to -4\omega^2/gh$. Let $\kappa \to ik$, so that $k^2 = 4\omega^2/gh$; then (33) becomes

$$\xi = 4C - \frac{2C(ka)}{I_0'(ka)} I_0(kr), \quad (34)$$

where $I_0(z) = J_0(iz)$. This, then, is the limiting form towards which the forced tide corresponding to

$$\bar{\xi} = -C(kr)^2 e^{i\sigma t} \quad (35)$$

tends, as the period tends to become infinite.

Now the "equilibrium tide," in which the elevation of the free surface is ξ_0, say, is given by $\xi_0 = \bar{\xi} + A$, where A is a constant so chosen as to make $\iint \xi_0 dS = 0$, the integral being taken over the area of the sheet. For the area in question, viz, that bounded by $r = a(1+\epsilon\cos n\theta)$, this gives

$$\xi_0 = \tfrac{1}{2}C(ka)^2 - C(kr)^2. \quad (36)$$

Thus, ξ and ξ_0 are different, or, in other words, the limiting form of the forced tide as the period tends to become infinite is not the same as that calculated on the equilibrium theory, when the sheet of water is rotating.

It is immediately verified, however, that $\xi/\xi_0 \to 1$, as $k \to 0$, *i.e.*, as the angular velocity tends to become zero, the case of infinite depth being excluded owing to the *tidal* nature of the motion.

The expressions (34) and (36) do not involve ϵ, so that both the limiting form and the "equilibrium value" of the forced symmetrical tides on a flat rotating sheet of water are not sensibly affected by a small departure from the circular form of boundary.

We shall return later to the subject of the present section in a more general manner.

Limiting Forms of Forced Tides of Long Period on Flat Rotating Sheets of Water.

7. It was mentioned in Section 2 that the equations there stated do not necessarily hold when $\sigma = 0$. In fact, when we start with $\sigma = 0$ we have the theory of steady motions * under the disturbing force (which is then constant as regards time). These are largely arbitrary, and include as a particular case, viz., that in which the relative motion vanishes, the ordinary equilibrium tides.

However, as we shall see, if we take the limiting forms of the equations in Section 2 as $\sigma \to 0$, we obtain conditions which give a determinate value for the tide in any particular instance. A special case was calculated in the last section.

Equations (1) and (3) become respectively

$$\nabla_1^2 \zeta - \frac{4\omega^2}{gh} \zeta = \nabla_1^2 \bar{\zeta}, \tag{37}$$

and

$$\frac{\partial}{\partial s} (\zeta - \bar{\zeta}) = 0, \tag{38}$$

at the boundary. This last condition may be written in the form

$$\zeta = \bar{\zeta} + \text{constant}, \tag{39}$$

where the "constant" is the same for each distinct portion of the boundary, but may be different for different distinct portions if the sheet be not simply connected.

The solution of (37) and (39) is, however, not determinate; for we may add, without violating the equations, any solution of (37) with $\bar{\zeta} = 0$, which takes constant values over the separate portions of the boundary. An additional necessary condition is taken to be given by (5) for each distinct portion of the boundary. This means that the relative circulation

---

* Lamb, *Hydrodynamics*, 2nd ed. (1895), p. 334; 3rd ed., p. 303.

round each closed boundary line vanishes, as it does so long as σ is not actually zero. The "limiting form" is defined to be such that this condition remains satisfied, so that the results may have approximate application to forced tides of very long, but not actually infinite, period. Of course, steady motions (*i.e.*, motions with $\sigma = 0$) for which (5) is not satisfied, or for which the relative circulation round each boundary line does not vanish, may exist at any time.

We can now shew that a unique solution of (37), (39), and (5) exists—at least, for any region, whatever the connectivity, for which the ordinary problem of Dirichlet can be proved to have a solution. However, to avoid a somewhat lengthy discussion, we will here only consider the case in which there is a single closed boundary.

Consider a solution ζ_1 of (37) which takes the value $\bar{\zeta}$ on the boundary, and also a solution ζ_2 of (37) with $\bar{\zeta} = 0$, which takes the value unity on the boundary. The question of the existence of ζ_1 and ζ_2 can be reduced to that of a solution of Dirichlet's problem.* We now show that

$$\int \frac{\partial \zeta_2}{\partial n}\, ds \neq 0\,;$$

for,

$$\begin{aligned}\int \frac{\partial \zeta_2}{\partial n}\, ds &= \int \zeta_2 \frac{\partial \zeta_2}{\partial n} ds \\ &= \iint \zeta_2 \nabla_1^2 \zeta_2\, dS + \iint \left\{ \left(\frac{\partial \zeta_2}{\partial x}\right)^2 + \left(\frac{\partial \zeta_2}{\partial y}\right)^2 \right\} dS \\ &= k^2 \iint \zeta_2^2\, dS + \iint \left\{ \left(\frac{\partial \zeta_2}{\partial x}\right)^2 + \left(\frac{\partial \zeta_2}{\partial y}\right)^2 \right\} dS,\end{aligned}$$

where we have used Green's formula. Since k is real, and ζ_2 not everywhere zero, we obtain the desired result.

We now add to ζ_1 such a constant multiple C of ζ_2 as will make $\zeta = \zeta_1 + C\zeta_2$ satisfy the conditions (37), (39), and (5).

We can easily shew that the solution is unique. For, if there were two solutions, their difference ζ' would satisfy

$$(\nabla_1^2 - k^2)\zeta' = 0,$$

and

$$\int \frac{\partial \zeta'}{\partial n}\, ds = 0,$$

at the boundary and be a constant there. By substituting ζ' for ζ_2 in the equations given above we see that ζ' must be everywhere zero.

* See, for instance, Heywood and Fréchet, *L'Equation de Fredholm*, chap. iii.

As far as the uniqueness of the solution in the general case is concerned this proof can be immediately extended.

Since in the present problem

$$\nabla_1^2(\zeta-\bar{\zeta}) = k^2\zeta, \tag{40}$$

where $$k^2 = 4\omega^2/gh,$$

we have $$\iint \zeta\, dS = \frac{1}{k^2}\iint \nabla_1^2(\zeta-\bar{\zeta})\, dS = \frac{1}{k^2}\Sigma\int \frac{\partial}{\partial n}(\zeta-\bar{\zeta})\, ds, \tag{41}$$

where the surface integrals extend over the whole sheet, and the line integrals round all the portions of the boundary, and we see that the condition of constancy of total volume of water is satisfied by (5). For the case of a single boundary the condition

$$\iint \zeta\, dS = 0, \tag{42}$$

may be taken instead of (5), to which it is then equivalent.

The reason why the necessary condition (42) is not included in (37) and (38), whereas it is included in the equations of which they are the limiting forms, can be seen from (4) and (41).

Since the "equilibrium tide" ζ_0 is given by $\zeta_0 = \bar{\zeta}+$ constant, we immediately see from (40) that the equality $\zeta = \zeta_0$ requires $k^2\zeta = 0$, and therefore $k = 0$ or $\zeta = 0$. Now when $k = 0$, $\omega = 0$, since the depth may not be infinite; but in this case the condition (38) would have to be replaced by $\partial(\zeta-\bar{\zeta})/\partial n = 0$ at the boundaries, and then $(\zeta-\bar{\zeta})$ would be a potential function with a zero normal derivative at the boundary, and, therefore, a constant. Hence we see that, when there is no rotation, the limiting form of the forced tide is the "equilibrium form"; but that when there *is* rotation the limiting form can *never* be exactly equal to the "equilibrium form," if it is other than null.*

8. We now proceed to the calculation, in some particular cases, of the limiting forms of the forced tides corresponding to the special value of $\bar{\zeta}$ given by (35).

Let us consider first the case of the *nearly circular* sheet defined by (10) with the condition (12). Of course, our results will be accurately true for a sheet which is exactly circular if we put $f(\theta) = 0$.

We have to satisfy

$$(\nabla_1^2-k^2)\zeta = -4Ck^2, \tag{43}$$

* Cf. Lamb, *l.c.*, p. 300.

with the conditions that

$$\zeta = -C(kr)^2 + \text{constant}, \tag{44}$$

at the boundary $r = a\{1+f(\theta)\}$,

and that

$$\int \frac{\partial}{\partial n}\{\zeta + C(kr)^2\}\, ds = 0, \tag{45}$$

the integral being taken round the boundary.

We assume

$$\zeta = 4C + AI_0(kr), \tag{46}$$

A being a constant. This satisfies (43), and we shall see that A can be chosen so as to make it satisfy both (44) and (45).

On the boundary we may put

$$I_0(kr) = I_0(ka) + (ka)\, f(\theta)\, I_0'(ka),$$

and

$$(kr)^2 = (ka)^2\{1+2f(\theta)\},$$

so that condition (44) reduces to

$$AI_0'(ka) = -2C(ka). \tag{47}$$

In (45) we may replace $\partial/\partial n$ by $\partial/\partial r$, since the assumed value of ζ does not involve θ. This gives

$$\int \{AkI_0'(kr) + 2Ck^2 r\}\, ds = 0,$$

which, to our order, may be written

$$\int_0^{2\pi} \{AI_0'(ka) + 2C(ka) + Bf(\theta)\}\{1+f(\theta)\}\, d\theta = 0,$$

or

$$\int_0^{2\pi} \{AI_0'(ka) + 2C(ka) + Df(\theta)\}\, d\theta = 0, \tag{48}$$

where B and D are certain constants. By virtue of (12), (48) reduces to the same as (47), and gives, on substituting in (46), the result

$$\zeta = 4C - \frac{2C(ka)}{I_0'(ka)} I_0(kr), \tag{34}$$

which is independent of θ and is unaffected by the small departure from the circular boundary.

The expression (34) has already been obtained in Section 6 by other methods, but not with the same generality as at present.

The "equilibrium value" is again given by (36).

9. Let us now consider the case of the *rectangular* sheet of water. Using Cartesians we take the origin of coordinates at the centre. Let the length of the side parallel to the axis of x be $2a$, and of that parallel to the axis of y be $2b$.

The equations we have to satisfy are (43), (44), and (5); but we will take the last in the form (42), since the region is simply connected. In the first place, let us take the "constant" of (44) to be $4C+Ck^2(a^2+b^2)$, and call the corresponding solution of (43) and (44) ζ'.

Then on the sides $x = \pm a$ we shall have

$$\zeta' = 4C+Ck^2\{(a^2+b^2)-(a^2+y^2)\} = 4C+Ck^2(b^2-y^2), \tag{49}$$

and similarly on the sides $y = \pm b$,

$$\zeta' = 4C+Ck^2(a^2-x^2). \tag{50}$$

As we shall then have the condition (42) unsatisfied, we next obtain a solution ζ'' of

$$(\nabla_1^2-k^2)\zeta = 0, \tag{51}$$

which takes a constant value over the boundary, and then add to ζ' such a multiple A of ζ'' as will make $\zeta = \zeta'+A\zeta''$ satisfy the condition (42).

Since a particular solution of (43) is $\zeta = 4C$, besides this we are only concerned with solutions of (51).

Now $$\zeta_s = \cosh m_s x \cos\frac{s\pi y}{2b},$$

satisfies (51), provided $$m_s^2 = k^2+\frac{s^2\pi^2}{4b^2},$$

and on $x = \pm a$, it takes the form

$$\zeta_s = \cosh m_s a \cos\frac{s\pi y}{2b},$$

while on $y = \pm b$, it vanishes, provided s is an odd integer.

Now, by Fourier's theorem, or otherwise,

$$b^2-y^2 = \frac{32b^2}{\pi^3}\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)^3}\cos\frac{(2n+1)\pi y}{2b},$$

for $-b \leqslant y \leqslant b$, so that

$$\Sigma\zeta_s = \frac{32Ck^2b^2}{\pi^3}\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)^3}\frac{\cosh\mu_n x}{\cosh\mu_n a}\cos\frac{(2n+1)\pi y}{2b}, \tag{52}$$

where $$\mu_n^2 = k^2+\frac{(2n+1)^2\pi^2}{4b^2}, \tag{53}$$

which is uniformly convergent over the rectangle, satisfies (51) and is equal to $Ck^2(b^2-y^2)$ on $x=\pm a$, while it vanishes on $y=\pm b$.

Similarly,

$$\frac{32Ck^2a^2}{\pi^3}\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)^3}\frac{\cosh\nu_n y}{\cosh\nu_n b}\cos\frac{(2n+1)\pi x}{2a}, \tag{54}$$

where

$$\nu_n^2=k^2+\frac{(2n+1)^2\pi^2}{4a^2}, \tag{55}$$

which is also uniformly convergent over the rectangle, satisfies (51) and is equal to $Ck^2(a^2-x^2)$ on $y=\pm b$, while it vanishes on $x=\pm a$.

Adding (52) and (54) to $4C$, we obtain an expression satisfying the conditions necessary for ζ'. Thus

$$\zeta'=4C+\frac{32Ck^2}{\pi^3}\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)^3}\times\left\{b^2\frac{\cosh\mu_n x}{\cosh\mu_n a}\cos\frac{(2n+1)\pi y}{2b}+a^2\frac{\cosh\nu_n y}{\cosh\nu_n b}\cos\frac{(2n+1)\pi x}{2a}\right\}. \tag{56}$$

We now proceed to the determination of ζ'', for which we make use of the known expansion

$$\frac{\pi}{4}=\sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}\cos(2n+1)\theta,$$

which holds for $-\frac{1}{2}\pi<\theta<\frac{1}{2}\pi$. From this and what has been said above we see that

$$\zeta''=\sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}\left\{\frac{\cosh\mu_n x}{\cosh\mu_n a}\cos\frac{(2n+1)\pi y}{2b}+\frac{\cosh\nu_n y}{\cosh\nu_n b}\cos\frac{(2n+1)\pi x}{2a}\right\}, \tag{57}$$

which is convergent over the rectangle, satisfies (51) and takes a constant value, viz., $\frac{1}{4}\pi$, over the sides (with the exception of the corners, which, however, need not trouble us).

Now $\iint\zeta'\,dx\,dy$, when taken over the area of the rectangle, has the value

$$16Cab+\frac{256Ck^2}{\pi^4}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4}\left\{\frac{b^3}{\mu_n}\tanh\mu_n a+\frac{a^3}{\nu_n}\tanh\nu_n b\right\},$$

and similarly $\iint\zeta''\,dx\,dy$ has the value

$$\frac{8}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\left\{\frac{b}{\mu_n}\tanh\mu_n a+\frac{a}{\nu_n}\tanh\nu_n b\right\},$$

since term-by-term integration can be justified in each case, so that the required value of ζ is given by

$$\zeta = 4C + \frac{32Ck^2}{\pi^3} \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)^3} \times \left\{ b^2 \frac{\cosh \mu_n x}{\cosh \mu_n a} \cos \frac{(2n+1)\pi y}{2b} + a^2 \frac{\cosh \nu_n y}{\cosh \nu_n b} \cos \frac{(2n+1)\pi x}{2a} \right\} - AC \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} \times \left\{ \frac{\cosh \mu_n x}{\cosh \mu_n a} \cos \frac{(2n+1)\pi y}{2b} + \frac{\cosh \nu_n y}{\cosh \nu_n b} \cos \frac{(2n+1)\pi x}{2a} \right\}, \quad (58)$$

where the constant A is given by

$$A = \frac{2\pi^4 ab + 32k^2 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} \left\{ \frac{b^3}{\mu_n} \tanh \mu_n a + \frac{a^3}{\nu_n} \tanh \nu_n b \right\}}{\pi^3 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \left\{ \frac{b}{\mu_n} \tanh \mu_n a + \frac{a}{\nu_n} \tanh \nu_n b \right\}}, \quad (59)$$

and μ_n, ν_n are given by (53) and (55).

The "equilibrium value" is given by

$$\zeta_0 = \tfrac{4}{3} Ck^2 ab\,(a^2+b^2) - Ck^2(x^2+y^2). \quad (60)$$

10. Let us now discuss the case of a sheet of water which has the shape of a *circular sector*, whose sides are given by $\theta = \pm a$, in polar coordinates.

We know that, unless $\cos sa = 0$, the function

$$\zeta_s = \frac{\cos s\theta}{\cos sa} I_s(kr), \quad (61)$$

satisfies (51) and becomes equal to $I_s(kr)$ on the sides. When $\cos sa = 0$, we may replace this by

$$\frac{\frac{\partial}{\partial s} \{\cos s\theta\, I_s(kr)\}}{\frac{\partial}{\partial s} \cos sa} = \frac{\theta \sin s\theta\, I_s(kr) - \cos s\theta \frac{\partial}{\partial s} I_s(kr)}{a \sin sa}, \quad (62)$$

which then satisfies the same conditions.

By means of the known expansions

$$1 = I_0(kr) + 2 \sum_{n=1}^{\infty} (-1)^n I_{2n}(kr),$$

and $$(kr)^2 = 8 \sum_{n=1}^{\infty} (-1)^{n-1} n^2 I_{2n}(kr),$$

together with (61) and (62), we can form expressions ξ' and ξ'', each of which satisfies (51), whilst at the sides $\theta = \pm a$,

$$\xi' = -C(kr)^2, \quad \xi'' = 1. \tag{63}$$

Further, by means of Fourier's theorem and the assumption

$$\xi'_n = \cos \frac{(2n+1)\pi\theta}{2a} I_{[(2n+1)\pi]/2a}(kr), \tag{64}$$

which satisfies (51) and vanishes at the sides, we can satisfy the conditions (63) on the enclosed arc $r = a$.

Then taking $\xi = \xi' + A\xi''$, and determining A by means of the condition (42), we obtain the required solution; but it would, perhaps, be difficult to interpret. For the case of the semicircle the solutions (62) would not be required.

Some Partial Solutions.

11. Leaving now the question of limiting forms of forced tides, we proceed to investigate the solution of the general equations of free motion given in Section 2 for regions, however, which are not bounded in all directions. The results will only be applicable over limited portions of the regions considered, since the fundamental equations are really only valid for limited regions.

Let us first consider the case of the *semi-infinite plane* given in polar coordinates by $0 \leqslant \theta \leqslant \pi$. We have to satisfy (13), while condition (3) becomes

$$\left(i\sigma \frac{\partial}{r\partial\theta} - 2\omega \frac{\partial}{\partial r}\right) \xi = 0, \tag{65}$$

for $\theta = 0$ and $\theta = \pi$.

We assume $$\xi = \sum_n (A_n \cos n\theta + B_n \sin n\theta) J_n(kr), \tag{66}$$

where the values of n are as yet unrestricted. This satisfies (13), and on substituting in (65) we obtain

$$\sum_n \left\{ \frac{i\sigma n}{r} (-A_n \sin n\theta + B_n \cos n\theta) J_n(\kappa r) \right.$$
$$\left. -2\omega (A_n \cos n\theta + B_n \sin n\theta) \kappa J'_n(\kappa r) \right\} = 0, \tag{67}$$

for $\theta = 0$ and $\theta = \pi$.

Now $$J_n'(\kappa r) = \frac{n}{(\kappa r)} J_n(\kappa r) - J_{n+1}(\kappa r),$$

and $$\frac{n}{(\kappa r)} J_n(\kappa r) = \tfrac{1}{2}\{J_{n+1}(\kappa r) + J_{n-1}(\kappa r)\},$$

so that, on dividing through by $\frac{1}{2}\kappa$, (67) may be written in the form

$$\sum_n \{[i\sigma(-A_n \sin n\theta + B_n \cos n\theta) - 2\omega(A_n \cos n\theta + B_n \sin n\theta)] J_{n-1}(\kappa r) + [i\sigma(-A_n \sin n\theta + B_n \cos n\theta) + 2\omega(A_n \cos n\theta + B_n \sin n\theta)] J_{n+1}(\kappa r)\} = 0, \tag{68}$$

for $\theta = 0$ and $\theta = \pi$.

To satisfy this we take n in the form $s+2m$, where s is a constant and m takes positive integral values, and proceed to make the coefficients of the different Bessel's functions in equation (68) vanish for $\theta = 0$ and $\theta = \pi$. The coefficient of $J_{s-1}(\kappa r)$ gives, except for $s = 0$,

$$i\sigma B_s - 2\omega A_s = 0,$$

and $$i\sigma(-A_s \sin s\pi + B_s \cos s\pi) - 2\omega(A_s \cos s\pi + B_s \sin s\pi) = 0,$$

from which we deduce

$$\frac{A_s}{B_s} = \frac{i\sigma}{2\omega} = \frac{A_s \cos s\pi + B_s \sin s\pi}{B_s \cos s\pi - A_s \sin s\pi}. \tag{69}$$

From this we see that we must have $\sin s\pi = 0$, so that s must be an integer. The case $s = 0$ is excluded from (69), since $J_1(\kappa r)$ and $J_{-1}(\kappa r)$ are not independent; but, on looking back at (67) we see that zero is a possible value of s.

With this determination of s we observe that the two conditions involved in (68) now reduce to the same and become in fact

$$\sum_{m=0}^{\infty} \{(i\sigma B_{s+2m} - 2\omega A_{s+2m}) J_{s+2m-1}(\kappa r) + (i\sigma B_{s+2m} + 2\omega A_{s+2m}) J_{s+2m+1}(\kappa r)\} = 0, \tag{70}$$

so that the determination of the constants is partly arbitrary. We have the equation

$$(i\sigma B_{s+2m} - 2\omega A_{s+2m}) + (i\sigma B_{s+2(m-1)} + 2\omega A_{s+2(m-1)}) = 0, \tag{71}$$

for $m = 1, 2, 3, \ldots$, when $s \neq 0$, and $m = 2, 3, 4, \ldots$, when $s = 0$.

We can make $A_{s+2m} = B_{s+2m} = 0$, for $m > 1$ if we take

$$i\sigma B_{s-2} + 2\omega A_{s+2} = 0.$$

A_s and B_s have their ratio determined by (69), except for $s = 0$, when B_0 does not exist, so that we obtain, when $s \neq 0$,

$$A_{s+2} = A_s = 1$$

say. When $s = 0$, (70) gives, from the coefficient of $J_1(\kappa r)\,[= -J_{-1}(\kappa r)]$,

$$(i\sigma B_2 - 2\omega A_2) + 4\omega A_0 = 0,$$

so that again we have $$A_2 = A_0 = 1.$$

Therefore solutions of our problem are given by

$$\zeta = \left\{ \cos s\theta - \frac{2\omega}{\sigma} i \sin s\theta \right\} J_s(\kappa r) + \left\{ \cos (s+2)\,\theta + \frac{2\omega}{\sigma} i \sin (s+2)\,\theta \right\} J_{s+2}(\kappa r), \quad (72)$$

where s is any integer, including zero.

It was from expressions of this type that the solution mentioned in Section 1, for a slowly rotating semicircle of radius a, was obtained.

The solution given by Lord Kelvin * for a rotating canal, in which the relative motion of the water is everywhere parallel to the banks, can be constructed from the expressions (72) or obtained directly by a certain choice of the constants in (71).

12. Let us now consider the case of a *channel* bounded by $x = 0$ and $x = a$, using Cartesian coordinates.

We have to satisfy

$$(\nabla_1^2 + \kappa^2)\,\zeta = 0, \quad (73)$$

for $0 < x < a$, together with the condition

$$\left(i\sigma \frac{\partial}{\partial x} + 2\omega \frac{\partial}{\partial y} \right) \zeta = 0, \quad (74)$$

at $x = 0$ and $x = a$.

Let us consider

$$\zeta = (A_n e^{inx} + B_n e^{-inx})\, e^{imy}, \quad (75)$$

which satisfies (73), provided

$$m^2 + n^2 = \kappa^2.$$

* *L.c.*; see Lamb, *l.c.*, p. 304.

Now
$$i\sigma\frac{\partial\xi}{\partial x}+2\omega\frac{\partial\xi}{\partial y}=-n\sigma(A_ne^{inx}-B_ne^{-inx})\,e^{imy}$$
$$+2i\omega m\,(A_ne^{inx}+B_ne^{-inx})\,e^{imy}. \qquad (76)$$

For this to vanish at $x=0$ and $x=a$ we must have either $n=s\pi/a$, where s is any integer, and

$$-A_n(n\sigma-2i\omega m)+B_n(n\sigma+2i\omega m)=0,$$

or else $B_n=0$ and $n\sigma=2i\omega m$.

The second of these alternatives makes (74) to be satisfied for all values of x, and leads to Lord Kelvin's solution for the canal which was mentioned in the last section.

The first gives

$$\xi=\frac{1}{2\sigma}\left[\left(\frac{s\pi\sigma}{a}+2i\omega m_s\right)e^{is\pi x/a}+\left(\frac{s\pi\sigma}{a}-2i\omega m_s\right)e^{-is\pi x/a}\right]e^{im_sy},$$

or
$$\xi=\left(\frac{s\pi}{a}\cos\frac{s\pi x}{a}-\frac{2\omega}{\sigma}m_s\sin\frac{s\pi x}{a}\right)e^{im_sy}, \qquad (77)$$

where
$$m_s^2=\kappa^2-\frac{s^2\pi^2}{a^2},$$

A_n having been taken equal to $(s\pi\sigma/a+2i\omega m_s)/2\sigma$.

However, if m_s is imaginary (77) is not applicable, so that the only admissible values of the integer s are those which make m^2 positive. There need not be any if κ is small enough, as $s=0$ does not lead to a solution.

Symmetrical Tides of a Small Circular Sea on a Sphere.

13. Finally, we consider the free tidal motion of a sea which is situated on a rotating sphere and bounded by a small circle with its centre at one of the poles.

Let R be the radius of the sphere, h the (uniform) depth of the sea, and ω the angular velocity of the sphere which is supposed to be rotating about the axis through the pole P of the small circle. Let r be the distance (measured along a great circle) of any point on the sphere from P, and let $r=a$ be the equation of the small circle.

We suppose that we may neglect a^3/R^3. Now the equation of *free*

motion (symmetrical about the axis) of the water in such a sea is *

$$\frac{1}{R^2 \sin\theta}\frac{\partial}{\partial\theta}\left\{\frac{\sin\theta}{\frac{\sigma^2}{4\omega^2}-\cos^2\theta}\frac{\partial\zeta}{\partial\theta}\right\}+\frac{4\omega^2}{gh}\zeta=0, \tag{78}$$

where ζ and σ have their previous meanings and $\theta = r/R$.

The condition to be satisfied at the boundary $r = a$ is that $\partial\zeta/\partial\theta = 0$ there.

We now put $\sin\theta = \frac{r}{R}$, $\cos\theta = 1-\frac{1}{2}\frac{r^2}{R^2}$,

and the equation (78) becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left\{\frac{r}{\frac{\sigma^2}{4\omega^2}-1+\frac{r^2}{R^2}}\frac{\partial\zeta}{\partial r}\right\}+\frac{4\omega^2}{gh}\zeta=0. \tag{79}$$

Now to our order of small quantities

$$\left(\frac{\sigma^2}{4\omega^2}-1+\frac{r^2}{R^2}\right)^{-1}=\left(\frac{\sigma^2}{4\omega^2}-1\right)^{-1}\left(1-\frac{r^2}{b^2}\right),$$

where $$b^2=\left(\frac{\sigma^2}{4\omega^2}-1\right)R^2, \tag{80}$$

and so (79) becomes $$\frac{1}{r}\frac{\partial}{\partial r}\left\{\left(r-\frac{r^3}{b^2}\right)\right\}\frac{\partial\zeta}{\partial r}+\kappa^2\zeta=0, \tag{81}$$

where $$\kappa^2=\left(\frac{\sigma^2}{4\omega^2}-1\right)\frac{4\omega^2}{gh}=\frac{\sigma^2-4\omega^2}{gh}. \tag{82}$$

The equation to be satisfied at $r = a$ is $\partial\zeta/\partial r = 0$. To solve (81) we assume

$$\zeta = J_0(\kappa r)+\frac{1}{b^2}F(\kappa r). \tag{83}$$

Then to our order we have

$$\frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}F(\kappa r)\right\}+\kappa^2F(\kappa r)=\frac{1}{r}\frac{d}{dr}\left\{r^3\frac{d}{dr}J_0(\kappa r)\right\}. \tag{84}$$

It is easily verified that a particular solution of this equation is given by

$$F(\kappa r)=\frac{1}{6\kappa^2}\left[2(\kappa r)^2J_0(\kappa r)+\left\{4(\kappa r)+(\kappa r)^3\right\}J_0'(\kappa r)\right]; \tag{85}$$

* See Lamb, *l.c.*, p. 316. The mutual attraction of the particles of water is here neglected.

and, as any other admissible solution can be obtained from this by adding a constant multiple of $J_0(\kappa r)$, this is the one we require.

The condition to be satisfied at the boundary $r = a$ gives

$$J_0'(\kappa a) + \frac{1}{b^2} F'(\kappa a) = 0. \tag{86}$$

Let us assume for a moment that a solution of (86) differs from one of $J_0'(\kappa a) = 0$ by a quantity of the order a^2/b^2. Suppose $J_0'(a) = 0$; then to our order (86) may be written

$$(\kappa a - a) J_0''(a) + \frac{1}{b^2} F'(a) = 0, \tag{87}$$

in which we may put

$$b^2 = \frac{\kappa^2 ghR^2}{4\omega^2} = \frac{a^2 ghR^2}{4\omega^2 a^2}, \tag{88}$$

so that

$$\kappa a = a - \frac{4\omega^2 a^2}{a^2 ghR^2} \frac{F'(a)}{J_0''(a)}. \tag{89}$$

The above assumption will be justified if the last term in (89) is of the order a^2/R^2.

Since $J_0'(a) = 0$, we may put in (89) $J_0'(a) = -J_0(a)$, and we find that $F'(a) = -\frac{1}{6} a^2 a J_0(a)$, so that κa is given by

$$\kappa a = a - \frac{2\omega^2 a^4}{3agh R^2}. \tag{90}$$

The elevation of the water ξ is then given by

$$\xi = J_0(\kappa r) + \frac{2\omega^2 a^4}{3a^4 ghR^2} [2(\kappa r)^2 J_0(\kappa r) + \{4(\kappa r) + (\kappa r)^3\} J_0'(\kappa r)], \tag{91}$$

where the time factor $e^{i\sigma t}$ is omitted, σ being given by (82).

The first three values of a are 3·8317, 7·0156, 10·1735 approximately.*

It is interesting to notice that, to our order of approximation, when $\omega = 0$, the results are the same as for a plane.

At the boundary the ratio of the second term in (91) to the first is

$$\frac{4\omega^2 a^4}{3a^2 ghR^2},$$

* See Gray and Mathews, *Treatise on Bessel Functions*, p. 280.

which is positive, so that the effect of the curvature is to make the rise and fall of the water near the boundary slightly more marked, as compared with that at the pole.

Again, since the second term on the right-hand side of (90) is small compared with the first and of opposite sign, we see that the value of κ^2 and therefore of σ^2, is diminished by the curvature. This means that the periods are slightly increased.

The *forced* symmetrical tides on such a sea may be easily calculated to the same order of approximation; the results are a little more complicated.

FIGURES IN n-DIMENSIONAL SPACE ANALOGOUS TO ORTHOCENTRIC TETRAHEDRA

By T. C. LEWIS.

[Received and read February 13th, 1913.—Received, in revised form, March 28th, 1913.]

1. It is proposed to investigate the properties of rectilineal figures in space of n-dimensions formed by joining $n+1$ vertices, subject to the condition that the perpendiculars from the vertices upon the opposite $(n-1)$-dimensional faces meet in a point, which is called the orthocentre of the figure, or the orthocentre of the $n+1$ vertices.

(*a*) Let A, B be any two vertices, and P the orthocentre of the figure. Now the straight line CP, which joins any other vertex C to P, is by hypothesis at right angles to every straight line in the opposite $(n-1)$-dimensional face, and it is therefore at right angles to AB. Therefore AB is at right angles to the $(n-1)$-dimensional space which contains the remaining $n-1$ vertices and the orthocentre P; and it meets that space at a point P_1 which is the foot of the perpendicular from any point of that space upon AB. It thus appears that any two non-intersecting edges are at right angles to one another. The point P_1 may be regarded as the orthocentre of the edge AB of the figure.

(*b*) Take the triangle ABC. Its plane is at right angles to the $(n-2)$-dimensional space which contains the remaining $n-2$ vertices and P; and it meets it at a point P_2 which is the foot of the perpendicular from any point of that space upon ABC. Since DP_1 and CP_1 are at right angles to AB, P_2 must lie on CP_1, *i.e.*, it must lie on the perpendicular from any angular point of the triangle ABC upon the opposite side, therefore it is the orthocentre of the triangle.

(*c*) Proceeding in the same manner it may be demonstrated that every figure formed with any given number of the given vertices has an orthocentre which is the foot of the perpendicular from any of the remaining vertices and from P upon the space of lowest dimensions which contains that figure; and is, reciprocally, the foot of the perpendicular from any

point of that figure or space upon the space defined by the remaining vertices and P.

The letter P will be used in general to indicate an orthocentre, and any suffix to it will denote the space-dimensions of the figure to which it relates; the suffix n, however, will generally be omitted, only smaller suffixes being shewn.

2. If the process be continued by taking one more vertex at each operation until the last vertex N is included, NP will be at right angles to the space of $n-1$ dimensions containing the n vertices $A, B, \ldots, M$, and will meet it in the point P_{n-1}, viz., the orthocentre of the figure formed with those n vertices, which therefore lies on the line joining P_{n-2} to M (Fig. 1), so that

$$P_{n-1}P.P_{n-1}N = MP_{n-1}.P_{n-1}P_{n-2}.$$

Therefore N and P are interchangeable, *i.e.*, if P is a vertex instead of N, then N is the orthocentre of the figure. Hence the $n+1$ vertices and P constitute $n+2$ points such that each is the orthocentre of all the rest.

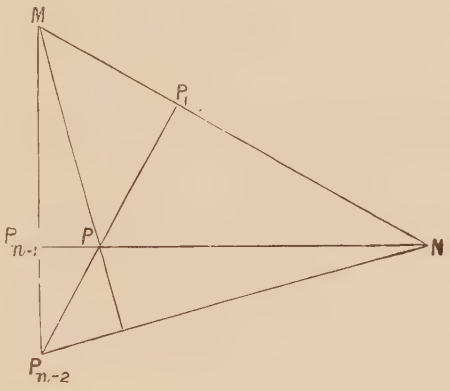


FIG. 1.

Also P_1 is at once the orthocentre of two points, say AB, and of the system $C, \ldots, N, P$; P_2 is the orthocentre of ABC and of the system $D, \ldots, N, P$; and so on. Therefore, in general, the $n+2$ points may be divided in any way into two parts, and the two parts or sets will have a common orthocentre.

Moreover the orthocentres of two sets which include $n+1$ of the $n+2$ points, none being repeated, will lie on a line through the remaining point.

If three sets be made from the $n+2$ points their three orthocentres will lie on a straight line, so that any one of them is the foot of the perpendiculars from the vertices with which it is associated upon the line joining the other two, of which line it is therefore the orthocentre. So with four sets the four orthocentres will lie in a plane and each will be the orthocentre of the triangle formed by the other three.

And in general the orthocentres of any $r+2$ sets into which the $n+2$ points may be divided lie in a space of r dimensions, and each is the

orthocentre of the figure of which the remaining $r+1$ orthocentres are the vertices.

3. The figure with $n+1$ vertices is self-conjugate with respect to a hypersphere of n dimensions (which may be called an n-sphere) having its centre at the orthocentre of the figure. For, as in Fig. 1, $NP.PP_{n-1}$ is constant for any two vertices M and N; therefore it is constant for all vertices. Also on any line P_1PP_{n-2}, $P_1P.PP_{n-2}$ is equal to the same constant. Moreover on any line $P_{r-1}PP_{n-r}$ (Fig. 2), the product $P_{r-1}P.PP_{n-r}$ is seen to be equal to the same constant.

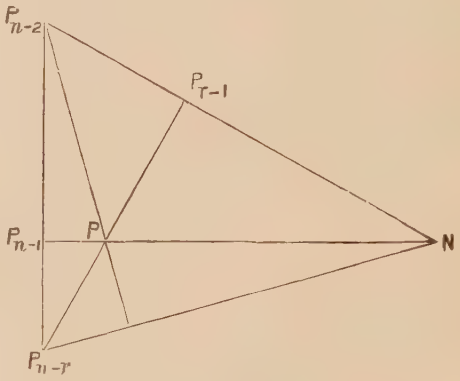


FIG. 2.

Thus there is a constant associated with each of the $n+2$ points; this may be taken to be the square of the radius of the hypersphere with respect to which the figure having vertices at the remaining $n+1$ points is self-conjugate.

These $n+2$ fundamental hyperspheres constitute the single, definite orthotomic system having centres at the $n+2$ given points, for, in Fig. 1, the sum of the constants for M and N is MN^2.

Let $\alpha, \beta, \gamma, \ldots, \nu, \varpi$ be the constants for the points $A, B, C, \ldots, N, P$ respectively.

If P_1 is the orthocentre of AB,

$$\alpha : \beta :: AP_1 : BP_1.$$

Hence it follows that P is the centroid of particles proportional to $1/\alpha, 1/\beta, \ldots, 1/\nu$ placed at $A, B, \ldots, N$ respectively. Therefore the centroid of masses proportional to $1/\alpha, 1/\beta, \ldots, 1/\nu, 1/\varpi$ at $A, B, \ldots, N, P$ respectively is also at P.

Similarly, it is also at any other of the given $n+2$ points. We thence deduce the following relation between the constants,

$$1/\alpha+1/\beta+1/\gamma+\ldots+1/\nu+1/\varpi = 0,$$

or

$$\Sigma\, 1/\alpha+1/\varpi = 0.$$

4. Let AP_r be the line joining A to the orthocentre P_r of the figure of $r+1$ vertices of which A is one, but not P. Then for all values of r

less than n, AP_r and PP_r are at right angles to one another. Therefore the hypersphere in n dimensions on AP as diameter passes through the orthocentres of all edges, triangles, tetrahedra, &c., of which A is a vertex, but not P; A, P being any two of the $n+2$ given points.

5. A hypersphere in n dimensions, being determined in general by $n+1$ points, can be drawn through A and B to pass through the orthocentres of the lines CA, CB, *or* through the orthocentres of the triangles ABC, where C is any one of the remaining $n-1$ vertices. In the latter case it also intersects CA, CB at twice the distance from C of the orthocentre of CA, CB respectively.

6. Similarly a hypersphere can be drawn through the angular points of a triangle ABC to pass through

(i) the orthocentres of the edges DA, DB, DC, where D is any other vertex;

or (ii) the orthocentres of all triangles DAB, DBC, DCA, and to intersect DA, ... at twice the distance of its orthocentre from D;

or (iii) the orthocentres of all tetrahedra $ABCD$, and to intersect DA, ... at three times the distance of its orthocentre from D;

or to satisfy (i) with regard to some of the vertices D, (ii) with regard to others, and (iii) with regard to the rest.

And similar statements may be made with regard to analogous figures in higher space.

7. Let ABC ... be a figure with $r+1$ vertices in r dimensions, P_r its orthocentre, G_r its centre of gravity. Draw G_rH at right angles to AP_r meeting it in the point H; and let AP_r produced meet the face BC, ... opposite to A in the point P_{r-1} (Fig. 3), and let G_{r-1} be the centroid of that face. The

$$AH.AP_r = \frac{r}{r+1} AP_r.AP_{r-1}$$

$$= \frac{r}{r+1} a.$$

This is therefore the square of the tangent from A to the hypersphere of not less than r dimensions on P_rG_r as diameter; and similarly for other vertices.

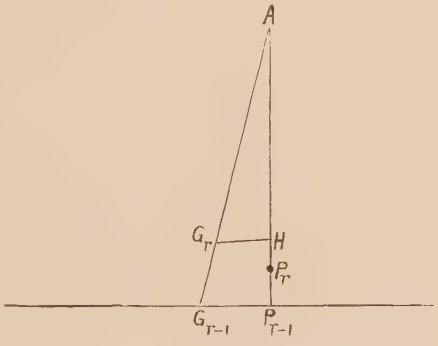


FIG. 3.

8. Let O be the circumcentre and G the centroid of the figure.

All the perpendiculars upon AB from P and the remaining vertices, from their centroid G_{n-2} and their orthocentre P_{n-2} meet it in the point P_1.

Let G_1 be the middle point of AB. Then G lies on G_1G_{n-2} and divides it in the ratio $n-1 : 2$.

And OG_1 is at right angles to AB, though not necessarily parallel to PP_1. Therefore OP is divided in the same ratio $n-1 : 2$ by the $(n-1)$-space through G at right angles to AB.

And the same is true for each of the $\frac{1}{2}n(n+1)$ cases arising from the different pairs of points A, B that may be taken.

But in general in n-space, n spaces of $n-1$ dimensions determine a point by their intersection. Now all these $(n-1)$-spaces to the number of $\frac{1}{2}n(n+1)$ intersect in G, and it has been shewn that they all intersect OP in a definite point. Therefore that point must be G. Therefore G lies on OP and divides it in the ratio $n-1 : 2$.

9. It is obvious that the circumhypersphere and the n-sphere circumscribing the figure whose vertices are the points G_{n-1} have G as their internal centre of similitude. The centre of the latter hypersphere will therefore be on OP, at O_{n-1} (Fig. 4) such that

$$GO_{n-1} : GO :: 1 : n,$$

but $$PG : PO :: 2 : n+1;$$

therefore

$$PO_{n-1} : PO :: 1 : n.$$

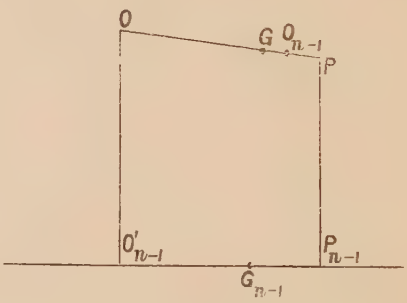


FIG. 4.

Therefore P is the external centre of similitude, and the hypersphere through the points G_{n-1} will intersect PA at $1/n$-th of its length measured from P.

Also since $$P_{n-1}G_{n-1} = 2/n . P_{n-1}O'_{n-1},$$

O'_{n-1} being the circumcentre of the figure to which P_{n-1} belongs, it follows that O_{n-1} is equidistant from G_{n-1} and P_{n-1}, and therefore that the n-sphere through the points G_{n-1} will pass through the points P_{n-1}; and $P_{n-1}G_{n-1}$ is the diameter of the section of this n-sphere by the $(n-1)$-face of P_{n-1}, G_{n-1}. Also if an n-sphere has this section on one $(n-1)$-face and passes through G_{n-1} of any other such face, it will have the same characteristic section by all the $(n-1)$-faces.

10. Thus, if $r+2$ of the $n+1$ vertices be taken they will constitute an orthocentric figure of $r+1$ dimensions, and an $(r+1)$-sphere will pass through the $r+2$ r-spheres on the diameters P_rG_r of the r-faces.

Consider this as part of the full figure of $n+1$ vertices.

Describe an n-sphere whose section by any r-face is the r-sphere on the diameter P_rG_r in that face (determining $r+1$ independent points) and also passing through the point G_r of one other r-face of each of the $(r+1)$-figures whose vertices are the $r+1$ vertices in the chosen r-face and one of the remaining $n-r$ vertices (determining in all $r+1+n-r$, *i.e.*, $n+1$ points, as required). Then it is apparent from what has been established in the preceding article that the n-sphere so described will intersect every r-face of the n-figure in the r-sphere whose diameter is the line P_rG_r of that face.

Hence for all values of $r < n$, the r-spheres on the diameters P_rG_r of all r-faces lie on an n-sphere.

11. If G_r, G_{n-r-1} are the centroids, and P_r, P_{n-r-1} the orthocentres of two sets of $r+1$ and $n-r$ vertices respectively into which the given $n+1$ vertices may be divided, G_rG_{n-r-1} is divided at G in the ratio $n-r : r+1$ (Fig. 5).

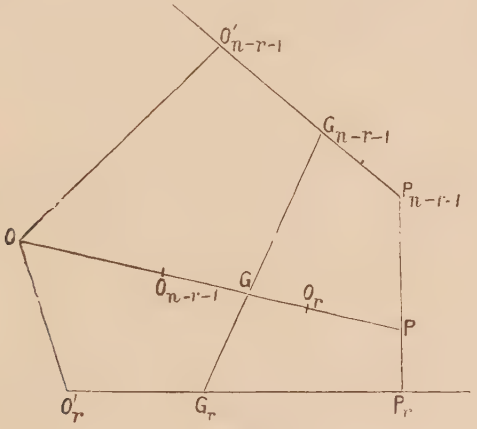


FIG. 5.

Also G lies on PO and divides it in the ratio $2 : n-1$; and P lies on P_rP_{n-r-1}, which it divides into two segments whose product is $-\varpi$.

Now G_r divides $P_rO'_r$ in the ratio $2 : r-1$, and G_{n-r-1} divides $P_{n-r-1}O'_{n-r-1}$ in the ratio $2 : n-r-2$, where O'_r, O'_{n-r-1} are respectively the circumcentres of the $r+1$ and $n-r$ points of the two sets under consideration.

A hypersphere can be described with centre O_r on OP so as to be intersected by the r-space of P_r in the r-sphere on $P_r G_r$ as diameter when

$$PO_r : PO :: 1 : r+1.$$

Therefore the centre of the n-sphere which, as has been proved, passes through all points P_r, G_r and has its centre on OP is at O_r, where

$$PO_r = \frac{1}{r+1} PO.$$

Similarly all points G_{n-r-1}, P_{n-r-1} lie on the n-sphere with centre at O_{n-r-1}, where

$$PO_{n-r-1} = \frac{1}{n-r} PO.$$

Thus, as before, G and P are the centres of similitude of the two hyperspheres. Therefore, the radii being ρ_r and ρ_{n-r-1} respectively,

$$(r+1)\rho_r = (n-r)\rho_{n-r-1},$$

and the hypersphere through G_r, P_r cuts PP_{n-r-1} at Q, when

$$PQ = \frac{n-r}{r+1} PP_{n-r-1},$$

and the hypersphere through G_{n-r-1}, P_{n-r-1} cuts PP_r at a distance from P equal to $\frac{r+1}{n-r} PP_r$.

12. Also $$\rho_r^2 - PO_r^2 = QP.PP_r = -\frac{n-r}{r+1}\varpi;$$

therefore $$(r+1)^2\rho_r^2 = PO^2 - (n-r)(r+1)\varpi;$$

therefore $$R^2 = PO^2 - n\varpi;$$

therefore $$(r+1)^2\rho_r^2 = R^2 - r(n-r-1)\varpi.$$

But since the fundamental spheres constitute an orthotomic system, it is known that

$$\frac{\Sigma\alpha}{4R^2} + \frac{(n-1)^2\varpi}{4R^2} = 1;$$

therefore $$4R^2 = \Sigma\alpha + (n-1)^2\varpi,$$

whence $$4.OP^2 = \Sigma\alpha + (n+1)^2\varpi.$$

Substituting the value of R^2 in the expression for ρ_r^2,

$$4(r+1)^2\rho_r^2 = \Sigma\alpha + (n-2r-1)^2\varpi,$$

which gives a simple expression for ρ_r in terms of the fundamental constants.

13. The series of hyperspheres with radius ρ_r, where r is any integer $\geqslant n$, belong to a system of such figures, in which r may vary continuously, to which the remarkable property appertains that they have a common section of $n-1$ dimensions, *i.e.*, they have a common radical $(n-1)$ dimensional space.

The equation to any hypersphere may be written, with P as origin and OP produced as axis of x, y being the radius of the $(n-1)$-section at right angles to OP,

$$\left(x+\frac{l}{r+1}\right)^2+y^2=\frac{l^2}{(r+1)^2}-\frac{n-r}{r+1}\varpi,$$

where $$l=OP.$$

The intersection of this with the hypersphere for which r' takes the place of r, whether the difference between r and r' be infinitesimal or not, is such that

$$2lx=-(n+1)\varpi,$$

which is independent of r and r'. Therefore all the hyperspheres, whatever value be given to r will have a common section at right angles to OP through the point determined by this value of x. The corresponding value of y is given by

$$4l^2y^2=\varpi\Sigma a.$$

14. Let T be the point on OP determined by the relation

$$2lx=-(n+1)\varpi.$$

Then $$GP.PT=P_rP.PP_{n-r-1}.$$

Therefore the circle in the plane P_rPG which passes through P_r, G and P_{n-r-1} also passes through T. And there are 2^n-1 such circles which have a common chord GT.

Thus in two dimensions the circumscribing circle, the nine-points circle, the circle on PG as diameter, and the fundamental circle at P have a common radical axis (and chord) through T at right angles to OP; and the three circles passing each through G, a vertex, and the foot of the perpendicular on the opposite side have a common radical axis (and chord) GT.

Similarly in three dimensions there are five spheres having a common circular section with centre at T, and seven circles with a common chord GT.

15. It may be shewn that any centre O_r is the centroid of a set of masses at the vertices, viz.,

$$\Sigma \frac{1}{\alpha} - \frac{n-2r-1}{\alpha} \text{ at } A, \ldots,$$

and O_{n-r-1} is the centroid of a set of masses, viz.,

$$\Sigma \frac{1}{\alpha} + \frac{n-2r-1}{\alpha} \text{ at } A, \ldots .$$

Thus, if n be odd, and $r = \frac{1}{2}(n-1)$,

O_r will coincide with the centre of gravity G, as, for example, in three dimensions; and in this case the hyperspheres with centres O_r and P cut orthogonally, for

$$\rho_r^2 + \varpi = PO_r^2.$$

If n be even, and $r = \frac{1}{2}n$,

O_r will coincide with the centre of gravity of the $n+2$ given points, *i.e.*, of equal particles placed at those points; and therefore remains in the same position for all the figures, whichever of the given points is considered as orthocentre of the remaining vertices.

16. If Δ_n be the n-dimensional contents of the figure

$$n!\,\Delta_n = \frac{\rho_\alpha \rho_\beta \ldots \rho_r}{\rho_\varpi},$$

when ρ_α is the radius of the fundamental sphere at A, omitting any irrational factor $\sqrt{(-1)}$ where it occurs.

Also the sum of the squares of the edges of any set of polygons having between them only one angular point at each vertex is constant, and equal to $2\Sigma\alpha$.

If P be added to the number of vertices the sum is still constant and equal to $2\Sigma\alpha + 2\varpi$.

In the case of the four triangles that can be drawn from four vertices A, B, C, P in two dimensions, it is known that R is constant for them all. A corresponding property in n dimensions is

$$(n+2)R^2 - (n-2)\,OP^2 = \Sigma\alpha + \varpi,$$

which is constant for the $n+2$ figures for which any one of the given $n+2$ points is orthocentre.

When n is even there is another interesting general property of which the same two-dimensional property is a particular case. In two dimensions both R and ρ_1 are constant for all the four triangles involved, and the

circle with ρ_1 as radius is identically the same, in position as well as in magnitude, in the four cases.

The corresponding general property is that if n is even, and $r = \frac{1}{2}n$, ρ_r and ρ_{r-1} are constant for all the $n+2$ figures formed by taking the $n+2$ points one by one as orthocentre, and the n-sphere with radius ρ_r is identically the same for these figures.

If r is the radius of the inscribed hypersphere

$$\frac{1}{r} = \Sigma \sqrt{\left(\frac{\varpi}{\alpha \varpi_\alpha}\right)},$$

where ϖ_α is the same as ϖ_{n-1} for the face opposite to the vertex A.

Other formulæ may be easily proved, for example,

$$\Sigma PA^2 - 4R^2 = -n(n-3)\varpi,$$

which vanishes when $n = 3$.

INDEX.

INDEX OF PROCEEDINGS.

[*The numbers refer to pages.*]

Admissions of Members, iv, xx, xxxiv, xliii.

Annual General Meeting (November, 1912), iv.

Auditor's Report, xiii.

Council and Officers for the Session 1912–1913, iv.

Elections of Members, xii, xx, xl.

Exchanges of Publications, xlix.

Number of Members, xx.

Presents to Library, xlix.

Treasurer's Report, iv, xiii.

INDEX OF PAPERS READ AND COMMUNICATIONS MADE DURING THE SESSION NOVEMBER, 1912—JUNE, 1913.

[*The names of Authors are given and the numbers of the pages on which the titles of the papers appear. The Abstract of a paper will be found on a page soon after that containing the title.*]

Baker, H. F., xiii, xl.

Bateman, H., xx.

Berwick, W. E. H., xx.

Bohr, H., xxiv.

Bromwich, T. J. I'A., xlvi.

Burnside, W., v, xlii.

Carslaw, H. S., xlii.

Cunningham, A., xxxiv.

Cunningham, E., v.

Curzon, H. E. J., xiii.

Dickson, L. E., xl.

Fields, J. C., xx.

Grieve, A. B., v.

Hammond, J., xlvi.

Hardy, G. H., xiii, xxiv.

Hardy, G. H., and Littlewood, J. E., xxiv, xlvi.

Hassé, H. R., xiii.

Heawood, P. J., xxv.

Hobson, E. W., v, xiii.

Holt, J. B., v, xxv.

Jolliffe, A. E., and Shovelton, S. T., xl.

Landau, E., xlvi.

Larmor, Sir J., xlv.

Lees, S., xxxiv, xlii.

Lewis, T. C., xxiv.

Littlewood, J. E., xxiv.

Love, A. E. H., xx.

Macdonald, H. M., xxxiv.

McDonnell, J., xiv.

Mordell, L. J., xiv, xxxiv.

Piaggio, H. T. H., xxv.

Proudman, J., xxxiv, xxxiv.

Rose-Innes, J. R., xl.

Sheppard, W. F., xlii, xlii.

Watson, G. N., v.

Young, W. H., v, xiv, xx, xxv, xlii.

INDEX OF PAPERS PUBLISHED IN THIS VOLUME ARRANGED ACCORDING TO NAMES OF AUTHORS.

[*The numbers refer to pages.*]

Baker, H. F. On some Recent Advances in the Theory of Algebraic Surfaces, 1.

Berwick, W. E. H. The Classification of Ideal Numbers that depend on a Cubic Irrationality, 393.

Bromwich, T. J. I'A. Certain Potential Functions and a New Solution of Laplace's Equation, 100.

Burnside, W. On Groups of Linear Substitutions of Finite Order which possess Quadratic Invariants, 89.

Cunningham, E. The Theory of Functions of a Real Vector, 133.

Curzon, H. E. J. On a Connexion between the Functions of Hermite and the Functions of Legendre, 236.

Fields, J. C. Proofs of certain General Theorems relating to Orders of Coincidence, 218.

Grieve, A. B. Some Points in the Geometry of Cubic Surfaces, 315.

Hardy, G. H. An Extension of a Theorem on Oscillating Series, 174.

——— On the Summability of Fourier Series, 365.

Hassé, H. R. The Equations of the Theory of Electrons transformed relative to a System in Accelerated Motion, 181.

Heawood, P. J. On a Graphical Demonstration of the Fundamental Properties of Quadratic Residues, 373.

Hilton, H. Some Properties of Symmetric and Orthogonal Substitutions, 94.

Hobson, E. W. On the Representation of a Summable Function by a Series of Finite Polynomials, 156.

——— On the Convergence of Series of Orthogonal Functions, 297.

Holt, J. B. On the Irreducibility of Legendre's Polynomials, 126.

Lewis, T. C. Figures in n-Dimensional Space analogous to Orthocentric Tetrahedra, 474.

Love, A. E. H. Notes on the Dynamical Theory of the Tides, 309.

Macdonald, H. M. The Diffraction of Light by an Opaque Prism, 430.

Piaggio, H. T. H. Some Non-Primary Perpetuant Syzygies of the Second Kind, 377.

Proudman, J. On some Cases of Tidal Motion of Rotating Sheets of Water, 453.

Watson, G. N. Some Properties of the Extended Zeta Function, 288.

Young, W. H. On the Fourier Series of Bounded Functions, 41.

——— On the Determination of the Summability of a Function by Means of its Fourier Constants, 71.

——— On Derivates and their Primitive Functions, 207.

——— On Functions and their Associated Sets of Points, 260.

——— On Uniform Oscillation of the First and Second Kind, 340.

——— On the Mode of Oscillation of a Fourier Series and its Allied Series, 433.

INDEX ACCORDING TO SUBJECTS.

[*The numbers refer to pages.*]

Equations, Theory of: Holt, J. B., 126.
Fourier's Series: Hardy, G. H., 365; Young, W. H., 41, 71, 433.
Forms, Algebraic, Theory of: Piaggio, H. T. H., 377.
Functions of a Complex Variable: Fields, J. C., 218.
Functions of a Real Variable: Young, W. H., 207, 260, 340.
Functions, Orthogonal, Theory of: Hobson, E. W., 297.
Functions, Special, Theory of: Curzon, H. E. J., 236; Watson, G. N., 288.
Geometry of n-Dimensions: Lewis, T. C., 474.
Groups of Finite Order: Burnside, W., 89.
Light, Electromagnetic, Theory of: Macdonald, H. M., 430.
Numbers, Theory of: Berwick, W. E. H. 393; Heawood, P. J., 373.
Potential, Theory of: Bromwich, T. J. I'A., 100.
Relativity, Theory of: Hassé, H. R., 181.
Series, Theory of: Hardy, G. H., 174; Hobson, E. W., 156.
Substitutions, Linear: Burnside, W., 89: Hilton, H., 94.
Surfaces, Theory of: Baker, H. F., 1.
Surfaces, Special: Grieve, A. B., 315.
Tides, Theory of: Love, A. E. H., 309; Proudman, J., 453.
Vectors, General Theory of: Cunningham, E., 133.

BIBL.
LONDIN.
UNIV.

Printed by C. F. Hodgson & Son, 2 Newton Street, Kingsway, W.C.

PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY.

SERIES 2.—VOL. 12.—PART 2.

CONTENTS.

ON THE DETERMINATION OF THE SUMMABILITY OF A FUNCTION BY MEANS OF ITS FOURIER CONSTANTS (*continued*): PROF. W. H. YOUNG 81

ON GROUPS OF LINEAR SUBSTITUTIONS OF FINITE ORDER WHICH POSSESS QUADRATIC INVARIANTS: PROF. W. BURNSIDE 89

SOME PROPERTIES OF SYMMETRIC AND ORTHOGONAL SUBSTITUTIONS: MR. H. HILTON 94

CERTAIN POTENTIAL FUNCTIONS AND A NEW SOLUTION OF LAPLACE'S EQUATION: DR. T. J. I'A. BROMWICH 100

ON THE IRREDUCIBILITY OF LEGENDRE'S POLYNOMIALS (SECOND PAPER): MR. J. B. HOLT 126

THE THEORY OF FUNCTIONS OF A REAL VECTOR: MR. E. CUNNINGHAM ... 133

ON THE REPRESENTATION OF A SUMMABLE FUNCTION BY A SERIES OF FINITE POLYNOMIALS: PROF. E. W. HOBSON 156

Issued April 9th, 1913.

LONDON:

PRINTED BY C. F. HODGSON & SON,

2 NEWTON STREET, KINGSWAY, W.C.;

AND PUBLISHED FOR THE SOCIETY BY

FRANCIS HODGSON, 89 FARRINGDON STREET, E.C.

THE LONDON MATHEMATICAL SOCIETY is instituted for the promotion and extension of Mathematical Knowledge.

It was founded in 1865, and incorporated under Section 23 of the Companies Act 1867 in 1894.

Every Candidate for Membership must be proposed and recommended, according to a form, which the Secretaries will supply, by not less than three Members, of whom one at least, except in special cases to be submitted for the decision of the Council, must certify his personal knowledge of the Candidate.

This form is read at one of the Ordinary or Annual General Meetings, and the Candidate is balloted for at the next ensuing meeting, provided that seven Members are present thereat.

The Candidate, if elected, is informed of his election by one of the Secretaries, and supplied with a copy of the Memorandum and Articles of Association and By-Laws. He must pay the contribution which is due from him within six months after the day of his election, otherwise his election shall be void.

An entrance fee of one guinea is required to be paid by each newly elected Member.

The Annual Subscription to be paid by each Member is one guinea: any Member may compound for his *annual* subscriptions by the payment of fifteen (or in certain cases ten) guineas in one sum.

Every Member is considered liable for his annual subscription until he has signified in writing his desire to resign, and has returned all books and property belonging to the Society.

Annual subscriptions are due at the beginning of each Session (the second Thursday in November of each year), and may be paid to the account of the London Mathematical Society, London County and Westminster Bank, 1 St. James's Square, London, S.W., or to the Treasurer, Sir J. Larmor, at St. John's College, Cambridge.

The affairs of the Society are directed by the Council and Officers.

The Council consists of sixteen Members, including the Officers, and is chosen from among the Ordinary Members of the Society at the Annual General Meeting, held on the second Thursday in November.

The Officers are a President, Vice-President, a Treasurer, and Secretaries.

The Ordinary Meetings of the Society are held at its Rooms, 22 Albemarle Street, and commence at 5.30 o'clock in the evening. The dates of meeting for the Session 1912–1913 are the *second* Thursdays in November, December, January, February, March, May, June, and the *fourth* Thursday in *April*. Cards giving the Programmes of the meetings as they occur can be had by Members a few days beforehand on application to C. F. Hodgson & Son, 2 Newton Street, Kingsway, London, W.C.

At these meetings papers are read and communications made: upon each paper or communication the Chairman invites discussion.

The Council alone decides whether any paper proposed for reading shall or shall not be read.

After a paper has been presented to the Society, it is referred by the Council to two or more Members, who report to the Council on its fitness for publication in the *Proceedings*. After hearing the reports, the Council decides by ballot whether it shall be printed or not.

Authors of Papers intended for communication to the Society are requested to furnish to the Secretaries short abstracts of their Papers, indicating the nature of the methods employed and the character of the results obtained.

Communications for the Secretaries may be forwarded to them at the following addresses:—

London Mathematical Society, 22 Albemarle Street, W. { J. H. Grace. / T. J. I'A. Bromwich.

Peterhouse, Cambridge.—J. H. Grace.

1 Selwyn Gardens Cambridge.—T. J. I'A. Bromwich.

AUTHORS of Papers printed in the *Proceedings* are entitled to receive, free of cost, 25 separate copies of their communications, stitched in a paper cover. They can, however, by previous notice to the Printers, obtain as many copies as they wish, at the following rates:—

Of a Paper not exceeding	25 copies.	50 copies.	75 copies.	100 copies.
		s. *d.*	*s.* *d.*	*s.* *d.*
4 pages	Free	3 6	5 0	6 0
8 ,, ...	,,	4 6	6 6	8 0
12 ,, ...	,,	7 0	10 6	13 0
16 ,, ...	,,	8 0	12 0	15 0
32 ,, ...	,,	14 0	21 0	26 0

Complete Volumes of the *Proceedings*, bound in cloth, 8vo, can be purchased by the public from the Publisher, Mr. FRANCIS HODGSON, 89 Farringdon Street, E.C., at the following rates: –

Vol.	*I.	Nos.	1–8	...	Jan. 1865	to Nov. 1866	...	10*s.*
,,	*II.	,,	9–20	...	Nov. 1866	,, ,, 1869	...	16*s.*
,,	III.	,,	21–40	...	,, 1869	,, ,, 1871	...	20*s.*
,,	IV.	,,	41–66	...	,, 1871	,, ,, 1873	...	31*s.* 6*d.*
,,	V.	,,	67–75	...	,, 1873	,, ,, 1874	...	15*s.*
,,	VI.	,,	76–88	...	,, 1874	,, ,, 1875	...	21*s.*
,,	VII.	,,	89–103	...	,, 1875	,, ,, 1876	...	21*s.*
,,	VIII.	,,	104–123	...	,, 1876	,, ,, 1877	...	21*s.*
,,	IX.	,,	124–140	...	,, 1877	,, ,, 1878	...	21*s.*
,,	X.	,,	141–155	...	,, 1878	,, ,, 1879	...	18*s.*
,,	XI.	,,	156–165	...	,, 1879	,, ,, 1880	...	12*s.* 6*d.*
,,	XII.	,,	166–179	...	,, 1880	,, ,, 1881	...	16*s.*
,,	XIII.	,,	180–194	...	,, 1881	,, ,, 1882	...	18*s.*
,,	XIV.	,,	195–214	...	,, 1882	,, ,, 1883	...	20*s.*
,,	XV.	,,	215–232	...	,, 1883	,, ,, 1884	...	20*s.*
,,	XVI.	,,	233–252	...	,, 1884	,, ,, 1885	...	20*s.*
,,	XVII.	,,	253–279	...	,, 1885	,, ,, 1886	...	25*s*
,,	XVIII.	,,	280–304	...	,, 1886	,, ,, 1887	...	25*s.*
,,	XIX.	,,	305–342	...	,, 1887	,, ,, 1888	...	30*s.*
,,	XX.	,,	343–369	...	,, 1888	,, ,, 1889	...	25*s.*
,,	XXI.	,,	370–398	...	,, 1889	,, ,, 1890	...	25*s.*
,,	XXII.	,,	399–429	...	,, 1890	,, ,, 1891	...	27*s.* 6*d.*
,,	XXIII.	,,	430–449	...	,, 1891	,, ,, 1892	...	20*s.*
,,	XXIV.	,,	450–474	...	,, 1892	,, ,, 1893	...	22*s.*
,,	XXV.	,,	475–499	...	,, 1893	,, ,, 1894	...	22*s.*
,,	XXVI.	,,	500–534	...	,, 1894	,, ,, 1895	...	30*s.*
,,	XXVII.	,,	535–574	...	,, 1895	,, ,, 1896	...	35*s.*
,,	XXVIII.	,,	575–611	...	,, 1896	,, ,, 1897	...	32*s.* 6*d.*
,,	XXIX.	,,	612–657	...	,, 1897	,, ,, 1898	Pt. I. 20*s.* Pt. II. 20*s.*	
,,	XXX.	,,	658–678	...	,, 1898	,, Mar. 1899	...	20*s.*
,,	XXXI.	,,	679–709	...	April 1899	,, Dec. 1899	...	20*s.*
,,	XXXII.	,,	710–737	...	Jan. 1900	,, June 1900	...	20*s.*
,,	XXXIII.	,,	738–763	...	Nov. 1900	,, Feb. 1901	...	20*s.*
,,	XXXIV.	,,	764–789	...	Mar. 1901	,, April 1902	...	20*s.*
,,	XXXV.	,,	790–819	...	May 1902	,, Jan. 1903	...	20*s.*

* It was ordered by the Council (June 8, 1911) that Vols. I and II of the first Series should not be sold except to purchasers of not less than twenty volumes of that Series.

Cases for binding the above Volumes can also be procured from the Publisher, price 1*s.* each.

A complete Index of all the papers printed in Vols. I to XXX of the *Proceedings* of the Society (112 pp.) and a List of Members of the Society from the foundation (1865) to Nov. 9th, 1399 (16 pp.), can be obtained gratis from the Society's Publisher, at the above address.

The price of the Second Series is 25*s.* each Volume (of about 500 pp.). The volumes may be subscribed for in advance by the public at the above price, and may be obtained for this subscription either in separate Parts immediately on publication or in bound Volumes. Cases for binding the Volumes can be procured from the Publisher, price 1*s.* 3*d.* each.

Vols. 1 to 11 of Series 2 (1904–1912) have been published.

It was ordered by the Council (Feb. 13, 1896), "That separate Parts of the *Proceedings* may be obtained from the Publisher at 1*s.* 6*d.* per sheet, when the number in stock exceeds 100."

"That Members be allowed to complete Sets at trade price" [*i.e.*, at 1*s.* 4*d.* per sheet].

It was further ordered by the Council (March 9, 1899), "That the Society allow 5 per cent. extra discount to each purchaser of a *complete* set of *Proceedings up to the last completed volume.*"